Stable Hypersurfaces in a 4-Dimensional Sphere

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Abstract We study complete noncompact 1-minimal stable hypersurfaces in a 4-dimensional sphere $S^4$. We show that there is no complete noncompact 1-minimal stable hypersurfaces in $S^4$ with polynomial volume growth and the restriction of the mean curvature and Gauss-Kronecker curvature. These results are partial answers to the conjecture of Alencar, do Carmo and Elbert when the ambient space is a 4-dimensional sphere.

Keywords constant scalar curvature; 1-minimal stable hypersurfaces in space forms

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1. Introduction

Cheng and Yau [1] proved that any complete noncompact hypersurface in the Euclidean space with constant scalar curvature and nonnegative sectional curvature must be a generalized cylinder. It is natural to study the global properties of hypersurfaces in space forms with constant scalar curvature. Alencar, do Carmo and Elbert posed the following question: Is there any complete 1-minimal stable hypersurfaces in $\mathbb{R}^4$ with nonzero Gauss-Kronecker curvature? In [2], it was proved that there is no complete noncompact 1-minimal stable hypersurface $M$ in $\mathbb{R}^4$ with nonzero Gauss-Kronecker curvature and finite total curvature. Silva Neto [3] showed that there is no complete 1-minimal stable hypersurface in $\mathbb{R}^4$ with zero scalar curvature, polynomial volume growth and the restriction of the mean curvature and the Gauss-Kronecker curvature.

Motivated by our recent work of hypersurfaces in spheres in [4,5], we study the global properties of complete noncompact 1-minimal stable hypersurfaces in a 4-dimensional sphere $S^4$ in this paper. A Riemannian manifold $M^3$ has polynomial volume growth, if there exists $\gamma \in (0, 3]$ such that $\lim_{r \to \infty} \frac{\text{vol} B_r(p)}{r^\gamma} < +\infty$, for all $p \in M$, where $B_r(p)$ is the geodesic ball of radius $r$ in $M$. We show two non-existence results as follows:

**Theorem 1.1** There is no stable complete noncompact 1-minimal hypersurface $M^3$ in $S^4$ with polynomial volume growth and such that the mean curvature $H$ satisfying

$$|H| \leq \delta_1, \quad |\nabla \left( \frac{1}{H} \right)| \leq \delta_2,$$
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for any positive constants $\delta_1$ and $\delta_2$.

**Theorem 1.2** There is no stable complete noncompact 1-minimal hypersurface $M^3$ in $S^4$ with polynomial volume growth and such that

$$\frac{-K}{H^3} \geq \delta_1, |\nabla(\frac{1}{H})| \leq \delta_2,$$

for any positive constants $\delta_1$ and $\delta_2$, where $H$ and $K$ are the mean curvature and the Gauss-Kronecker curvature, respectively.

2. Preliminaries

Let $M^3$ be a complete Riemannian manifold and let $x : M^3 \to S^4$ be an isometric immersion into the sphere $S^4$ with constant scalar curvature. We choose a unit normal field $N$ to $M$ and define the shape operator $A$ associated with the second fundamental form of $M$, i.e., for any $p \in M$

$$A : T_pM \to T_pM$$

satisfies $(A(X), Y) = -(\nabla_X N, Y)$, where $\nabla$ is the Riemannian connection in $S^4$. Let $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of $A$. The $r$-th symmetric function of $\lambda_1, \lambda_2, \lambda_3$, denoted by $S_r$, is defined by

$$S_1 = \lambda_1 + \lambda_2 + \lambda_3,$$

$$S_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$$

$$S_3 = \lambda_1\lambda_2\lambda_3.$$

With the above notations, we call $H_r = \frac{S_r}{C_r}$ the $r$-mean curvature of the immersion. Obviously, $H_1 = H$ is the mean curvature and $K = H_3$ is the Gauss-Kronecker curvature. $H_2$ is, modulo a constant 1, the scalar curvature of $M$. The hypersurface $M$ is called $r$-minimal if $H_{r+1}\equiv 0$.

It is well known that hypersurfaces with constant scalar curvature in space forms are critical point for a geometric variational problem, namely, that associated to the functional

$$A_1(M) = \int_M S_1$$

under compactly supported variations that preserves the volume. Let

$$P_1 = S_1 Id - A : T_pM \to T_pM.$$

Obviously,

$$\text{trace}(P_1) = 2S_1.$$

We obtain the second variational formula for hypersurfaces in $S^4$ with constant 2-mean curvature [6]:

$$\frac{d^2A_1}{dt^2}\bigg|_{t=0} = \int_M (P_1(\nabla f), \nabla f) - \int_M (S_1S_2 - 3S_3 + 2S_1)f^2,$$

for each $f \in C_c^\infty(M)$. It is known that $M^3$ is stable if and only if

$$\int_M (S_1S_2 - 3S_3 + 2S_1)f^2 \leq \int_M (P_1(\nabla f), \nabla f),$$

(2.1)
for each \( f \in C_c^\infty(M) \).

3. Proof of main results

In this section, we will give the proofs of Theorems 1.1 and 1.2.

**Proof of Theorem 1.1** Suppose by contradiction there exists a complete noncompact stable hypersurface satisfying the condition of Theorem 1.1. By assumption, \( S_1 = 3H \) is nonzero. We can choose an orientation such that \( S_1 = 3H > 0 \). There is a fact that \( 2S_1S_3 \leq S_2^2 \) which implies that \( S_3 \leq 0 \). The operator \( P_1 \) is positive definite since \( H \) is positive [7]. Stability and 1-minimality of the hypersurface \( M \) imply that there is the following inequality:

\[
\int_M (2S_1 - 3S_3)f^2 \leq \int_M \langle P_1(\nabla f), \nabla f \rangle,
\]

for each \( f \in C_c^\infty(M) \). Choose \( f = S_1^q \varphi \) for a positive constant \( q \) to be determined and \( \varphi \in C_c^\infty(M) \).

We get that

\[
\langle P_1(\nabla f), \nabla f \rangle = \langle qS_1^{q-1} \varphi P_1(\nabla S_1), \nabla \rangle + (1 + q)S_1^{q-1} \varphi \nabla S_1 + S_1^q \varphi
\]

\[
= q^2S_1^{2q-2} \varphi^2 \langle P_1(\nabla S_1), \nabla S_1 \rangle + 2qS_1^{2q-1} \varphi \langle P_1(\nabla S_1), \nabla \varphi \rangle + S_1^{2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle.
\]

Since \( P_1 \) is positive definite, we obtain that

\[
2qS_1^{2q-1} \varphi \langle P_1(\nabla S_1), \nabla \varphi \rangle = S_1^{2q-2} \langle P_1(\varphi \nabla S_1), S_1 \nabla \varphi \rangle
\]

\[
= 2qS_1^{2q-2} \langle \sqrt{P_1}(\varphi \nabla S_1), \sqrt{P_1}(S_1 \nabla \varphi) \rangle
\]

\[
\leq qS_1^{2q-2} \langle \sqrt{P_1}(\varphi \nabla S_1) \rangle \leq qS_1^{2q-2} \varphi \langle P_1(\nabla S_1), S_1 \nabla \varphi \rangle + qS_1^{2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle.
\]

By (3.1)–(3.3) and the fact \( \langle P_1(X), X \rangle \leq 2S_1|X|^2 \), we get the following inequality:

\[
\int_M (2S_1 - 3S_3)S_1^{2q} \varphi^2 \leq (q^2 + q) \int_M S_1^{2q-2} \varphi^2 \langle P_1(\nabla S_1), \nabla S_1 \rangle + (1 + q)S_1^{2q} \langle P_1(\nabla \varphi), \nabla \varphi \rangle
\]

\[
\leq 2(q^2 + q) \int_M S_1^{2q-1} \varphi^2 |\nabla S_1|^2 + 2(1 + q) \int_M S_1^{2q+1} |\nabla \varphi|^2.
\]

We choose \( \varphi = \phi^{\frac{1+2q}{4}} \) and get that

\[
|\nabla \varphi|^2 = \frac{(3 + 2q)^2}{4} \phi^{1 + 2q} |\nabla \phi|^2.
\]

Combining (3.4) with (3.5), we obtain that

\[
\int_M (2S_1 - 3S_3)S_1^{2q} \phi^{1+2q} \leq 2(q^2 + q) \int_M S_1^{2q-1} \phi^{3+2q} |\nabla S_1|^2 + \frac{1 + q}{2} \int_M S_1^{1+2q} \phi^{1+2q} |\nabla \phi|^2.
\]

\[ \text{(3.6)} \]
Using Young’s inequality, we have
\[
S_1^{1+2q} \phi^{1+2q} |\nabla \phi|^2 = (bS_1^{1+2q} \phi^{1+2q}) \cdot (|\nabla \phi|^2 / b) \\
\leq \frac{1}{3+2q} \frac{2^{3+2q} S_1^{3+2q} \phi^{3+2q}}{3+2q} + \frac{2}{3+2q} b^{\frac{3+2q}{2}} |\nabla \phi|^{3+2q},
\]
for a positive constant $b$ to be determined. Combining with (3.6), we have
\[
\int_M (2S_1 - 3S_3) S_1^{2q} \phi^{3+2q} - 2(q^2 + q) \int_M S_1 S_1^{2q-1} \phi^{3+2q} |\nabla S_1|^2 \\
\leq \frac{(1 + q)(3 + 2q)}{2} \int_M ((1 + 2q)b^{\frac{3+2q}{2}} S_1^{3+2q} \phi^{3+2q} + 2b^{\frac{3+2q}{2}} |\nabla \phi|^{3+2q}).
\]
That is,
\[
\int_M A S_1^{3+2q} \phi^{3+2q} \leq B \int_M |\nabla \phi|^{3+2q},
\]
where
\[
A = \frac{2}{S_1^2} + \frac{3S_3}{S_1^2} - 2(q^2 + q)|\nabla (\frac{1}{S_1})|^2 - \frac{(1 + q)(3 + 2q)(1 + 2q)b^{\frac{3+2q}{2}}}{2} \\
\]
and
\[
B = (1 + q)(3 + 2q)b^{\frac{3+2q}{2}} > 0.
\]
Since
\[
|H| \leq \delta_1, |\nabla (\frac{1}{H})| \leq \delta_2,
\]
we have
\[
|S_1| \leq 3\delta_1, |\nabla (\frac{1}{S_1})| \leq \frac{\delta_2}{3},
\]
which imply that
\[
A \geq \frac{2}{9\delta_1^2} + \frac{3S_3}{S_1^2} - 2(q^2 + q)\delta_2^2 / 9 - \frac{(1 + q)(3 + 2q)(1 + 2q)b^{\frac{3+2q}{2}}}{2}.
\]
Choosing $q$ and $b$ sufficiently small such that
\[
\frac{2}{9\delta_1^2} - \frac{2(q^2 + q)\delta_2^2}{9} - \frac{(1 + q)(3 + 2q)(1 + 2q)b^{\frac{3+2q}{2}}}{2} > 0.
\]
Combining (3.10) with the fact that $\frac{3S_3}{S_1^2} \geq 0$, we get
\[
A > 0.
\]
Let $\phi$ be a function depending on the distance $r$ with respect to a fixed point $p$,
\[
\phi(x) = \begin{cases} 
1, & \text{on } B(R), \\
\frac{2R - r}{R}, & \text{on } B(2R) \setminus B(R), \\
0, & \text{on } M \setminus B(2R).
\end{cases}
\]
Combining with (3.9), we obtain that
\[
\int_{B(R)} AS_1^{3+2q} \leq B \int_{B(2R) \setminus B(R)} \frac{1}{R^{3+2q}} \leq B \frac{\text{vol}(B(2R))}{R^{3+2q}}.
\]
Noting that $M$ has polynomial volume growth and taking $R \to +\infty$, we obtain that $S_1 = 0$. This contradicts $S_1 \neq 0$. □

**Proof of Theorem 1.2** Suppose by contradiction there exists a complete noncompact stable hypersurface satisfying the condition of Theorem 1.2. Following the same step as proof of Theorem 1.1, we still obtain the inequality (3.9). Since

\[-\frac{K}{H^3} \geq \delta_1, \left| \nabla \left( \frac{1}{H} \right) \right| \leq \delta_2,\]

we get

\[-\frac{S_1}{S_1^3} \geq \frac{\delta_1}{27}, \left| \nabla \left( \frac{1}{S_1} \right) \right| \leq \frac{\delta_2}{3}.\]

Thus,

\[A = \frac{2}{S_1^3} \left( \frac{\delta_1}{9} - \frac{2(q^2 + aq)\delta_2}{3} - \frac{(a + q)(3 + 2q)(1 + 2q)\delta^3_{3+2q}}{2a} \right).\]

Choosing $q$ and $b$ sufficiently small such that

\[\frac{\delta_1}{9} - \frac{2(q^2 + aq)\delta_2}{3} - \frac{(a + q)(3 + 2q)(1 + 2q)\delta^3_{3+2q}}{2a} > 0.\]

Thus $A > 0$. Let $\phi$ be a function depending on the distance $r$ with respect to a fixed point $p$,

\[\phi(x) = \begin{cases} 1, & \text{on } B(R), \\ \frac{2R - r}{R}, & \text{on } B(2R) \setminus B(R), \\ 0, & \text{on } M \setminus B(2R). \end{cases}\]

Combining with (3.9), we obtain that

\[\int_{B(R)} A S_1^{3+2q} \leq B \int_{B(2R) \setminus B(R)} \frac{1}{R^{3+2q}} \leq B \frac{\text{vol}(B(2R))}{R^{3+2q}}. \tag{3.12}\]

Noting that $M$ has polynomial volume growth and taking $R \to +\infty$, we obtain that $S_1 = 0$. This contradicts $S_1 \neq 0$. □

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**References**


