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# An Equivalent of Algebraic Ω-Categories

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**Abstract** In this paper, based on Ω-categories, some properties of (continuous) $\mathcal{I}$-cocomplete Ω-categories are studied. Then, we introduce the concepts of bicomplete Ω-category and approximable bimodule, discuss their properties and we also show any $\mathcal{I}$-cocomplete Ω-category is a bicomplete Ω-category. Finally, it is proved that the category of algebraic Ω-categories is equivalent to the category of bicomplete Ω-categories.

**Keywords** Quantitative domain; (algebraic) Ω-category; bicomplete Ω-category; equivalent category

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## 1. Introduction

Domain theory [1,2] was introduced by Scott [3,4] in the 1970s and has been playing a fundamental role in the semantics of programming languages. In the early eighties, de Bakker and Zucker [5] presented a quantitative model of concurrent processes based on metric spaces. From then on, many researchers have been searching for a class of mathematical structures that can serve as (quantitative) domains of computation. Quantitative domain theory (QDT for short) [6–12], which refines ordinary domain theory by replacing the qualitative notion of approximation by a quantitative one of degree of approximation, has undergone active research in the past three decades, and forms a new branch of domain theory. QDT is concerned with models of computation that, in addition to qualitative information, allows also for the extraction of quantitative information, such as determining the speed of convergence or the complexity of a program.

The theory of categories enriched in a commutative unital quantale Ω, or Ω-categories for short, providing quantitative models for semantic of program languages, has received wide attention in quantitative domain theory, including Waszkewicz [13], Hamann and Waszkewicz [14], Lai and Zhang [15]. And a kind of Lawson duality in framework of Ω-categories has been studied by Hofmanna and Waszkewicz [16].

In this paper, based on Ω-categories [15–17], we define bicomplete Ω-categories and mainly

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prove that the category of algebraic $\Omega$-categories is equivalent to the category of bicomplete $\Omega$-categories.

The contents of the paper are organized as follows. In Section 2, we list some preliminary notions and results about $\Omega$-categories. In Section 3, we study some properties of (continuous) $\mathcal{I}$-cocomplete $\Omega$-categories. In Section 4, we propose the definition of bicomplete $\Omega$-category and study its properties. In Section 5, we show that if $\Omega$ satisfies conditions $I \leq \bigvee A$ implies $I \leq x$ for some $x \in A \subseteq \Omega$, $I \leq x * y$ implies $I \leq x$ and $I \leq y$ for any $x, y \in \Omega$, then the category of algebraic $\Omega$-categories is equivalent to the category of bicomplete $\Omega$-categories.

## 2. Preliminaries

Quantale was introduced by Mulvey in [18]. We give some basic properties of quantale that will be used throughout the paper. For general category theory please refer to [19] and for $\Omega$-categories to [15,16,20].

A quantale is a complete lattice $Q$ with an associative binary operation $*$ satisfying $x * (\bigvee_{i\in J} x_i) = \bigvee_{i\in J}(x * x_i)$ and $(\bigvee_{i\in J} x_i) * x = \bigvee_{i\in J}(x_i * x)$, for all $x, x_i \in Q, i \in J$ (here $J$ is any index set).

1 denotes the greatest element of $Q$ and 0 is the least element of $Q$. Let $Q$ be a quantale. Then (1) $Q$ is unital if there is an element $I \in Q$ such that $I * a = a = a * I$ for all $a \in Q$. (2) $Q$ is commutative if $a * b = b * a$ for all $a, b \in Q$. Generally, in a commutative unital quantale, unit element $I$ need not be the greatest element of $\Omega$. When the unit element $I$ is the greatest element of $\Omega$, we call $\Omega$ a complete residuated lattice [21,22].

In this paper, $(\Omega, *, I)$, or just $\Omega$ will always denote a commutative unital quantale if not otherwise specified.

For all $x \in \Omega$, since $x*\_$ preserves arbitrary sups, it has a right adjoint, which we will denote by $x \rightarrow \_$. Thus, $x * y \leq z$ iff $y \leq x \rightarrow z$ iff $x \leq y \rightarrow z$.

**Proposition 2.1** *Suppose that $(\Omega, *, I)$ is a commutative unital quantale. Then*

*(1)* $0 * p = 0$;

*(2)* $p * (\bigvee_i q_i) = \bigvee_i (p * q_i)$;

*(3)* $I \leq p \rightarrow q \Longleftrightarrow p \leq q$;

*(4)* $I \rightarrow p = p, 0 \rightarrow p = 1$;

*(5)* $(p \rightarrow q) * (q \rightarrow r) \leq p \rightarrow q$;

*(6)* $(\bigvee_i p_i) \rightarrow q = \bigwedge_i (p_i \rightarrow q)$;

*(7)* $p \rightarrow (\bigwedge_i q_i) = \bigwedge_i (p \rightarrow q_i)$;

*(8)* $(r \rightarrow p) \rightarrow (r \rightarrow q) \geq p \rightarrow q$;

*(9)* $(p \rightarrow r) \rightarrow (q \rightarrow r) \geq q \rightarrow p$;

*(10)* $p \rightarrow (q \rightarrow r) = (p * q) \rightarrow r$;

*(11)* $\bigwedge_{r\in\Omega}((r \rightarrow p) \rightarrow (r \rightarrow q)) = p \rightarrow q$;

*(12)* $\bigwedge_{r\in\Omega}((p \rightarrow r) \rightarrow (q \rightarrow r)) = q \rightarrow p$.

**Example 2.2** (1) Every complete residuated lattice$(L, *, 1)$ (particularly, every complete Heyting algebra with the binary meet operation $\wedge$) is a commutative unital quantale with the greatest element 1 being the unit.

(2) Let $\Omega = [0, \infty]^{op}$ denote the extended interval of all non-negative real numbers with the opposite ordering as real numbers (so 0 is the greatest element). Let $+$ be the usual addition on real numbers extended to cope with infinity, such that $x + \infty = \infty$ for every $x \in [0, \infty]$. Then $(\Omega, +, 0)$ is a commutative unital quantale, which will be often denoted $[0, \infty]^{op}$ in the sequel [15].

We recall that a category enriched in a commutative unital quantale $\Omega$, or an $\Omega$-category for short, is a set $X$ with a map $X : X \times X \longrightarrow \Omega$, called the structure of $X$, with two properties: $I \leq X(a, a)$ for all $a \in X$ (reflexivity), and $X(a, b) * X(b, c) \leq X(a, c)$ for all $a, b, c \in X$ (transitivity). In this paper, $\Omega$-**Cat** denotes the category of $\Omega$-categories, where morphisms, called $\Omega$-functors, are maps $f : X \longrightarrow Y$ such that $X(x, y) \leq Y(f(x), f(y))$ for all $x, y \in X$. For example: for all $a, b \in \Omega$, let $\Omega(a, b) = a \rightarrow b$. Then $(\Omega, \rightarrow)$ becomes an $\Omega$-category. For an ordinary set $X$, $\Omega^X$ is the set of all maps from $X$ to $\Omega$, the members are called $\Omega$-sets of $X$, subsethood operation sub [23]: $\Omega^X \times \Omega^X \longrightarrow \Omega$ is defined by $\forall\, \phi, \psi \in \Omega^X$, $\mathrm{sub}(\phi, \psi) = \bigwedge_{x \in X}(\phi(x) \rightarrow \psi(x))$, then $(\Omega^X, \mathrm{sub})$ is an $\Omega$-category.

**Definition 2.3** ([15,24]) *Let $X$ be an $\Omega$-category and $x \in X$. Define $\downarrow x \in \Omega^X (\uparrow x \in \Omega^X)$ by $\forall\, y \in X, \downarrow x(y) = X(y, x) (\uparrow x(y) = X(x, y))$.*

**Definition 2.4** ([15,20]) *Let $X$ be an $\Omega$-category and $\phi \in \Omega^X$. An element $a \in X$ is called a supremum (resp., infimum) of $\phi$, in symbols $a = \sqcup\phi$ (resp., $a = \sqcap\phi$), if $X(a, y) = \bigwedge_{x \in X}(\phi(x) \rightarrow X(x, y))$ for all $y \in X$ (resp., $X(y, a) = \bigwedge_{x \in X}(\phi(x) \rightarrow X(y, x))$ for all $y \in X$).*

An $\Omega$-category $X$ is said to be complete if for any $\Omega$-functor $f : K \longrightarrow A$ and any $\psi \in [K, \Omega]$, the weighted limit $\lim_{\psi} f$ exists. $X$ is said to be cocomplete if for any $\Omega$-functor $f : K \longrightarrow A$ and any $\psi \in [K^{op}, \Omega]$ the weighted colimit $\mathrm{colim}_{\phi} f$ exists.

**Proposition 2.5** ([15]) *An $\Omega$-category $X$ is said to be complete, if for all $\phi \in \Omega^X$, $\sqcap\phi$ exists. $X$ is said to be cocomplete if for all $\phi \in \Omega^X$, $\sqcup\phi$ exists.*

**Proposition 2.6** ([15,20,25]) *Given an $\Omega$-category $X$. The following conditions are equivalent:*

*(1) $X$ is complete;*

*(2) $X$ is cocomplete;*

*(3) $\forall\, \phi \in \Omega^X, \sqcap\phi$ exists;*

*(4) $\forall\, \phi \in \Omega^X$, $\sqcup\phi$ exists.*

**Proposition 2.7** ([24]) *Let $X$ be an $\Omega$-category, $x_0 \in X$ and $\phi \in \Omega^X$. Then $x_0$ is the supremum of $\phi$, iff*

*(1) $\forall\, x \in X$, $\phi(x) \leq X(x, x_0)$;*

*(2) $\forall\, y \in X$, $\bigwedge_{x \in X}(\phi(x) \rightarrow X(x, y)) \leq X(x_0, y)$.*

**Definition 2.8** ([15]) *Let $X$ be an Ω-category. $\phi \in \Omega^X$ is called an upper Ω-set(a lower Ω-set) if $\forall\, x,\ y \in X,\ X(x,y) * \phi(x) \le \phi(y)(X(x,y) * \phi(y) \le \phi(x))$.*

Let $X$ be an Ω-category. An Ω-set $\phi$ of $X$ is called a directed set on $X$ if $\bigvee_{x\in X} \phi(x) \ge I$; $\phi(x) * \phi(y) \le \bigvee_{z\in X}(\phi(z) * X(x,z) * X(y,z))$ for all $x, y \in X$. A directed set is called an *ideal* if it is a lower Ω-set additionally. Dually, we can give the concept of filter on $X$. The set of all ideals (filters) on $X$ is denoted by $\mathcal{I}(X)(\mathcal{F}(X))$. Clearly, for all $x \in X, \downarrow x \in \mathcal{I}(X)$ and $\uparrow x \in \mathcal{F}(X)$.

**Definition 2.9** ([24,26]) *Let $X$ be an Ω-category. For all $\phi, \psi \in \Omega^X$, define $\phi^{\uparrow}$, $\psi^{\downarrow}$ by*

$$\forall\, y \in X, \phi^{\uparrow}(y) = \bigwedge_{x\in X}(\phi(x) \to X(x,y)),\ \ \psi^{\downarrow}(y) = \bigwedge_{x\in X}(\psi(x) \to X(y,x)).$$

Suppose $f : X \longrightarrow Y$ is a function, $f^{\rightarrow} : \Omega^X \longrightarrow \Omega^Y$ is defined by $\forall\, \phi \in \Omega^X,\ y \in Y$, $f^{\rightarrow}(\phi)(y) = \bigvee_{x\in X}(\phi(x) * Y(y, f(x)))$.

**Lemma 2.10** ([15]) *Suppose $f : X \longrightarrow Y$ is an Ω-functor and $\phi$ is an ideal on $X$. Then $f^{\rightarrow}(\phi)$ is an ideal on $Y$.*

**Lemma 2.11** ([9]) *Define $\eta : X \longrightarrow \mathcal{I}(X)$ by $\forall\, x \in X,\ \eta(x) = \downarrow x$. Then $\forall\, \psi \in \mathcal{I}(X)$, $\psi = \bigsqcup \eta^{\rightarrow}(\psi)$.*

**Corollary 2.12** ([27]) *An Ω-category $X$ is $\mathcal{I}$-cocomplete iff $\sqcup\phi$ exists for all $\phi \in \mathcal{I}(X)$. An Ω-functor $f$ is $\mathcal{I}$-cocontinuous iff $f(\sqcup\phi) = \sqcup f^{\rightarrow}(\phi)$ for all $\phi \in \mathcal{I}(X)$.*

Suppose $X$ is an $\mathcal{I}$-cocomplete Ω-category, $x \in X$, define $\Downarrow x \in \Omega^X$ by $\forall\, y \in X$, $\Downarrow x(y) = \bigwedge_{I\in\mathcal{I}(X)}(X(x, \sqcup I) \to I(y))$. For all $x \in X$, if $\Downarrow x(x) \ge I$, then we call $x$ a compact element in $X$. The set of all compact elements in $X$ is denoted by $K(X)$.

**Proposition 2.13** ([27]) *Let $X$ be an $\mathcal{I}$-cocomplete Ω-category and $x \in K(X)$. Then $\Downarrow x(x) \ge I$ iff $\forall\, \phi \in \mathcal{I}(X)$, $X(x, \sqcup\phi) = \phi(x)$.*

**Proposition 2.14** ([27]) *For any Ω-category $X$, $\mathcal{I}(X)$ is $\mathcal{I}$-cocomplete.*

**Proposition 2.15** ([27]) *Suppose that $\Phi \in \mathcal{I}(\mathcal{I}(X))$. Then*

*(1)* $\bigsqcup \Phi = \bigvee_{J\in\mathcal{I}(X)} \Phi(J) * J$;

*(2)* $\forall\ x \in X,\ (\bigsqcup \Phi)(x) = \Phi(\downarrow x)$.

**Proposition 2.16** ([27]) *In an $\mathcal{I}$-cocomplete Ω-category $X$, for all $J \in \mathcal{I}(X)$, we have $\Downarrow J(J) = \bigvee_{x\in X} J(x) * \mathrm{sub}(J, \downarrow x)$. It follows that for all $x \in X, \downarrow x$ is a compact element in $\mathcal{I}(X)$.*

## 3. $\mathcal{I}$-cocomplete Ω-category

In this section, we show the set of all ideals on an Ω-category is a continuous $\mathcal{I}$-cocomplete Ω-category.

**Definition 3.1** *Let $X$ be an Ω-category. A mapping $\nu : X \times X \longrightarrow \Omega$ is auxiliary if*

*(i)* $\nu(x,y) \le X(x,y)$;

*(ii)* $X(x,y) * \nu(y,z) * X(z,t) \leq \nu(x,t)$ *for all* $x,y,z,t \in X$.

The collection of auxiliary mappings on $X$ is denoted by $\mathrm{Aux}(X)$.

**Remark 3.2** Auxiliary maps are transitive: $\forall\, x,y,z \in X$, $\nu(x,y)*\nu(y,z) \leq \nu(x,y)*X(y,z) \leq \nu(x,z)$.

**Proposition 3.3** $\mathrm{Aux}(X) \cong \{s | s : X \longrightarrow [X,\Omega]$ *is* $\Omega$*-functor and* $s \leq \downarrow x\}$.

By Proposition 3.3, we will treat $\nu \in$ Aux(X) either as $\nu(\_) : X \longrightarrow [X,\Omega]$ or as $\nu(\_,\_) : X \times X \longrightarrow \Omega$ without further explanation.

**Example 3.4** $\downarrow x, \Downarrow x$ are auxiliary maps.

**Definition 3.5** *An auxiliary map* $\nu : X \longrightarrow [X,\Omega]$ *is approximating if for all* $x \in X : \nu(x) \in \mathcal{I}(X)$ *and* $x = \sqcup \nu(x)$. *Let* $\mathrm{App}(X)$ *denote all the approximating maps.*

The way-below mapping $\Downarrow$ is below all approximating maps. That is to say, for all $\nu \in \mathrm{App}(X)$, $\Downarrow \leq \nu$.

**Definition 3.6** *An* $\Omega$*-category is continuous if its way-below map is approximating.*

**Theorem 3.7** ([13]) *For an auxiliary map* $\nu : X \longrightarrow \mathcal{I}(X)$ *the following statements are equivalent:*

*(1)* $\nu$ *is approximating and* $\mathcal{I}$*-cocontinuous,*

*(2)* $\nu$ *is approximating and coincides with the way-below map,*

*(3)* $\mathrm{sub}(\nu(y),\phi) = X(y,\sqcup\phi)$ *for all* $y \in X$ *and* $\phi \in \mathcal{I}(X)$ *which has supremum.*

**Proposition 3.8** *For any* $\Omega$*-category* $X$, $\mathcal{I}(X)$ *is a continuous* $\mathcal{I}$*-cocomplete* $\Omega$*-category.*

**Proof** It suffices to prove the continuity of $\mathcal{I}(X)$. Clearly, $\downarrow : X \longrightarrow \mathcal{I}(X)$ is a map. We demonstrate that $\downarrow^{\rightarrow} : \mathcal{I}(X) \longrightarrow \mathcal{I}\mathcal{I}(X)$ is approximating and $\mathcal{I}$-cocontinuous.

(1) $\downarrow^{\rightarrow} : \mathcal{I}(X) \longrightarrow \mathcal{I}\mathcal{I}(X)$ is approximating. Firstly, $\downarrow^{\rightarrow}$ is an auxiliary map. For all $\phi,\psi,\eta,\phi \in \mathcal{I}(X)$, $\mathrm{sub}(\phi,\psi) \leq \mathrm{sub}(\downarrow^{\rightarrow}(\phi),\downarrow^{\rightarrow}(\psi))$. $\mathrm{sub}(\phi,\psi)* \downarrow^{\rightarrow}(\psi,\eta) * \mathrm{sub}(\eta,\varphi) = \mathrm{sub}(\phi,\psi)* \downarrow^{\rightarrow}(\eta)(\psi) * \mathrm{sub}(\eta,\varphi) \leq \downarrow^{\rightarrow}(\varphi)(\phi)$. Secondly, for all $\phi \in \mathcal{I}(X), z \in X, \downarrow^{\rightarrow}(\phi) \in \mathcal{I}\mathcal{I}(X)$. $\bigsqcup(\downarrow^{\rightarrow}(\phi))(z) = \bigvee_{\eta\in\mathcal{I}(X)}(\eta(z)* \downarrow^{\rightarrow}(\phi)(\eta)) = \bigvee_{\eta\in\mathcal{I}(X)}(\eta(z) * (\bigvee_{x\in X}\phi(x) * \mathrm{sub}(\eta,\downarrow x))) = \bigvee_{x\in X}(\phi(x) * (\bigvee_{\eta\in\mathcal{I}(X)}\eta(z) * \mathrm{sub}(\eta,\downarrow x)) = \bigvee_{x\in X}(\phi(x) * X(z,x)) = \phi(z)$.

(2) $\downarrow^{\rightarrow} : \mathcal{I}(X) \longrightarrow \mathcal{I}\mathcal{I}(X)$ is $\mathcal{I}$-cocontinuous. For all $\phi \in \mathcal{I}(X), \Phi \in \mathcal{I}\mathcal{I}(X)$,

$\downarrow^{\rightarrow}(\bigsqcup\Phi)(\phi) = \bigvee_{x\in X}(\bigsqcup\Phi(x) * \mathrm{sub}(\phi,\downarrow x)) = \bigvee_{x\in X}(\bigvee_{\eta\in\mathcal{I}(X)}\Phi(\eta) * \eta(x) * \mathrm{sub}(\phi,\downarrow x)) = \bigvee_{\eta\in\mathcal{I}(X)}(\Phi(\eta) * \bigvee_{x\in X}\eta(x) * \mathrm{sub}(\phi,\downarrow x)) = \bigvee_{\eta\in\mathcal{I}(X)}(\Phi(\eta)* \downarrow^{\rightarrow}(\phi,\eta)) = \bigvee_{\eta\in\mathcal{I}(X)}(\Phi(\eta) * \mathrm{sub}(\downarrow\phi,\downarrow^{\rightarrow}(\eta))) = (\downarrow^{\rightarrow})^{\rightarrow}(\Phi)(\downarrow\phi) = \bigsqcup(\downarrow^{\rightarrow})^{\rightarrow}(\Phi)(\phi)$. By Theorem 3.7, $\downarrow^{\rightarrow} = \Downarrow$, and $\Downarrow$ is approximating. $\square$

## 4. Bicomplete $\Omega$-category

In this section, we propose the definitions of bicomplete $\Omega$-category and approximable bimodule. We also study some properties of them.

**Definition 4.1** ([27]) *Let $X$ be an $\mathcal{I}$-cocomplete $\Omega$-category and $x \in X$. Define $k_x : X \longrightarrow \Omega$ by $\forall\, y \in X$,*

$$k_x(y) = \begin{cases} X(y,x), & y \in K(X), \\ 0, & \textit{otherwises.} \end{cases}$$

*If $k_x$ is a directed set on $X$ and $\bigsqcup k_x = x$, then we call $X$ an algebraic $\Omega$-category.*

**Proposition 4.2** *An algebraic $\Omega$-category is a continuous $\Omega$-category.*

**Proof** It follows from Proposition 2.13 and Definitions 3.6, 3.7 and 4.1. □

**Definition 4.3** *Assume $X$ is an $\mathcal{I}$-cocomplete $\Omega$-category, $Y \subseteq X$. Then $Y$ generates $X$ if $\forall\, x \in X$, $\exists\, \phi \in \mathcal{I}(Y)$ such that $x = \bigsqcup i_Y^{\rightarrow}(\phi)$, where $i_Y : Y \longrightarrow X$ is an inclusion mapping.*

**Proposition 4.4** *Let $X$ be an $\mathcal{I}$-cocomplete $\Omega$-category. Then $X$ is an algebraic $\Omega$-category iff $K(X)$ generates $X$.*

**Proof** Necessity. Let $k_x$ be a directed set on $X$ and $\bigsqcup k_x = x$. Then we can easily get $k_x|_{K(X)} \in \mathcal{I}(K(X))$ and $\bigsqcup k_x = \bigsqcup k_x|_{K(X)} = x$. For all $y \in K(X)$, $k_x|_{K(X)}(y) = i_{K(X)}^{\rightarrow}(k_x|_{K(X)})(y)$. Thus, for all $x \in X$, $\exists\, \phi = k_x|_{K(X)} \in \mathcal{I}(K(X))$ such that $x = \bigsqcup i_{K(X)}^{\rightarrow}(\phi)$.

Sufficiency. $\forall\, x \in X$, $\exists\, \phi \in \mathcal{I}(K(X))$ such that $x = \bigsqcup i_{K(X)}^{\rightarrow}(\phi)$. For all $y \in K(X)$,

$$k_x|_{K(X)}(y) = X(y,x) = X(y, \bigsqcup i_{K(X)}^{\rightarrow}(\phi)) = X(y, \bigsqcup \phi) = \phi(y).$$

Hence, $\phi = k_x|_{K(X)}$, $x = \bigsqcup k_x|_{K(X)} = \bigsqcup k_x$. Since $\phi \in \mathcal{I}(K(X))$, we have $k_x$ is a directed set on $X$. □

Let $\Omega$-**ACAT** denote the category of algebraic $\Omega$-categories and $\mathcal{I}$-cocontinuous mappings.

**Proposition 4.5** *Let $X$ be an $\mathcal{I}$-cocomplete $\Omega$-category, $Y \subseteq K(X)$ and $Y$ generates $X$. Then the following statements are true:*

*(1) $X$ is an algebraic $\Omega$-category;*

*(2) If $\phi \in \mathcal{I}(Y)$, and $x = \bigsqcup i^{\rightarrow}(\phi)$. Then $\phi = \downarrow x|_Y$.*

**Proof** (1) Let $i_Y : Y \longrightarrow K(X)$, $i_{K(X)} : K(X) \longrightarrow X$ be inclusion mappings. Since $Y$ generates $X$, then $\forall\, x \in X$, $\exists\, \phi \in \mathcal{I}(Y)$ such that $x = \bigsqcup (i_{K(X)} \circ i_Y)^{\rightarrow}(\phi)$. Because $(i_{K(X)} \circ i_Y)^{\rightarrow}(\phi) = i_{K(X)}^{\rightarrow}(i_Y^{\rightarrow}(\phi))$, $x = \bigsqcup i_{K(X)}^{\rightarrow}(i_Y^{\rightarrow}(\phi))$ and $i_Y^{\rightarrow}(\phi) \in \mathcal{I}(K(X))$. Thus, $X$ is an algebraic $\Omega$-category.

(2) For all $y \in Y$, $i^{\rightarrow}(\phi)(y) = \bigvee_{z \in Y} \phi(z) * X(y,z) = \phi(y)$. Since $Y \subseteq K(X)$, then $y \in K(X)$ is a compact element, $\phi(y) = i^{\rightarrow}(\phi)(y) = X(y, \bigsqcup i^{\rightarrow}(\phi)) = X(y,x)$. Therefore, $\phi = \downarrow x|_Y$. □

**Theorem 4.6** *Let $X$ be an $\mathcal{I}$-cocomplete $\Omega$-category, $Y \subseteq K(X)$ and $Y$ generates $X$. Define $\eta : X \longrightarrow \mathcal{I}(Y)$, $\varepsilon : \mathcal{I}(Y) \longrightarrow X$ by $\forall\, x \in X$, $\eta(x) = \downarrow x|_Y$, $\forall\, \phi \in \mathcal{I}(Y)$, $\varepsilon(\phi) = \bigsqcup i^{\rightarrow}(\phi)$, respectively. Then $X \cong \mathcal{I}(Y)$.*

**Proof** By Proposition 4.5, we know that $X$ is an algebraic $\Omega$-category. First we will prove that the definitions are reasonable. For all $x \in X$, since $Y$ generates $X$, then there exists $\phi \in \mathcal{I}(Y)$ such that $x = \sqcup i_Y^{\rightarrow}(\phi)$, by Proposition 4.5, $\phi = \downarrow x|_Y$. So $\downarrow x|_Y = \phi \in \mathcal{I}(Y)$. Because

$i^{\rightarrow}(\phi) \in \mathcal{I}(X)$ and $X$ is an $\mathcal{I}$-cocomplete category, we have $\bigsqcup i^{\rightarrow}(\phi) \in X$, that is to say, $\varepsilon$ is well defined.

(i) $\eta$, $\varepsilon$ are $\Omega$-functors. For all $x, y \in X$, $X(x,y) = \mathrm{sub}(\downarrow x, \downarrow y) \leq \mathrm{sub}(\downarrow x|_Y, \downarrow y|_Y)$. For all $\phi$, $\psi \in \mathcal{I}(Y)$,

$$\begin{aligned} X(\bigsqcup i^{\rightarrow}(\phi), \bigsqcup i^{\rightarrow}(\psi)) &= \bigwedge_{x\in X} (i^{\rightarrow}(\phi)(x) \rightarrow X(x, \bigsqcup i^{\rightarrow}(\psi))) \\ &= \bigwedge_{x\in X} ((\bigvee_{y\in Y} \phi(y) * X(x,y)) \rightarrow X(x, \bigsqcup i^{\rightarrow}(\psi))) \\ &= \bigwedge_{y\in Y} (\phi(y) \rightarrow X(y, \bigsqcup i^{\rightarrow}(\psi))) \\ &\geq \bigwedge_{y\in Y} (\phi(y) \rightarrow i^{\rightarrow}(\psi)(y)) \\ &= \bigwedge_{y\in Y} (\phi(y) \rightarrow \psi(y)) = \mathrm{sub}(\phi, \psi). \end{aligned}$$

(ii) By Proposition 4.5, for all $x \in X$, $\varepsilon \circ \eta(x) = \varepsilon(\downarrow x|_Y) = \bigsqcup i^{\rightarrow}(\downarrow x|_Y) = x$. Since $\forall \phi \in \mathcal{I}(Y)$, $y \in Y \subseteq K(X)$, $X(y, \bigsqcup i^{\rightarrow}(\phi)) = i^{\rightarrow}(\phi)(y) = \phi(y)$, $\eta \circ \varepsilon(\phi) = \eta(\bigsqcup i^{\rightarrow}(\phi)) = (\downarrow \bigsqcup i^{\rightarrow}(\phi))|_Y = \phi$. Then $X \cong \mathcal{I}(Y)$. □

**Theorem 4.7** *Let $X$ be an $\Omega$-category, define $\eta : X \longrightarrow \mathcal{I}(X)$ by $\forall x \in X$, $\eta(x) =\downarrow x$. Then $\eta(X) \subseteq K(\mathcal{I}(X))$, $\eta(X)$ generates $\mathcal{I}(X)$. Therefore, $\mathcal{I}(X)$ is algebraic [27].*

**Proof** For all $\Phi \in \mathcal{I}(\mathcal{I}(X))$, $x \in X$, $\mathrm{sub}(\downarrow x, \bigsqcup \Phi) = (\bigsqcup \Phi)(x) = \Phi(\downarrow x)$, that is, $\Downarrow\downarrow x(\downarrow x) \geq I$, so $\eta(X) \subseteq K(\mathcal{I}(X))$. By Lemma 2.11, for all $\phi \in \mathcal{I}(X)$, $\phi = \bigsqcup \eta^{\rightarrow}(\phi) = \bigsqcup i^{\rightarrow}(\eta^{\rightarrow}(\phi))$, note that $\eta^{\rightarrow}(\phi) \in \mathcal{I}(\eta(X))$, then $\eta(X)$ generates $\mathcal{I}(X)$, so by Proposition 4.5, $\mathcal{I}(X)$ is algebraic. □

**Definition 4.8** *Let $X$ be an $\Omega$-category, $A \subseteq X$ and $cl(A) = \{x \in X | I \leq \bigvee_{a\in A}(X(x,a) * X(a,x))\}$. If $cl(A) = A$, then we call $A$ a symmetrically closed set.*

**Lemma 4.9** *Let $X$ be an algebraic $\Omega$-category. Then $K(X)$ is a symmetrically closed set.*

**Proof** It suffices to show $\mathrm{cl}(K(X)) \subseteq K(X)$. For all $x \in \mathrm{cl}(K(X))$ and $\phi \in \mathcal{I}(X)$,

$$\begin{aligned} X(x, \sqcup\phi) = X(x, \sqcup\phi) * I &\leq X(x, \sqcup\phi) * \bigvee_{a\in K(X)} (X(x,a) * X(a,x)) \\ &= \bigvee_{a\in K(X)} (X(x, \sqcup\phi) * X(x,a) * X(a,x)) \\ &\leq \bigvee_{a\in K(X)} (X(a, \sqcup\phi) * X(x,a)) \leq \phi(x). \end{aligned}$$

Hence, $X(x, \sqcup\phi) = \phi(x)$, $\Downarrow x(x) \geq I$ for all $x \in K(X)$. □

**Definition 4.10** *Let $X$ be an $\Omega$-category, the pair $(\phi, \psi)$ will be called an ideal-filter if $\phi \in \mathcal{I}(X)$, $\psi \in \mathcal{F}(X)$ and the following conditions hold:*

*(1)* $\bigvee_{x\in X}(\phi(x) * \psi(x)) \geq I$;

*(2)* $\forall x, y \in X$, $\phi(x) * \psi(y) \leq X(x, y)$.

An ideal-filter must be a small projective [15].

**Example 4.11** $\forall a \in X$, $(\downarrow a, \uparrow a)$ ia an ideal-filter of $X$.

**Example 4.12** Let $\Omega = \{0, a, b, 1\}$ be the diamond lattice, that is $0 \leq a, b \leq 1$ and $a \parallel b$. Define $* : \Omega \times \Omega \to \Omega$ by

| $*$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | $b$ | 1 |
| $b$ | 0 | $b$ | $a$ | 1 |
| 1 | 0 | 1 | 1 | 1 |

It is easy to show that Ω is a commutative unital quantale. Clearly,

| $\to$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | 0 | $a$ | $b$ | 1 |
| $b$ | 0 | $b$ | $a$ | 1 |
| 1 | 0 | 0 | 0 | 1 |

Let $X = \Omega$. We have $(X, \to)$ is an Ω-category. We can easily prove $(\Downarrow a, \Uparrow a)$ is a small projective weight but not an ideal-filter.

**Definition 4.13** *Let $X$ be an Ω-category. If for each ideal-filter $(\phi, \psi)$ of $X$, there exists $x \in X$ such that $\phi = \downarrow x$, $\psi = \uparrow x$, then we call $X$ a bicomplete Ω-category.*

**Proposition 4.14** *Let $X$ be an Ω-category and $(\phi, \psi)$ be an ideal-filter of X. Then $\phi = \psi^{\downarrow}$, $\psi = \phi^{\uparrow}$.*

**Lemma 4.15** *Let $X$ be an $\mathcal{I}$-cocomplete Ω-category. Then $X$ is a bicomplete Ω-category.*

**Proof** Let $(\phi, \psi)$ be an ideal-filter of $X$. Then $\sqcup\phi = a$. For all $x \in X$, $X(a, x) = X(\sqcup\phi, x) = \bigwedge_{y \in X}(\phi(y) \to X(y, x)) = \psi(x)$, $\psi = \uparrow a$. By Proposition 4.14, $\phi(x) = \psi^{\downarrow}(x) = \bigwedge_{y \in X}(\psi(y) \to X(x, y)) = \bigwedge_{y \in X}(\uparrow a(y) \to X(x, y)) = \bigwedge_{y \in X}(X(a, y) \to X(x, y)) = X(x, a) = \downarrow a(x)$, so $\phi = \downarrow a$. Hence, $X$ is a bicomplete Ω-category. □

**Proposition 4.16** *Let $X$ be a bicomplete Ω-category and $A \subseteq X$. Then $A$ is a symmetrically closed set iff $A$ is a bicomplete Ω-category.*

**Proof** Necessity. Let $(\phi,\psi)$ be an ideal-filter of $A$. Define $\phi',\psi'$ as follows:

$$\forall\, x\in X,\ \phi'(x)=\bigvee_{a\in A}(\phi(a)*X(x,a)),\ \ \psi'(x)=\bigvee_{a\in A}(\psi(a)*X(a,x)).$$

Then it is easy to check that $(\phi',\psi')$ is an ideal-filter of $X$.

Because $X$ is a bicomplete $\Omega$-category, we obtain that there is $x\in X$ such that $\phi'=\downarrow x$ and $\psi'=\uparrow x$. Note that $I\le\phi'(x)*\psi'(x)=\bigvee_{a,a'\in A}(\phi(a)*\psi(a')*X(x,a)*X(a',x))\le\bigvee_{a,a'\in A}(X(a,a')*X(x,a)*X(a',x))\le\bigvee_{a\in A}(X(x,a)*X(a,x))$, then $x\in cl(A)=A$.

Sufficiency. Let $A$ be a bicomplete $\Omega$-category. We will prove $A$ is a symmetrically closed set. It suffices to show $\mathrm{cl}(A)\subseteq A$. $\forall\, a\in\mathrm{cl}(A)$, $I\le\bigvee_{x\in A}X(a,x)*X(x,a)$, by $\downarrow a(x)*\uparrow a(y)\le X(x,y)$, $(\downarrow a,\uparrow a)$ is an ideal-filter of $A$, so there is a $b\in A$ such that $\downarrow a=\downarrow b$, $\uparrow a=\uparrow b$. Thus, $a=b\in A$. $\square$

**Definition 4.17** *Let $X,Y$ be $\Omega$-categories. A bimodule $\alpha:X\multimap Y$ is a mapping $\alpha:Y\times X\to\Omega$ such that $\alpha(y,x)*Y(y',y)\le\alpha(y',x)$; $\alpha(y,x)*X(x,x')\le\alpha(y,x')$ for all $x,x'\in X$, $y,y'\in Y$.*

**Remark 4.18** It can be easily seen that a map $\alpha:Y\times X\to\Omega$ is a bimodule iff $\alpha:Y^{op}\times X\to\Omega$ is an $\Omega$-functor.

**Example 4.19** Define $\mathrm{Id}_X:X\multimap X$ by $\forall\,(x',x)\in X\times X$, $\mathrm{Id}_X(x',x)=X(x',x)$. Then $\mathrm{Id}_X$ is a bimodule, we call $\mathrm{Id}_X$ an identity bimodule.

**Proposition 4.20** *Let $\alpha:X\multimap Y$, $\beta:Y\multimap Z$ be bimodule functions. Define $\beta\circ\alpha:X\multimap Z$ by*

$$\forall\,(z,x)\in Z\times X,\ \beta\circ\alpha(z,x)=\bigvee_{y\in Y}\beta(z,y)*\alpha(y,x).$$

*Then $\beta\circ\alpha$ is a bimodule.*

**Definition 4.21** *Let $\alpha:X\multimap Y$ be bimodule. If $\forall\,x\in X$, $\alpha(\_,x):Y\to\Omega$ is an ideal of $Y$, then we call $\alpha$ an approximable bimodule.*

**Definition 4.22** *Let $X,Y$ be $\Omega$-categories and $\alpha:X\multimap Y$ be an approximable bimodule. We say that $\alpha$ is an isomorphism between $X$ and $Y$ if there is an approximable bimodule $\alpha':Y\multimap X$ such that $\alpha'\circ\alpha=\mathrm{Id}_X$, $\alpha\circ\alpha'=\mathrm{Id}_Y$.*

**Theorem 4.23** *Let $X,Y$ and $Z$ be $\Omega$-categories. Then*

*(1) $\mathrm{Id}_X:X\multimap X$ is an approximable bimodule.*

*(2) If $\alpha:X\multimap Y$, $\beta:Y\multimap Z$ are approximable bimodules, then $\beta\circ\alpha:X\multimap Z$ is an approximable bimodule.*

**Proof** (1) It suffices to show that for all $x\in X$, $\mathrm{Id}_X(\_,x):X\to\Omega$ is an ideal of $X$. Since $\bigvee_{x'\in X}\mathrm{Id}_X(x',x)=\bigvee_{x'\in X}X(x',x)\ge X(x,x)\ge I$, $\forall\,x_1,x_2\in X,\mathrm{Id}_X(x_1,x)*\mathrm{Id}_X(x_2,x)=\mathrm{Id}_X(x,x)*X(x_1,x)*X(x_2,x)\le\bigvee_{x'\in X}\mathrm{Id}_X(x',x)*X(x_1,x')*X(x_2,x')$, then $\mathrm{Id}_X:X\multimap X$ is an approximable bimodule.

(2) It suffices to show that for all $x \in X$, $\beta\circ\alpha(\_, x) : Z \to \Omega$ is an ideal of $Z$. Since $\bigvee_{z\in Z} \beta\circ\alpha(z,x) = \bigvee_{z\in Z}\bigvee_{y\in Y}\beta(z,y)*\alpha(y,x) = \bigvee_{z\in Z}((\bigvee_{y\in Y}\beta(z,y))*\alpha(y,x)) = \bigvee_{y\in Y}((\bigvee_{z\in Z}\beta(z,y))*\alpha(y,x)) \geq \bigvee_{y\in Y}(I*\alpha(y,x)) = \bigvee_{y\in Y}(\alpha(y,x)) \geq I$ and for all $z_1, z_2 \in Z$,

$$\begin{aligned}
&\beta\circ\alpha(z_1,x)*\beta\circ\alpha(z_2,x)\\
&= (\bigvee_{y_1\in Y}\beta(z_1,y_1)*\alpha(y_1,x))*\bigvee_{y_2\in Y}(\beta(z_2,y_2)*\alpha(y_2,x))\\
&\leq \bigvee_{y_1,y_2\in Y}(\beta(z_1,y_1)*\beta(z_2,y_2)*(\bigvee_{y\in Y}\alpha(y,x)*Y(y_1,y)*Y(y_2,y)))\\
&\leq \bigvee_{y\in Y}(\beta(z_1,y)*\beta(z_2,y)*\alpha(y,x))\\
&\leq \bigvee_{y\in Y}\bigvee_{z\in Z}(\beta(z,y)*Z(z_1,z)*Z(z_2,z)*\alpha(y,x))\\
&= \bigvee_{z\in Z}(\beta\circ\alpha(z,x)*Z(z_1,z)*Z(z_2,z)).
\end{aligned}$$

Therefore, $\beta\circ\alpha : X \multimap Z$ is an approximable bimodule. □

Let Ω-**BCAT** denote the category of bicomplete Ω-categories and approximable bimodules.

## 5. Equivalent category

In this section, we mainly prove that if Ω satisfies conditions $I \leq \bigvee A$ implies $I \leq x$ for some $x \in A \subseteq \Omega$, and $I \leq x*y$ implies $I \leq x$ and $I \leq y$ for any $x, y \in \Omega$, then the category of algebraic Ω-categories is equivalent to the category of bicomplete Ω-categories.

**Proposition 5.1** *Let $X$ be an algebraic Ω-category. Then $K(X)$ is a bicomplete Ω-category.*

**Proof** It follows from Lemmas 4.9, 4.15 and Proposition 4.16. □

**Proposition 5.2** *Suppose $f : X \longrightarrow Y \in \mathcal{M}$ or (Ω-**BCAT**), define $K(f) : K(X) \multimap K(Y)$ as follows:*

$$\forall\,(b,a) \in K(Y)\times K(X),\ K(f)(b,a) = Y(b, f(a)).$$

*Then $K$ : Ω-**ACAT** $\longrightarrow$ Ω-**BCAT** is a functor.*

**Proof** Clearly, $K(f)$ is a bimodule function. It suffices to show that $K(f)$ is approximable, that is, $\forall\, a \in K(X)$, $K(f)(\_, a) \in \mathcal{I}(K(Y))$. Because $Y \in \mathcal{O}b$(Ω-**ACAT**), $f(a) \in Y$, by Definition 4.3 and Proposition 4.4 we know that there exists a $\phi \in \mathcal{I}(K(Y))$ such that $f(a) = \bigsqcup(i_{K(Y)})^{\rightarrow}(\phi)$. $\downarrow f(a)|_{K(Y)} = \downarrow \bigsqcup(i_{K(Y)})^{\rightarrow}(\phi)|_{K(Y)} = \phi \in \mathcal{I}(K(Y))$. Thus $K(f)(\_, a) \in \mathcal{I}(K(Y))$. Clearly, $K$ preserves identity morphisms and composition. □

**Proposition 5.3** *Suppose $\alpha : X \multimap Y$ is an approximable bimodule, $X, Y$ are bicomplete Ω-categories, define $\mathcal{I}(\alpha) : \mathcal{I}(X) \longrightarrow \mathcal{I}(Y)$ by*

$$\forall\,\phi \in \mathcal{I}(X),\ b \in Y,\ \mathcal{I}(\alpha)(\phi)(b) = \bigvee_{a\in X}\alpha(b,a)*\phi(a).$$

*Then $\mathcal{I}(\alpha):\mathcal{I}(X)\longrightarrow\mathcal{I}(B)$ is an $\mathcal{I}$-cocontinuous mapping.*

**Proof** Step 1. $\mathcal{I}(\alpha)(\phi)\in\mathcal{I}(Y)$.

(i) For all $b,b'\in B$, $\mathcal{I}(\alpha)(\phi)(b)*Y(b',b)=\bigvee_{a\in X}\alpha(b,a)*\phi(a)*Y(b',b)\le\mathcal{I}(\alpha)(\phi)(b')$;

(ii) For all $a\in X$, $\alpha(\_,a)$ is an ideal of $Y$, so $\bigvee_{b\in Y}\alpha(b,a)\ge I$. Since $\phi\in\mathcal{I}(A)$, $\bigvee_{a'\in X}\phi(a')\ge I$, then $\bigvee_{b\in Y}\mathcal{I}(\alpha)(\phi)(b)=\bigvee_{b\in Y}\bigvee_{a\in X}\alpha(b,a)*\phi(a)\ge I$.

(iii) For all $b_1,b_2\in Y$,

$$\begin{aligned}\mathcal{I}(\alpha)(\phi)(b_1)*\mathcal{I}(\alpha)(\phi)(b_2)&=(\bigvee_{a_1\in X}\alpha(b_1,a_1)*\phi(a_1))*(\bigvee_{a_2\in X}\alpha(b_2,a_2)*\phi(a_2))\\&=\bigvee_{a_1,a_2\in X}(\alpha(b_1,a_1)*\alpha(b_2,a_2)*\phi(a_1)*\phi(a_2))\\&\le\bigvee_{a_1,a_2,a\in X}(\alpha(b_1,a_1)*\alpha(b_2,a_2)*\phi(a)*X(a_1,a)*X(a_2,a))\\&\le\bigvee_{a\in X}(\alpha(b_1,a)*\alpha(b_2,a)*\phi(a))\\&\le\bigvee_{b\in Y}\bigvee_{a\in X}(\alpha(b,a)*Y(b_1,b)*Y(b_2,b)*\phi(a))\\&=\bigvee_{b\in Y}(\bigvee_{a\in X}\alpha(b,a)*\phi(a))*Y(b_1,b)*Y(b_2,b)\\&=\bigvee_{b\in Y}\mathcal{I}(\alpha)(\phi)(b)*Y(b_1,b)*Y(b_2,b).\end{aligned}$$

Therefore, $\mathcal{I}(\alpha)$ is well defined.

Step 2. $\mathcal{I}(\alpha):\mathcal{I}(A)\longrightarrow\mathcal{I}(B)$ is an $\mathcal{I}$-cocontinuous mapping.

(i) $\mathcal{I}(\alpha)$ is an $\Omega$-functor. For all $\phi,\ \psi\in\mathcal{I}(X)$,

$$\begin{aligned}\mathrm{sub}(\mathcal{I}(\alpha)(\phi),\mathcal{I}(\alpha)(\psi))&=\bigwedge_{b\in Y}(\mathcal{I}(\alpha)(\phi)(b)\to\mathcal{I}(\alpha)(\psi)(b))\\&=\bigwedge_{b\in Y}\big(\bigvee_{a\in X}\alpha(b,a)*\phi(a)\big)\to\big(\bigvee_{c\in X}\alpha(b,c)*\psi(c)\big)\\&=\bigwedge_{b\in Y}(\bigwedge_{a\in X}(\alpha(b,a)*\phi(a)\to(\bigvee_{c\in X}\alpha(b,c)*\psi(c))))\\&\ge\bigwedge_{b\in Y}\bigwedge_{a\in X}(\alpha(b,a)*\phi(a)\to\alpha(b,a)*\psi(a))\\&\ge\bigwedge_{a\in X}(\phi(a)\to\psi(a))=\mathrm{sub}(\phi,\psi).\end{aligned}$$

(ii) For all $\Phi\in\mathcal{I}\mathcal{I}(A)$, $b\in Y$,

$$\begin{aligned}\mathcal{I}(\alpha)(\bigsqcup\Phi)(b)&=\bigvee_{a\in X}\alpha(b,a)*(\bigsqcup\Phi)(a)=\bigvee_{a\in X}\bigvee_{\phi\in\mathcal{I}(X)}\alpha(b,a)*\Phi(\phi)*\phi(a)\\&=\bigvee_{\phi\in\mathcal{I}(X)}\Phi(\phi)*\mathrm{sub}(\downarrow b,\mathcal{I}(\alpha)(\phi))=\mathcal{I}(\alpha)^{\rightarrow}(\Phi)(\downarrow b)\\&=(\bigsqcup\mathcal{I}(\alpha)^{\rightarrow}(\Phi))(b).\end{aligned}$$

Hence, $\mathcal{I}(\alpha): \mathcal{I}(X) \longrightarrow \mathcal{I}(Y)$ is an $\mathcal{I}$-cocontinuous mapping. □

By Proposition 5.3, we have the following Corollary.

**Corollary 5.4** $\mathcal{I}$: Ω-**BCAT**⟶ Ω-**ACAT** *is a functor.*

In order to show the equivalent of Ω-**BCAT** and Ω-**ACAT**, we need two additional conditions for Ω:

(i) $I \leq \bigvee A$ implies $I \leq x$ for some $x \in A \subseteq \Omega$,

(ii) $I \leq x * y$ implies $I \leq x$ and $I \leq y$ for any $x, y \in \Omega$.

Here, for sake of completeness, we give the following example which has appeared in [27] to show Ω which satisfies the conditions (i), (ii) is nontrivial, $* \neq \wedge$ and $I \neq 1$.

**Example 5.5** Let $\Omega = \{0, a, b, 1\}$ be the diamond lattice, that is $0 \leq a, b \leq 1$ and $a \parallel b$. Define $* : \Omega \times \Omega \to \Omega$ by

| $*$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $b$ | $1$ |
| $b$ | $0$ | $b$ | $b$ | $b$ |
| $1$ | $0$ | $1$ | $b$ | $1$ |

Let $X$ be an Ω-category. We can easily get $K(\mathcal{I}(X)) = \{\downarrow x \mid x \in X\}$.

**Proposition 5.6** $\eta : I_{\Omega-\mathbf{BCAT}} \longrightarrow K \circ \mathcal{I}$ *is a natural isomorphism.*

**Proof** For all $X \in \mathcal{O}b(\Omega\text{-}\mathbf{BCAT})$, $\eta_X : A \multimap K\mathcal{I}(X)$ is defined by

$$\forall\, (\phi, a) \in K\mathcal{I}(X) \times X,\ \eta_X(\phi, a) = \mathrm{sub}(\phi, \downarrow a).$$

Step 1. $\eta_X$ is an approximable bimodule.

(i) For all $\phi,\ \psi \in K(\mathcal{I}(X)),\ a,\ a' \in X$, $\eta_X(\phi, a) * \mathrm{sub}(\psi, \phi) \leq \eta_X(\psi, a)$; $\eta_X(\phi, a) * X(a, a') \leq \eta_X(\phi, a')$;

(ii) For all $a \in X$, $\phi_1, \phi_2 \in K(\mathcal{I}(X))$, $\eta_X(\downarrow a, a) = \mathrm{sub}(\downarrow a, \downarrow a) \geq I$;

$$\begin{aligned}\eta_X(\phi_1, a) * \eta_X(\phi_2, a) =& \mathrm{sub}(\phi_1, \downarrow a) * \mathrm{sub}(\phi_2, \downarrow a)\\ &\leq \mathrm{sub}(\downarrow a, \downarrow a) * \mathrm{sub}(\phi_1, \downarrow a) * \mathrm{sub}(\phi_2, \downarrow a)\\ &\leq \bigvee_{\phi \in K(\mathcal{I}(X))} \eta_X(\phi, a) * \mathrm{sub}(\phi_1, \phi) * \mathrm{sub}(\phi_2, \phi).\end{aligned}$$

Hence, $\eta_X(\_, a) \in \mathcal{I}(K(\mathcal{I}(X)))$.

Step 2. $\eta'_X : K\mathcal{I}(X) \multimap A$ is defined by

$$\forall\, (a, \phi) \in X \times K\mathcal{I}(X),\ \eta'_X(a, \phi) = \mathrm{sub}(\downarrow a, \phi).$$

Easily, we can prove $\eta'_X$ is an approximable bimodule. For all $(a, a') \in X \times X$, $(\phi, \phi') \in$

$\mathcal{I}(X) \times K\mathcal{I}(X)$,

$$\begin{aligned}\eta'_X \circ \eta_X(a,a') &= \bigvee_{\phi \in K\mathcal{I}(X)} \eta'_X(a,\phi) * \eta_X(\phi,a') \\ &= \bigvee_{\phi \in K\mathcal{I}(X)} \mathrm{sub}(\downarrow a,\phi) * \mathrm{sub}(\phi,\downarrow a') \\ &= X(a,a') = \mathrm{Id}_X(a,a'). \\ \eta_X \circ \eta'_X(\phi,\phi') &= \bigvee_{a \in X} \eta_X(\phi,a) * \eta'_X(a,\phi') \\ &= \bigvee_{a \in X} \mathrm{sub}(\phi,\downarrow a) * \mathrm{sub}(\downarrow a,\phi') \\ &= \mathrm{sub}(\phi,\phi') = \mathrm{Id}_{K\mathcal{I}(X)}(\phi,\phi').\end{aligned}$$

Therefore, $\eta'_X \circ \eta_X = \mathrm{Id}_X$, $\eta_X \circ \eta'_X = \mathrm{Id}_{K\mathcal{I}(X)}$.

Step 3. $(K \circ \mathcal{I}) \circ \eta_X = \eta_Y \circ \alpha$. For all $(\downarrow b, a) \in K \circ \mathcal{I}(Y) \times X$,

$$\begin{aligned}K \circ \mathcal{I}(\alpha) \circ \eta_X(\downarrow b,a) &= \bigvee_{\downarrow c \in K\mathcal{I}(X)} K\mathcal{I}(\alpha)(\downarrow b,\downarrow c) * \eta_X(\downarrow c,a) \\ &= \bigvee_{\downarrow c \in K\mathcal{I}(X)} \mathrm{sub}(\downarrow b,\mathcal{I}(\alpha)(\downarrow c)) * \mathrm{sub}(\downarrow c,\downarrow a).\end{aligned}$$

However, $\mathcal{I}(\alpha)(\downarrow c)(a') = \bigvee_{a' \in X} \alpha(a',a'') * X(a'',c) = \alpha(a',c)$. Thus

$$(K \circ \mathcal{I}) \circ \eta_X(\downarrow b,a) = \bigvee_{c \in X} \bigwedge_{a' \in X} (X(a',b) \to \alpha(a',c)) * X(c,a) = \alpha(b,a).$$

Since

$$(\eta_Y \circ \alpha)(\downarrow b,a) = \bigvee_{c \in X} \eta_Y(\downarrow b,c) * \alpha(c,a) = \bigvee_{c \in X} X(b,c) * \alpha(c,a) = \alpha(b,a),$$

we have $(K \circ \mathcal{I}) \circ \eta_X = \eta_Y \circ \alpha$. $\square$

**Proposition 5.7** $\varepsilon : \mathcal{I} \circ K \longrightarrow \mathrm{Id}_{\Omega\text{-}\mathbf{ACAT}}$ *is a natural isomorphism.*

**Proof** Suppose $\phi \in \mathcal{I}(K(X))$, $X \in \mathcal{O}b(\Omega\text{-}\mathbf{ACAT})$. Define $\varepsilon_X : \mathcal{I}(K(X)) \longrightarrow X$ by

$$\forall\, \phi \in \mathcal{I}(K(X)),\ \varepsilon_X(\phi) = \bigsqcup i^{\rightarrow}(\phi).$$

Step 1. $\mathrm{sub}(\phi,\psi) = X(\bigsqcup \phi, \bigsqcup \psi) = X(\bigsqcup i^{\rightarrow}(\phi), \bigsqcup i^{\rightarrow}(\psi))$, $\varepsilon_X$ is an $\Omega$-functor.

Step 2. For all $\Phi \in \mathcal{I}(\mathcal{I}(K(X)))$, we will prove $\varepsilon_X(\bigsqcup \Phi) = \bigsqcup \varepsilon_X^{\rightarrow}(\Phi)$.

$$\begin{aligned}X(\varepsilon_X(\bigsqcup \Phi),y) &= \bigwedge_{x \in X} ((\bigsqcup \Phi)(x) \to X(x,y)) \\ &= \bigwedge_{x \in X} ((\bigvee_{\phi \in \mathcal{I}K(X)} \Phi(\phi) * \phi(x)) \to X(x,y)) \\ &= \bigwedge_{\phi \in \mathcal{I}K(X)} (\Phi(\phi) \to X(\bigsqcup \phi, y)) \\ &= \bigwedge_{x \in X} ((\bigvee_{\phi \in \mathcal{I}K(X)} \Phi(\phi) * X(x,\bigsqcup \phi)) \to X(x,y))\end{aligned}$$

$$= \bigwedge_{x \in X} \varepsilon_X^{\rightarrow}(\Phi)(x) \to X(x, y).$$

Step 3. Define $\varepsilon'_X : X \longrightarrow \mathcal{I}(K(X))$ by $\forall\, x \in X$, $\varepsilon'_X(x) = \downarrow x|_{K(X)}$. Then $\varepsilon'_X$ is well defined and an Ω-functor. For all $J \in \mathcal{I}(X)$, $\psi \in \mathcal{I}(K(X))$,

$$\begin{aligned}
&\bigwedge_{\phi \in \mathcal{I}(K(X))} ((\varepsilon'_X)^{\rightarrow}(J)(\phi) \to \mathrm{sub}(\phi, \psi)) \\
&= \bigwedge_{\phi \in \mathcal{I}(K(X))} ((\bigvee_{x \in X} J(x) * \mathrm{sub}(\phi, \downarrow x|_{K(X)})) \to \mathrm{sub}(\phi, \psi)) \\
&= \bigwedge_{x \in X} (J(x) \to (\bigwedge_{\phi \in \mathcal{I}(K(X))} \mathrm{sub}(\phi, \downarrow x|_{K(X)}) \to \mathrm{sub}(\phi, \psi))) \\
&= \bigwedge_{x \in X} (J(x) \to \mathrm{sub}(\downarrow x|_{K(X)}, \psi)) \\
&= \bigwedge_{x \in X} (J(x) \to (\bigwedge_{a \in K(X)} X(a, x) \to \psi(a))) \\
&= \bigwedge_{a \in K(X)} ((\bigvee_{x \in X} J(x) * X(a, x)) \to \psi(a)) \\
&= \bigwedge_{a \in K(X)} (J(a) \to \psi(a)) = \bigwedge_{a \in K(X)} (X(a, \bigsqcup J) \to \psi(a)) \\
&= \bigwedge_{a \in K(X)} ((\downarrow (\bigsqcup J)|_{K(X)})(a) \to \psi(a)) \\
&= \mathrm{sub}(\varepsilon'_X(\bigsqcup J), \phi).
\end{aligned}$$

We obtain that $\varepsilon'_X$ is an $\mathcal{I}$-cocontinuous function. For all $x \in X, J \in \mathcal{I}(X), \varepsilon_X \circ \varepsilon'_X(x) = \bigsqcup(\downarrow x|_{K(X)}) = x$, $\varepsilon'_X \circ \varepsilon_X(J) = \varepsilon'_X(\bigsqcup J) = \downarrow (\bigsqcup J)|_{K(X)} = J$.

Step 4. Suppose $f : X \longrightarrow Y \in \mathcal{M}$ or (Ω-**ACAT**), we will prove the following diagram is commutative.

$$\begin{array}{ccc}
\mathcal{I} \circ K(X) & \xrightarrow{\ \varepsilon_X\ } & X \\
{\scriptstyle \mathcal{I} \circ K(f)} \big\downarrow & & \big\downarrow {\scriptstyle f} \\
\mathcal{I} \circ K(Y) & \xrightarrow[\ \varepsilon_{X'}\ ]{} & Y
\end{array}$$

Diagram 1 Commutative diagram

For all $\phi \in \mathcal{I} \circ K(X), f \circ \varepsilon_X(\phi) = f(\bigsqcup i^{\rightarrow}(\phi)) = f(\bigsqcup \phi) = \bigsqcup f^{\rightarrow}(\phi)$. $\forall\, b \in K(Y)$, $f^{\rightarrow}(\phi)(b) = \bigvee_{a \in K(X)} Y(b, f(a)) * \phi(a)$, $\mathcal{I} \circ K(f)(\phi)(b) = \bigvee_{a \in K(X)} K(f)(b, a) * \phi(a) = \bigvee_{a \in K(X)} Y(b, f(a)) * \phi(a)$. Therefore, $f \circ \varepsilon_X = \varepsilon_{X'} \circ (\mathcal{I} \circ K)(f)$. □

By Propositions 5.6 and 5.7, we can get the following Theorem.

**Theorem 5.8** Ω-**BCAT** *is equivalent to* Ω-**ACAT**.

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