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Direct GBQ Algorithm for Solving Mixed Trigonometric Polynomial Systems

Yan YU[1,2], **Bo DONG**[1,*], **Bo YU**[1]
1. *School of Mathematical Sciences, Dalian University of Technology, Liaoning* 116024*, P. R. China;*
2. *College of Sciences, Shenyang Agricultural University, Liaoning* 110866*, P. R. China*

Abstract In many fields of science and engineering, it is needed to find all solutions of mixed trigonometric polynomial systems. Commonly, mixed trigonometric polynomial systems are transformed into polynomial systems by variable substitution and adding some quadratic equations, and then solved by some numerical methods. However, transformation of a mixed trigonometric polynomial system into a polynomial system will increase the dimension of the system and hence induces extra computational work. In this paper, we consider to solve the mixed trigonometric polynomial systems by homotopy method directly. Homotopy with the start system constructed by GBQ-algorithm is presented and homotopy theorems are proved. Preliminary numerical results show that our constructed direct homotopy method is more efficient than the existent direct homotopy methods.

Keywords mixed trigonometric polynomial system; polynomial system; homotopy method; GBQ algorithm; upper bound

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1. Introduction

Mixed trigonometric polynomial systems (abbreviated by MTPS) of the following form:

$$P(x,y)=(p_1(x,y),\dots,p_{n+m}(x,y))^T=0 \tag{1}$$

where $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_m)$, and

$$p_i(x,y)=\sum_{j=1}^{k_i} b_{ij}x_1^{\alpha_{ij1}}\cdots x_n^{\alpha_{ijn}}(\sin y_1)^{\beta_{ij1}}\cdots(\sin y_m)^{\beta_{ijm}}(\cos y_1)^{\gamma_{ij1}}\cdots(\cos y_m)^{\gamma_{ijm}} \tag{2}$$

arise in many fields of science and engineering, such as six-revolute-joint problem of mechanics [1], neurophysiology problem [2], kinematics problem [2], PUMA robot [1,3], etc..

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* Corresponding author
E-mail address: yyansyau@gmail.com (Yan YU); dongbo@dlut.edu.cn (Bo DONG); yubo@dlut.edu.cn (Bo YU)

Example 1.1 The following MTPS [4] arises in the field of signal processing.

$$F(x,y)=\begin{cases} x_1+x_2=0.75,\\ x_1^2+x_2^2=0.60,\\ x_1\sin y_1+x_2\sin y_2=0.35,\\ x_1\cos y_1+x_2\cos y_2=0.80.\end{cases} \tag{3}$$

An MTPS can often be converted to a polynomial system by replacing $\sin y_i$ and $\cos y_i$ with new variables x_{n+i} and x_{n+m+i}, respectively and adding the polynomial relations $x_{n+i}^2+x_{n+m+i}^2=1$, $i=1,\dots,m$. Therefore, we can get solutions of an MTPS through solving the converted polynomial systems by the existent methods for polynomial systems.

Homotopy methods have been proved to be reliable and efficient numerical methods to approximate all isolated solutions of polynomial systems. The homotopy methods can find all isolated nonsingular solutions of a polynomial system through tracking the homotopy paths, and the number of paths is the so-called upper bound. The tighter bound implies less homotopy paths should be tracked. For polynomial systems, the best known upper bound is the total degree [5] and the corresponding homotopy method is the standard homotopy method [6–9]. Theoretically, this method can be applied to solve all polynomial systems, however, it is unsuitable to be applied to solve the deficient polynomial systems, which is more common in practical applications, because too many unnecessary paths have to be traced. Some tighter upper bounds and the corresponding efficient homotopy methods for deficient polynomial systems have been presented, for example, the multihomogeneous Bézout number and the corresponding multihomogeneous product homotopy method [10,11], the generalized Bézout number, which is a tighter upper bound than the multihomogeneous Bézout number, and the GBQ-algorithm [12]. The tightest upper bound in $(\mathbb{C}^*)^n$, where $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$, is the BKK bound [13–15]. And the upper bound in $\mathbb{C}^n$ was given in [16]. The corresponding homotopy method is the polyhedral homotopy method [17–19], see [20] for more details. The hybrid method, which is a symbolic-numerical method and is a combination of the product homotopy and the coefficient parameter homotopy, was presented in [4] to solve the polynomial systems transformed from the MTPSs.

Transformation from an MTPS to a polynomial system will increase the dimension of the problem by m. Therefore, it is reasonable to solve the MTPSs directly. There are also several direct homotopy methods. The total degree of an MTPS and the corresponding standard homotopy were presented in [21,22]. For deficient MTPSs, the multi-homogeneous homotopy Bézout number was presented in [22]. However, the multihomogeneous Bézout number is still too loose for approximating the number of isolated solutions of an MTPS. And the BKK bound cannot be defined directly due to the existence of the trigonometric functions. In this paper, we will present a tighter upper bound than the multihomogeneous Bézout number and construct the corresponding homotopy method.

The organization of this paper is as follows. Some basic definitions and the algorithm for constructing the support of a mixed trigonometric polynomial are presented in Section 2; In Section 3, the proof of our main theorem is presented; Numerical examples are given in Section

4 to illustrate the advantages of our method.

2. Direct linear product homotopy based on GBQ

To give a much tighter upper bound on the number of isolated solutions of an MTPS, instead of only one partition, a more refined data structure will be defined to present the structure of the MTPS $F(x, y) = 0$ in (1). Unlike the variable partition for the MTPS with multi-homogeneous structure, we take for different equations different variable partitions of the sets of the unknowns.

The followings are the definitions of the support of a mixed trigonometric polynomial $f(x, y) = 0$ in (2) and a supporting set structure of an MTPS $F(x, y) = 0$ in (1).

Definition 2.1 *Let T be an array of subsets of the set $\{x_1, \dots, x_n, y_1, \dots, y_m\}$. Then T is said to be the support of a mixed trigonometric polynomial $f(x, y)$ in (2) if it satisfies the following conditions:*

(i) For each term

$$c_d x_1^{d_1} \cdots x_n^{d_n} (\sin y_1)^{d_{n+1}} \cdots (\sin y_m)^{d_{n+m}} (\cos y_1)^{d_{n+m+1}} \cdots (\cos y_m)^{d_{n+2m}}$$

of $f(x, y)$, $c_d \neq 0$, there exist at least d_1 sets in T that contain x_1 such that, if they are removed from T, then the resulting array of subsets is the support of the term

$$c_d x_2^{d_2} \cdots x_n^{d_n} (\sin y_1)^{d_{n+1}} \cdots (\sin y_m)^{d_{n+m}} (\cos y_1)^{d_{n+m+1}} \cdots (\cos y_m)^{d_{n+2m}}.$$

(ii) For each term

$$c_d x_1^{d_1} \cdots x_n^{d_n} (\sin y_1)^{d_{n+1}} \cdots (\sin y_m)^{d_{n+m}} (\cos y_1)^{d_{n+m+1}} \cdots (\cos y_m)^{d_{n+2m}}$$

of $f(x, y)$, $c_d \neq 0$, there exist at least $d_{n+1} + d_{n+m+1}$ sets in T that contain y_1 such that, if they are removed from T, then the resulting array of subsets is the support of the term

$$c_d x_1^{d_1} \cdots x_n^{d_n} (\sin y_2)^{d_{n+2}} \cdots (\sin y_m)^{d_{n+m}} (\cos y_2)^{d_{n+m+2}} \cdots (\cos y_m)^{d_{n+2m}}.$$

Example 2.2 It is easy to verify that the supports of the first two equations in the MTPS (3) are $\sigma_1 := [\sigma_{11}] = [\{x_1, x_2\}]$, $\sigma_2 := [\sigma_{21}, \sigma_{22}] = [\{x_1, x_2\}, \{x_1, x_2\}]$. $\square$

Note that for a general mixed trigonometric polynomial, the number of sets in the supporting array is always not less than its degree.

Definition 2.3 *For the MTPS in (1), $\sigma = (\sigma_1, \dots, \sigma_{n+m})$ is called a supporting set structure of the MTPS $F(x, y)$ in (1) if σ_i is the support of the mixed polynomial $f_i(x, y)$ in (2).*

Example 2.4 The supports of the last two equations in the MTPS (3) are

$$\sigma_3 := [\sigma_{31}, \sigma_{32}] = [\{x_1, x_2\}, \{y_1, y_2\}], \quad \sigma_4 := [\sigma_{41}, \sigma_{42}] = [\{x_1, x_2\}, \{y_1, y_2\}].$$

Therefore, from Definition 2.3 and Example 2.2, we can get the supporting set structure of the MTPS (3) is $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. $\square$

In the following, we first give the definition of the acceptable tuple of the supporting set structure of an MTPS, and then define the generalized Bézout number of an MTPS.

Definition 2.5 *Let $Y = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ and σ be a supporting set structure. An acceptable tuple of σ, denoted by T_σ, is an $(n+m)$-tuple of subsets of the set Y such that the k-th subset belongs to σ_k and any union of k subsets of T_σ contains at least k elements of Y.*

Definition 2.6 *σ is the supporting set structure of an MTPS $F(x, y)$, the generalized Bézout number B_σ is defined by $2^m \sharp(T_\sigma)$, where $\sharp(T_\sigma)$ denotes the number of all acceptable tuples of σ.*

Example 2.7 For the MTPS in (3), corresponding to the supporting set structure in Example 2.4, there are $1 \times 2 \times 2 \times 2 = 8$ tuples. However, not all tuples are acceptable. For example, the tuple $[\sigma_{11}, \sigma_{21}, \sigma_{31}, \sigma_{41}]$ is not acceptable since the union of the subsets $\sigma_{11}, \sigma_{21}, \sigma_{31}, \sigma_{41}$ only contains 2 elements of the set $\{x_1, x_2, y_1, y_2\}$. An acceptable tuple should satisfy the condition: the numbers of subsets $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are both 2, therefore, the number of acceptable tuples is 2, and the generalized Bézout number is $2^2 \times 2 = 8$.

The following algorithm describes the method of constructing the support of a given mixed trigonometric polynomial.

Algorithm: Algorithm of the construction of the support

Input: $p(x, y) = \sum\limits_{i=1}^{k} b_i x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}} (\sin y_1)^{\beta_{i1}} \cdots (\sin y_m)^{\beta_{im}} (\cos y_1)^{\gamma_{i1}} \cdots (\cos y_m)^{\gamma_{im}}$.

Output: The set structure S of $p(x, y) = 0$.

Algorithm:

(i) Find the index t such that $t = \arg\max_{1 \le i \le k} (\sum_{l=1}^{n} \alpha_{il} + \sum_{j=1}^{m} \beta_{ij} + \sum_{j=1}^{m} \gamma_{ij})$. If there are more than one index satisfying the condition, then any one is acceptable.

(ii) Construct the sets

$$S_1 = \cdots = S_{\alpha_{t1}} = \{x_1\}, \ldots, S_{\sum\limits_{i=1}^{n-1} \alpha_{ti} + 1} = \cdots = S_{\sum\limits_{i=1}^{n} \alpha_{ti}} = \{x_n\},$$

$$S_{\sum\limits_{i=1}^{n} \alpha_{ti} + 1} = \cdots = S_{\sum\limits_{i=1}^{n} \alpha_{ti} + \beta_{t1} + \gamma_{t1}} = \{y_1\},$$

$$\cdots$$

$$S_{\sum\limits_{i=1}^{n} \alpha_{ti} + \sum\limits_{i=1}^{m-1} \beta_{ti} + \sum\limits_{i=1}^{m-1} \gamma_{ti} + 1} = \cdots = S_{\sum\limits_{i=1}^{n} \alpha_{ti} + \sum\limits_{i=1}^{m} \beta_{ti} + \sum\limits_{i=1}^{m} \gamma_{ti}} = \{y_m\}.$$

(iii) For the j-th $(1 \le j \le k)$ term, if $\alpha_{ji} > \alpha_{ti}$ (or $\beta_{ji} + \gamma_{ji} > \beta_{ti} + \gamma_{ti}$), then add x_i (or y_i) in the first $\alpha_{ji} - \alpha_{ti}$ (or $(\beta_{ji} + \gamma_{ji}) - (\beta_{ti} + \gamma_{ti})$) sets of maximal size and excluding y_i (or y_i).

Definition 2.8 *Let X, including s unknowns $x_1, \ldots, x_s$ and t angles $y_1, \ldots, y_t$, be a subset of Y. We define the mixed trigonometric polynomial with respect to X to be*

$$p(X) = a_1 x_1 + \cdots + a_s x_s + a_{s+1}(\sin y_1 + \cos y_1) + \cdots + a_{s+t}(\sin y_t + \cos y_t) \tag{4}$$

where $a_1, \ldots, a_{s+t}$ are random complex numbers in $\mathbb{C}^*$.

Example 2.9 Let $X = \{x_1, x_2, y_1, y_2\}$. The mixed trigonometric polynomial with respect to X is $p(X) = a_1 x_1 + a_2 x_2 + a_3(\sin y_1 + \cos y_1) + a_4(\sin y_2 + \cos y_2)$.

Suppose the target MTPS $P(x,y)=0$ is as (1), and the supporting set structure is $\sigma=(\sigma_1,\dots,\sigma_{n+m})$, we construct the start MTPS $Q(x,y)=0$ as follows:

$$Q(x,y)=\begin{cases}(q_{11}(\sigma_{11})+a_{11})\cdots(q_{1k_1}(\sigma_{1k_1})+a_{1k_1}),\\ \qquad\cdots\\ (q_{n+m,1}(\sigma_{n+m,1})+a_{n+m,1})\cdots(q_{n+m,k_{n+m}}(\sigma_{n+m,k_{n+m}})+a_{n+m,k_{n+m}}),\end{cases}\tag{5}$$

where $q_{ij}(\sigma_{ij})$ is the mixed trigonometric polynomial with respect to σ_{ij} as (4) and $a_{11},\dots,a_{n+m,k_{n+m}}$ are all random complex numbers.

Consider $\sin(y_i)+\cos(y_i)$ as a variable. Every polynomial of the start system can be transformed to the product of linear polynomials. For every random choice of the coefficients of $Q(x,y)=0$, except for a set of measure zero, the number of isolated solutions to the transformed start system is the number of all accepted tuples. Since $\sin y_i+\cos y_i=c$ has 2 isolated solutions with $\mathrm{Re}(y_i)\in[0,2\pi)$, we can get the following theorem.

Theorem 2.10 *Let $Q(x,y)=0$ be the start system based on the supporting set structure σ. Then for every random choice of the coefficients of $Q(x,y)=0$, except for a set of measure zero, the system $Q(x,y)=0$ has exactly B_σ regular solutions in $\mathbb{C}^{n+m}$.*

To solve the target system $P(x,y)=0$ in (1), we construct the following homotopy map

$$H(x,y,t)=\eta(1-t)Q(x,y)+tP(x,y),\tag{6}$$

where η is a random complex number. The following theorem states the homotopy in (6) is a good homotopy, i.e., it satisfies smoothness and accessibility. Hence by tracing the homotopy paths numerically, we can get all isolated solutions of $P(x,y)=0$.

Theorem 2.11 *$P(x,y),Q(x,y)$ are respectively as the systems in (1) and (5), then the homotopy map $H(x,y,t)$ in (6) possesses the following properties:*

(i) Smoothness: the solution set to $H(x,y,t)=0$ for all $t:0\le t\le 1$, consists of a finite number of smooth paths, each parameterized by t.

(ii) Accessibility: every isolated solution to $P(x,y)=0$ can be reached by some path originating at a solution to $Q(x,y)=0$.

3. Proof of theorems

With variable substitution, $P(x,y)=0$ and $Q(x,y)=0$ are respectively transformed to

$$\widehat{P}(z)=\begin{cases}\widehat{p}_1(z),\\ \quad\cdots\\ \widehat{p}_{n+m}(z),\\ z_{n+1}^2+z_{n+m+1}^2-1,\\ \quad\cdots\\ z_{n+m}^2+z_{n+2m}^2-1,\end{cases}\qquad \widehat{Q}(z)=\begin{cases}\widehat{q}_1(z)\\ \quad\cdots\\ \widehat{q}_{n+m}(z),\\ z_{n+1}^2+z_{n+m+1}^2-1,\\ \quad\cdots\\ z_{n+m}^2+z_{n+2m}^2-1,\end{cases}\tag{7}$$

where, for all $1 \le i \le n+m$,

$$\widehat{p}_i(z) = \sum_{j=1}^{k_i} b_{ij} z_1^{\alpha_{ij1}} \cdots z_n^{\alpha_{ijn}} z_{n+1}^{\beta_{ij1}} \cdots z_{n+m}^{\beta_{ijm}} z_{n+m+1}^{\gamma_{ij1}} \cdots z_{n+2m}^{\gamma_{ijm}},$$
$$\widehat{q}_i(z) = (\widehat{q}_{i1}(z) + a_{i1}) \cdots (\widehat{q}_{il_i}(z) + a_{ik_i}),$$

and $\widehat{q}_{ij}(z)$ is the transformed polynomial of $q_{ij}(\sigma_{ij})$.

Let $\langle f_1, \ldots, f_n \rangle$ be the ideal generated by the polynomials $f_1, \ldots, f_n$. The i-th equation of the start system belongs to $\langle s_{i1} \otimes \cdots \otimes s_{ik_i} \rangle$, where s_{ij} is the transformed set of σ_{ij} with variable substitution and $\otimes$ is the product of sets. Through expanding the product of each equation of the start system and collecting terms, we get an expansion of the start system. We consider a family of systems, parameterized by the coefficients of all the monomials that appear in the above expansion. Let

$$F(z;q) = \begin{cases} f_1(z_1, \ldots, z_{n+2m}, q_1, \ldots, q_r), \\ \qquad \cdots \\ f_{n+m}(z_1, \ldots, z_{n+2m}, q_1, \ldots, q_r), \\ z_{n+1}^2 + z_{n+m+1}^2 - 1, \\ \qquad \cdots \\ z_{n+m}^2 + z_{n+2m}^2 - 1, \end{cases}$$

be the family of above polynomial systems. Then there exist the parameters q_1, q_0 such that $\widehat{P}(z) = F(z;q_1), \widehat{Q}(z) = F(z;q_0)$.

Let $\mathcal{X}$ denote the nonreduced solution set of $F(x;q) = 0$, and $Z = V(F(x;q))$ denote the reduction of $\mathcal{X}$, and let $\pi : \mathcal{X} \to \mathcal{C}^{n+2m}$ be the map induced from $\mathcal{C}^{n+2m} \times \mathcal{C}^r \to \mathcal{C}^{n+m}$, and let $\mathcal{X}_0$ denote the union of irreducible components $\mathcal{Z}$ of Z such that $\pi_{\mathcal{Z}}$ is dominant and such that $\dim Z = r$. Then

Lemma 3.1 ([23]) *If there is an isolated solution (z^*, q^*) of $F(z;q^*) = 0$, then $(z^*, q^*) \in \mathcal{X}_0$. Moreover, there are arbitrarily small complex open sets $\mathcal{U} \subset \mathcal{C}^{n+2m} \times \mathcal{C}^r$ that contain (z^*, q^*) such that*

(i) (z^*, q^*) is the only solution of $F(z;q^*) = 0$ in $\mathcal{U} \cap (\mathcal{X} \times \{q^*\})$;

(ii) $f(z;q') = 0$ has only isolated solutions for $q' \in \pi(\mathcal{U})$ and $z \in \mathcal{U} \cap (\mathcal{X} \times \{q'\})$;

(iii) The multiplicity of (z^*, q^*) as a solution of $F(z;q^*) = 0$ equals the sum of the multiplicity of the isolated solutions of $F(z;q') = 0$ for $q' \in \pi(\mathcal{U})$ and $z \in \mathcal{U} \cap (\mathcal{X} \times \{q'\})$.

Let $\mathcal{N}(q)$ denote the number of nonsingular solutions as a function of q:

$$\mathcal{N}(q) = \sharp\big\{z \in \mathcal{C} | F(z;q) = 0, \det \frac{\partial F}{\partial z}(z;q) \neq 0\big\}.$$

Lemma 3.2 *The number of the isolated solutions of the start system is equal to that of systems in $F(z;q)$ with the generic choice of the parameters.*

Proof $\widetilde{Q}(\widehat{z}) = 0$ is the homogenization of $\widehat{Q}(z) = 0$, and when $\hat{z}_0 = 0$, the last m equations of $\widetilde{Q}(\widehat{z}) = 0$ are $\hat{z}_{n+i}^2 + \hat{z}_{n+m+i}^2 = 0$, which is equivalent to $\hat{z}_{n+m+i} = \pm \hat{z}_{n+i} I, I = \sqrt{-1}$. Substitute

$\hat{z}_{n+m+i}$ with $\pm\hat{z}_{n+i}I$ in the former $n+m$ equations of $\widetilde{Q}(\widehat{z}) = 0$, then solving $\widetilde{Q}(\widehat{z}) = 0$ is equivalent to solving many linear subsystems. Since the constant terms in the subsystems are zero and the coefficient matrix is regular, thus the solution of the subsystems are zero, but $[0, \dots, 0]$ is not a point in $\mathcal{P}^{n+m}$, hence $\widetilde{Q}(\widehat{z}) = 0$ has no solutions at infinity.

From Theorem 2.9, we know the solution set of $Q(x, y) = 0$ in (5) consists of finite isolated and nonsingular solutions. Furthermore, $\widehat{Q}(z) = 0$ in (5) is transformed from $Q(x, y) = 0$, and there is a one-one mapping between the solution sets of $\widehat{Q}(z) = 0$ and $Q(x, y) = 0$. Hence the solution set of $\widetilde{Q}(\widehat{z}) = 0$ consists of finite isolated and nonsingular solutions.

Since $\widehat{Q}(z) = F(z; q_0) \in F(z; q)$ and the number of solutions of $\widehat{Q}(z) = 0$ is B_σ, the number of the solutions of the polynomial system in $F(z; q) = 0$ with generic choice of the parameters is not less than B_σ. Suppose there exists a parameter q' in the neighborhood of q_0 such that $\mathcal{N}(q') > \mathcal{N}(q_0)$. Lemma 3.1 implies that nonsingular roots continue in an open neighborhood, therefore, since $\mathcal{P}^{n+m}$ is compact, the nonsingular along a path from q' to q_0 must have a limit in $\mathcal{P}^{n+m}$ as the path approaches q_0. Accordingly, some solution of $\widehat{Q}(z) = 0$ must have at least two solution paths approach it. But this contradicts Lemma 3.1, leaving $\mathcal{N} = B_\sigma$ as the only possible conclusion. Since $\widetilde{Q}(\widehat{z}) = 0$ has no infinity solutions, the conclusion follows. □

Lemma 3.3 ([23]) *Let $F(z; q)$ be a system of polynomials in n variables and m parameters,*

$$F(z; q) : \mathcal{C}^n \times \mathcal{C}^m \to \mathcal{C}^n$$

that is, $F(z; q) = \{f_1(z; q), \dots, f_n(z; q)\}$ and each $f_i(z; q)$ is polynomial in both z and q. Then,

(i) $\mathcal{N}(q)$ is finite, and it is the same, say $\mathcal{N}$, for almost all $q \in \mathcal{C}^m$;

(ii) For all $q \in \mathcal{C}^m$, $\mathcal{N}(q) \le \mathcal{N}$;

(iii) The subset of $\mathcal{C}^m$ where $\mathcal{N}(q) = \mathcal{N}$ is a Zariski open set.

(iv) The homotopy $F(z; tq_1 + (1-t)q_0) = 0$, where $t = \frac{\tau}{\tau + \gamma(1-\tau)}$ and $\gamma \notin [0, 1], \tau \in [0, 1]$, has $\mathcal{N}$ continuous nonsingular solution paths $z(t) \in \mathcal{C}^n$.

(v) As $t \to 0$, the limits of the solution paths of the homotopy $F(z; tq_1 + (1-t)q_0) = 0$ includes all the nonsingular roots of the target system $F(z; q_1) = 0$.

The following lemma states that the transformed homotopy map $\widehat{H}(z, t)$ satisfies the s-moothness and accessibility.

Lemma 3.4 *$\widehat{P}(z), \widehat{Q}(z)$ are respectively as the systems in (7), then the homotopy map*

$$\widehat{H}(z, t) = \gamma(1-t)\widehat{Q}(z) + t\widehat{P}(z)$$

is a good homotopy.

Proof From Lemma 3.3, the homotopy

$$\begin{aligned}
& F(z; tq_1 + (1-t)q_0) = F(z; \frac{\tau}{\tau + \gamma(1-\tau)} q_1 + \frac{\gamma(1-\tau)}{\tau + \gamma(1-\tau)} q_0) = 0 \\
\Longleftrightarrow\ & \frac{\tau}{\tau + \gamma(1-\tau)} F(z; q_1) + \frac{\gamma(1-\tau)}{\tau + \gamma(1-\tau)} F(z; q_0) = 0 \\
\Longleftrightarrow\ & \tau F(z; q_1) + \gamma(1-\tau) F(z; q_0) = 0
\end{aligned}$$

$$\Longleftrightarrow \gamma(1-\tau)\widehat{Q}(z)+\tau\widehat{P}(z)=0$$

is a good homotopy. □

Proof of Theorem 2.10 For every polynomial $h_i(x,y,t)$ in $H(x,y,t)$ and $\widehat{h}_i(z,t)$ in $\widehat{H}(z,t)$, the following equations hold:

$$\frac{\partial h_i}{\partial x_j}=\frac{\partial \widehat{h}_i}{\partial z_j},\ \frac{\partial h_i}{\partial t}=\frac{\partial \widehat{h}_i}{\partial t},\ \frac{\partial h_i}{\partial y_k}=\frac{\partial \widehat{h}_i}{\partial z_{n+k}}z_{n+m+k}-\frac{\partial \widehat{h}_i}{\partial z_{n+m+k}}z_{n+k}, \tag{8}$$

for $1\le j\le n,\ 1\le i\le n+m,\ 1\le k\le m$.

Suppose $\frac{\partial H(x,y,t)}{\partial(x,y,t)}$ is singular. We can find scalars $\beta_1,\dots,\beta_{n+m+1}$, at least one of which is nonzero, such that

$$\beta_1\frac{\partial H}{\partial x_1}+\cdots+\beta_n\frac{\partial H}{\partial x_n}+\beta_{n+1}\frac{\partial H}{\partial y_1}+\cdots+\beta_{n+m}\frac{\partial H}{\partial y_m}+\beta_{n+m+1}\frac{\partial H}{\partial t}=0. \tag{9}$$

From (8) and (9), there exist $\gamma_1,\dots,\gamma_{n+2m+1}$, one of which is nonzero, such that

$$\gamma_1\frac{\partial \widehat{H}}{\partial z_1}+\cdots+\gamma_n\frac{\partial \widehat{H}}{\partial z_n}+\gamma_{n+1}\frac{\partial \widehat{H}}{\partial z_{n+1}}+\cdots+\gamma_{n+2m}\frac{\partial \widehat{H}}{\partial z_{n+2m}}+\gamma_{n+2m+1}\frac{\partial \widehat{H}}{\partial t}=0,$$

which is inconsistent to Lemma 3.4. Thus the smoothness is proved. □

There exist finite solution paths of $\widehat{H}(z,t)=0$ intersecting with the plane $t=0$ and $t=1$. Applying the variables substitution, we can transform solution paths to piecewise smooth curves in x,y,t space and $\mathrm{Re}(y_i)\in[0,2\pi)$ $(1\le i\le m)$, which are solution curves of $H(x,y,t)=0$. Since the solutions of $\widehat{H}(z,t)=0$ are regular, the transformed curves do not intersect each other. The variable transformation between MTPS and polynomial system is one-one mapping, hence, all solution curves of $\widehat{H}(z,t)=0$ can be transformed to piecewise smooth curves, moreover, parallel moved $2k\pi(k\in\mathcal{N})$ along $\mathrm{Re}(y)$ axis, a piecewise smooth curve can turn to a smooth curve. On the one hand, for solution curves of $\widehat{H}(z,t)=0$ which intersect with the plane $t=0$ and $t=1$, the intersection also holds for piecewise smooth curves transformed from these solution curves. On the other hand, for solution curves which do not intersect with the plane $t=0$ and $t=1$, the intersection also does not hold for piecewise smooth curves transformed from these solution curves. The number of isolated solutions of $H(x,y,t)=0$ is equal to that of $\widehat{H}(z,t)=0$ due to one-one mapping between MTPS and polynomial system. Hence, following piecewise smooth curves in x,y,t space intersecting with the plane $t=0$ and $t=1$, we can get all isolated solutions of $P(x,y)=0$.

4. Numerical experiments

In this section, we apply our direct homotopy methods to solve PUMA MTPS, and through the comparison with homotopy method for polynomial systems, we can find that our method is more effective.

Example 4.1 Consider the PUMA MTPS

$$F(x,y)=\begin{cases} 0.004731\cos y_1\cos y_2-0.3578\sin y_1\cos y_2-0.1238\cos y_1- \\ \quad 0.001637\sin y_1-0.9338\sin y_2+\cos y_4=0.3571, \\ 0.2238\cos y_1\cos y_2+0.7623\sin y_1\cos y_2+0.2638\cos y_1- \\ \quad 0.07745\sin y_1-0.6734\sin y_2=0.6022, \\ \sin y_3\sin y_4+0.3578\cos y_1+0.004731\sin y_1=0, \\ -0.7623\cos y_1+0.2238\sin y_1=-0.3461. \end{cases}$$

The supporting set structure is

$$([\{y_1,y_4\},\{y_2\}],[\{y_1\},\{y_2\}],[\{y_1,y_3\},\{y_4\}],[\{y_1\}])\,.$$

There is only one acceptable tuple $[\{y_1,y_4\},\{y_2\},\{y_1,y_3\},\{y_1\}]$, and thus the generalized Bézout number is $2^4\times 1=16$. The constructed start system $G(x,y)=0$ based on the supporting set structure is

$$\begin{cases} (a_{11}(\sin y_1+\cos y_1)+a_{12}(\sin y_4+\cos y_4)+a_{10})\,(b_{11}(\sin y_2+\cos y_2)+b_{10})=0, \\ (a_{21}(\sin y_1+\cos y_1)+a_{20})\,(b_{21}(\sin y_2+\cos y_2)+b_{20})=0, \\ (a_{31}(\sin y_1+\cos y_1)+a_{32}(\sin y_3+\cos y_3)+a_{30})\,(b_{31}(\sin y_4+\cos y_4)+b_{30})=0, \\ a_{41}(\sin y_1+\cos y_1)+a_{40}=0, \end{cases}$$

where all coefficients are random complex numbers. Applying the homotopy in (6), we can find 16 solutions to $F(x,y)=0$ are

	y_1	y_2	y_3	y_4		y_1	y_2	y_3	y_4
1	4.8776	2.8395	2.7169	6.1514	9	4.8776	2.8395	5.8585	0.1318
2	4.8776	4.9544	0.0688	4.0494	10	0.8345	0.3021	4.8415	0.2484
3	4.8776	2.8395	3.5663	0.1318	11	0.8345	0.3021	1.6999	6.0348
4	0.8345	0.3021	1.4417	6.0348	12	0.8345	0.3021	4.5833	0.2484
5	0.8345	4.4704	5.9920	2.1269	13	4.8776	4.9543	3.0728	4.0494
6	4.8776	4.9544	6.2144	2.2338	14	0.8345	4.4704	0.2911	4.1563
7	4.8776	2.8395	0.4247	6.1514	15	0.8345	4.4704	2.8505	4.1563
8	4.8776	4.9544	3.2104	2.2338	16	0.8345	4.4704	3.4327	2.1269

Table 1 All isolated solutions

If we apply the direct standard homotopy to solve the PUMA MTPS, the total degree of the PUMA MTPS is $2^4\times 8=108$. If we apply the direct multi-homogeneous product homotopy method in [22] to solve the PUMA MTPS, the best partition of the variable is $\{y_1,y_3\},\{y_2,y_4\}$, and the corresponding multi-homogeneous Bézout number is $2^4\times 3=48$. Therefore, the numbers of paths have to be traced by our direct GBQ algorithm, the direct standard homotopy and the direct multi-homogeneous homotopy method are 16, 108 and 48, which shows our method is more efficient than the existent methods.

References

- [1] A. MORGAN, A. SOMMESE. *Computing all solutions to polynomial systems using homotopy continuation*. Appl. Math. Comput., 1987, **24**(2): 115–138.
- [2] P. VAN HENTENRYCK, D. MCALLESTER, D. KAPUR. *Solving polynomial systems using a branch and prune approach*. SIAM J. Numer. Anal., 1997, **34**(2): 797–827.
- [3] A. MORGAN, A. SOMMESE. *A homotopy for solving general polynomial systems that respects m-homogeneous structures*. Appl. Math. Comput., 1987, **24**(2): 101–113.
- [4] Bo YU, Bo DONG. *A hybrid polynomial system solving method for mixed trigonometric polynomial systems*. SIAM J. Numer. Anal., 2008, **46**(3): 1503–1518.
- [5] I. R. SHAFAREVICH. *Basic Algebraic Geometry*. Springer-Verlag, New York-Heidelberg, 1974.
- [6] T. Y. LI. *On Chow, Mallet-Paret and Yorke homotopy for solving system of polynomials*. Bull. Inst. Math. Acad. Sinica, 1983, **11**(3): 433–437.
- [7] A. MORGAN. *A homotopy for solving polynomial systems*. Appl. Math. Comput., 1986, **18**(1): 87–92.
- [8] ALDEN H. WRIGHT. *Finding all solutions to a system of polynomial equations*. Math. Comp., 1985, **44**(169): 125–133.
- [9] W. ZULEHNER. *A simple homotopy method for determining all isolated solutions to polynomial systems*. Math. Comp., 1988, **50**(181): 167–177.
- [10] T. Y. LI, T. SAUER, JAMES A. YORKE. *The random product homotopy and deficient polynomial systems*. Numer. Math., 1987, **51**(5): 481–500.
- [11] A. MORGAN, A. SOMMESE. *A homotopy for solving general polynomial systems that respects m-homogeneous structures*. Appl. Math. Comput., 1987, **24**(2): 101–113.
- [12] J. VERSCHELDE, A. HAEGEMANS. *The GBQ-algorithm for constructing start systems of homotopies for polynomial systems*. SIAM J. Numer. Anal., 1993, **30**(2): 583–594.
- [13] D. N. BERNSTEIN. *The number of roots of a system of equations*. Funkcional. Anal. i Priložen, 1975, **9**(3): 1–4. (in Russian)
- [14] A. G. HOVANSKIĬ, *Newton polyhedra, and the genus of complete intersections*. Funktsional. Anal. i Priložhen, 1978, **12**(1): 51–61. (in Russian)
- [15] A. G. KUŠNIRENKO. *Newton polyhedra and Bezout's theorem*. Funkcional. Anal. i Priložen, 1976, **10**(3): 82–83. (in Russian)
- [16] T. Y. LI, Xiaoshen WANG. *The BKK root count in $\mathbf{C}^n$*. Math. Comp., 1996, **65**(216): 1477–1484.
- [17] B. HUBER, B. STURMFELS. *A polyhedral method for solving sparse polynomial systems*. Math. Comp., 1995, **64**(212): 1541–1555.
- [18] T. Y. LI. *Solving polynomial systems by polyhedral homotopies*. Taiwanese J. Math., 1999, **3**(3): 251–279.
- [19] J. VERSCHELDE, P. VERLINDEN, R. COOLS. *Homotopies exploiting Newton polytopes for solving sparse polynomial systems*. SIAM J. Numer. Anal., 1994, **31**(3): 915–930.
- [20] T. Y. LI. *Numerical Solution of Polynomial Systems by Homotopy Continuation Methods*. North-Holland, Amsterdam, 2003.
- [21] Tang-an GAO, Zeke WANG. *A homotopy algorithm for finding the zeros of a class of trigonometric polynomials*. Numer. Math. J. Chinese Univ., 1990, **12**(4): 297–303.
- [22] Bo DONG, Bo YU. *Homotopy method for solving mixed trigonometric polynomial systems*. J. Inform. Comput. Sci., 2007, **4**(2): 505–513.
- [23] A. J. SOMMESE, C. W. WAMPLER. *The Numerical Solution of Systems of Polynomials*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.