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Some NP-Complete Results on Signed Mixed Domination Problem

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Abstract Let $G=(V,E)$ be a simple graph with vertex set V and edge set E. A signed mixed dominating function of G is a function $f\colon V\cup E\to\{-1,1\}$ such that $\sum_{y\in N_m(x)\cup\{x\}} f(y)\ge 1$ for every element $x\in V\cup E$, where $N_m(x)$ is the set of elements of $V\cup E$ adjacent or incident to x. The weight of f is $w(f)=\sum_{x\in V\cup E} f(x)$. The signed mixed domination problem is to find a minimum-weight signed mixed dominating function of a graph. In this paper we study the computational complexity of signed mixed domination problem. We prove that the signed mixed domination problem is NP-complete for bipartite graphs, chordal graphs, even for planar bipartite graphs.

Keywords signed mixed dominating function; signed mixed domination number; NP-completeness

MR(2010) Subject Classification 05C69; 05C85

1. Introduction

All graphs in this paper are simple, i.e., finite, undirected, loopless and without multiple edge. Domination is a core NP-complete problem in graph theory and combinatorial optimization. It has many applications in the real world such as location problems, sets of representatives, social network theory, etc; see [1] for further interesting applications. A dominating set of a graph $G=(V,E)$ is a subset D of V such that every vertex not in D is adjacent to at least one vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of all possible dominating sets in G. The domination problem is to find a minimum dominating set of G.

The notion of dominating (or covering, interchangeably) vertices or edges by vertices or edges has been widely studied in the literature. Traditional (vertex) domination problem asks for dominating vertices by vertices. Covering edges by vertices leads to the vertex cover problem.

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Covering vertices by edges results in the edge cover problem. When edges are to be dominated by edges, we obtain the edge domination problem.

The extension of the above notion is naturally considered. Dominating vertices and edges by vertices and edges is studied in [2–9]. Specifically, given a graph $G = (V, E)$, a vertex is said to mixed dominate itself, its neighbors and all edges incident to it; an edge uv is said to mixed dominate u and v and all edges incident to u or v. A mixed dominating set of G is a set $D \subseteq V \cup E$ such that every element in $V \cup E$ is mixed dominated by at least one element in D. In other words, every vertex and every edge not in D is adjacent or incident to at least one element in D. The mixed domination number $\gamma_m(G)$ of G is the minimum cardinality among all possible mixed dominating sets of G. The mixed domination problem is to find a minimum mixed dominating set of G.

A further generalization of mixed domination was done by Lv in 2007 (see [6]). A function $f : V \cup E \to \{-1, 1\}$ is called a signed mixed dominating function (SMDF for short) if $f[x] = \sum_{y \in N_m[x]} f(y) \geq 1$ for every element $x \in V \cup E$. In other words, for every list-assignment of two colors {red, blue} to every elements of $V \cup E$, there is a list-coloring of vertices and edges of G such that every mixed neighborhood contains more 'red' than 'blue'. The signed mixed domination number $\gamma_s^*(G)$ is the minimum weight among all signed mixed dominating functions of G. The signed mixed domination problem is to find a minimum signed mixed dominating function of G.

The aspect of complexity for mixed domination has been extensively studied by various authors, such as the NP-completeness for general graphs [10], for bipartite and chordal graphs [11], for split graphs [5,9], and for planar bipartite graphs of maximum degree four [7], a linear-time algorithm for trees [10,12].

However, since the concept of signed mixed domination was introduced by Lv in 2007, some works have concentrated on the computation of the exact values of signed mixed domination numbers [6,13,14], little works have been done on the complexity of this problem, except that in [14] Shan et al. showed the NP-completeness of this problem for planar graphs.

In this paper, we concentrate on the complexity aspects of the signed mixed domination problem. We prove that the signed mixed domination problem is NP-complete for bipartite graphs, chordal graphs, even for planar bipartite graphs.

2. Definitions

A practical application of signed mixed domination is stated as follows. Let $G = (V, E)$ be a graph representing an electronic network, where a vertex represents an electronic node and an edge represents a link of two electronic nodes. Assume that each element of G has an initial polarity. Also assume, however, that an element's polarity is affected by the polarities of its mixed neighbors. In particular, each element gives equal weight to its own polarity and to the polarities of its mixed neighbors. Then we try to find the minimum number of positive elements of G such that the mixed closed neighborhood of each element of G concludes more positive than negative elements.

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E, and let v be a vertex in V. The neighborhood $N_G(v)$ of v is the set of vertices adjacent to v. The closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. The degree of v is $d_G(v) = |N_G(v)|$. If no ambiguousness occurs, we simply write $N(v), N[v]$ and $d(v)$ in stead of $N_G(v), N_G[v]$ and $d_G(v)$, respectively. Similarly, for any edge $e \in E$, $N(e) = \{e' \in E | e'$ is adjacent to $e\}$ and $N[e] = N(e) \cup \{e\}$. For an element $x \in V \cup E$, the mixed neighborhood of x is the set $N_m(x) = \{y \in V \cup E | y$ is adjacent or incident to $x\}$. The mixed closed neighborhood of x is $N_m[x] = N_m(x) \cup \{x\}$. We also call the elements in $N_m(x)$ the mixed neighbors of x.

For elements ε_1 and ε_2, ε_1 is mixed dominated by ε_2 if and only if $\varepsilon_1 \in N_m[\varepsilon_2]$, and they are mixed neighbor of each other. The subgraph of G induced by $S \subseteq V$ is the graph $G[S]$ with vertex set S and edge set $\{uv \in E | u, v \in S\}$.

Let f be a function defined on $V \cup E$. For a subset $S \subseteq V \cup E$, we define $f(S) = \sum_{x \in S} f(x)$, write $f(N_m[x]) = f[x]$ for convenience, and let the weight of f be $w(f) = f(V \cup E)$.

Note that a signed mixed dominating function is also called a "signed total dominating function" in [6], but has different meaning from the same terminology in the earlier Refs [15–17].

3. NP-completeness of signed mixed domination

It is known that the mixed domination problem is NP-complete for chordal graphs [11], even for split graphs [5,9].

Mixed Domination Problem (MD)
Instance: A graph $G = (V, E)$ and a positive integer $p \leq |V| + |E|$.
Question: Is $\gamma_m(G) \leq p$?

We will demonstrate a polynomial-time reduction of this problem to our signed mixed domination problem, whose decision version is shown as follows.

Signed Mixed Domination Problem (SMD)
Instance: A graph $G = (V, E)$ and a positive integer $q \leq |V| + |E|$.
Question: Is $\gamma_s^*(G) \leq q$?

Theorem 3.1 *Problem SMD is NP-complete for chordal graphs.*

Proof It is obvious that SMD belongs to NP, since one can easily verify in polynomial time whether an arbitrary function $f : V \to (-1, 1)$ with $w(f) \leq q$ is a signed mixed dominating function.

We next construct a reduction from MD to SMD in polynomial time. Given a graph $G = (V, E)$ and a positive integer p, construct the graph H by attaching to each vertex v of G a set of $d_G(v)$ paths of length three. Suppose $n = |V(G)|$ and $m = |E(G)|$, then $|V(H)| = n + 3\sum_{v \in V(G)} d(v) = n + 6m$ and $|E(H)| = m + 3\sum_{v \in V(G)} d(v) = 7m$ and H can be constructed in polynomial time. Note that if G is a chordal graph, then so too is H.

For any element ε of G, we write $N_{Gm}[\varepsilon]$ to denote the set of closed mixed neighbors of ε

in G, and $N_{Hm}[\varepsilon]$ the set of closed mixed neighbors of ε in H. To proceed, we first give two lemmas as follows.

Lemma 3.2 *If g is a signed mixed dominating function of H, then $\{\varepsilon \in V(G) \cup E(G) | g(\varepsilon) = 1\}$ is a mixed dominating set of G.*

Proof Suppose that g is a γ_s^*-function of H and ε is any element of G. It is easy to see the fact that $|N_{Gm}[\varepsilon]| = 2d_G(v) + 1$ and $|N_{Hm}[\varepsilon]| - |N_{Gm}[\varepsilon]| = 2d_G(v)$ for $\varepsilon = v$ being a vertex of G; $|N_{Gm}[\varepsilon]| = d_G(v) + d_G(u) + 1$ and $|N_{Hm}[\varepsilon]| - |N_{Gm}[\varepsilon]| = d_G(v) + d_G(u)$ for $\varepsilon = vu$ being an edge of G. Now if $g(x) = -1$ for every $x \in N_{Gm}[\varepsilon]$, then we have $g(N_{Hm}[\varepsilon]) \leq 0$, contradicting the fact that g is a γ_s^*-function of H. This means that $\{\varepsilon \in V(G) \cup E(G) |\ g(\varepsilon) = 1\}$ is a mixed dominating set of G. $\square$

Lemma 3.3 *There exists a γ_s^*-function of H such that the function values of the elements in every added paths (not including v) are the same as in Figure 1.*

Proof Suppose that g is a γ_s^*-function of H. Let $A(v) = \{v_1, v_2, v_3, e_1, e_2, e_3\}$. Then by the condition $g(N_{Hm}[\varepsilon]) \geq 1, \varepsilon \in A(v)$, one can easily find that there are at most two elements of $A(v)$ being valued -1 under g (this fact can also be verified by a small program by computer). If at most one of $A(v)$ is valued -1 under g, then noting Lemma 3.2, one can easily see that the restriction of g on G together with the function as shown in Figure 1 is a signed mixed dominating function of H, with weight less than g, a contradiction. So exact two of $A(v)$ are valued -1 under g. Further, if the restriction of g on $A(v)$ is different from the function as shown in Figure 1, then the restriction of g on G together with the function as shown in Figure 1 is also a γ_s^*-function of H. $\square$

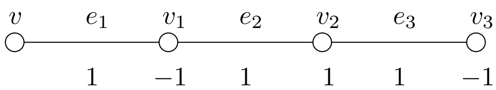

Figure 1 One of the paths attached to $v \in V(G)$

Now continue the proof of Theorem 3.1. We will show that graph G has a mixed dominating set with weight p if and only if the graph H has a signed mixed dominating function with weight $q = 2p - n + 4m$. Suppose first that M is any mixed dominating set of G with $|M| = p$. Then we construct a function f on G such that $f(\varepsilon) = 1$ for each $\varepsilon \in M$ and $f(\varepsilon) = -1$ otherwise. One can easily verify that f together with the function as shown in Figure 1 is a signed mixed dominating function of H with weight $q = p - (n - p) + \sum_{v \in V(G)} 2d_G(v) = 2p - n + 4m$. On the other hand, suppose g is a signed mixed dominating function of H with weight $q = 2p - n + 4m$. By Lemma 3.3, we may assume that the restriction of g on $A(v) = \{v_1, v_2, v_3, e_1, e_2, e_3\}$ is as shown in Figure 1. By Lemma 3.2, the set $M = \{\varepsilon \in V(G) \cup E(G) | g(\varepsilon) = 1\}$ is a mixed dominating set of G. Since $q = |M| - (n - |M|) + \sum_{v \in V(G)} 2d_G(v) = 2|M| - n + 4m$, we have $|M| = p$. This completes the proof of Theorem 3.1. $\square$

A set $F \subseteq E$ is an edge dominating set of a graph $G = (V, E)$ if every edge in $E \setminus F$ has at least one neighbor in F. The edge domination number of G, denoted by $\gamma'(G)$, is the minimum cardinality of an edge dominating set of G. The following decision version for the edge domination problem is known to be NP-complete, even when restricted to planar bipartite graphs and planar cubic graphs [18,19].

Edge Domination Problem (ED)
Instance: A graph $G = (V, E)$ and a positive integer $p \leq |E|$.
Question: Is $\gamma'(G) \leq p$?

We will demonstrate a polynomial-time reduction of this problem to our signed mixed domination problem.

Theorem 3.4 *Problem SMD is NP-complete for planar bipartite graphs.*

Proof It is obvious that SMD belongs to NP, since one can easily verify in polynomial time whether an arbitrary function $f : V \cup E \to (-1, 1)$ with $w(f) \leq q$ is a signed mixed dominating function.

We next construct a reduction from ED to SMD in polynomial time. Given a planar bipartite graph G, we can obviously suppose $\delta(G)) \geq 2$, since a graph with $\delta = 1$ is a tree or a forest, for which the ED problem is polynomially solvable. Construct the graph H by attaching to each vertex v of G a set of $d_G(v) - 2$ edges. Let $n = |V(G)|$ and $m = |E(G)|$. We have $|V(H)| = n+2\sum_{v\in V(G)}(d_G(v)-2) = 4m-3n$ and $|E(H)| = m+2\sum_{v\in V(G)}(d_G(v)-2) = 5m-4n$, and H can be constructed in polynomial time. Note that if G is planar bipartite, then so too is H.

Suppose f is a γ_s^*-function of H. For any vertex $v \in V(G)$, let v_1 be one of its neighbor in $V(H) \setminus V(G)$. Clearly, we may assume that $f(v_1) = -1$ and $f(v_1v) = f(v) = 1$. Further, we note that if there exists an edge $e = vu \in E(G)$ such that $f(e') = -1$ for every $e' \in N_G[e]$, then because $f(N_{Gm}[e]) = 2 - (d_G(v) + d_G(u) - 1) = 3 - d_G(v) - d_G(u)$ and $|N_{Hm}[e]| - |N_{Gm}[e]| = d_G(v) + d_G(u) - 4$, we would have $f(N_{Hm}[e]) \leq 0$, a contradiction. Thus, $\{e \in E(G) | \ f(e) = 1\}$ is an edge dominating set of G. It follows that if we let $j = n + m - 2(m - k) = n - m + 2k$, then $\gamma'(G) \leq k$ if and only if $\gamma_s^*(H) \leq j$. This completes the proof of Theorem 3.4. □

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