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On the Fourth Power Mean of Generalized Three-Term Exponential Sums

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Abstract The computational problem of fourth power mean of generalized three-term exponential sums is studied by using the trigonometric identity and the properties of the reduced residue system. Some explicit formulas for the fourth power mean of generalized three-term exponential sums under different conditions are given.

Keywords generalized three-term exponential sum; identity; fourth power mean formula

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1. Introduction

Let p be an odd prime, k,t be positive integers, and let χ denote any Dirichlet character mod p. For any integers m,s,n, the generalized three-term exponential sum $C(m,s,n,k,t,\chi;p)$ is defined as follows:

$$C(m,s,n,k,t,\chi;p)=\sum_{a=1}^{p-1}\chi(a)\mathrm{e}\Big(\frac{ma^k+sa^t+na}{p}\Big),$$

where $\mathrm{e}(y)=\mathrm{e}^{2\pi iy}$.

Many researchers have studied the various properties of this exponential sums and related sums, and obtained a series of results. For example, Yu and Zhang [1] studied the sixth power mean with the condition $(k,p-1)=1$, and obtained the result

$$\sum_{s=1}^{p}\sum_{n=1}^{p}\sum_{\chi \bmod p}|C(1,s,n,k,2,\chi;p)|^6=p^2(p-1)^2(6p^2-21p+19).$$

Du and Li [2] have given the following identities about the fourth power mean:

$$\sum_{s=1}^{p}\sum_{n=1}^{p}|C(1,s,n,k,2,\chi;p)|^4=2p^4-5p^3+3p^2,$$

$$\sum_{s=1}^{p}\sum_{n=1}^{p}|C(1,s,n,k,3,\chi;p)|^4=\begin{cases}3p^2(p-1)(p-2), & \chi \text{ is a real character mod } p,\\ p^2(p-1)(2p-5), & \chi \text{ is not a real character mod } p.\end{cases}$$

In this paper we further study the fourth power mean of generalized three-term exponential sums $|C(m,s,n,k,t,\chi;p)|^4$, and give some explicit formulas. Our main results are the following.

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Theorem 1.1 *Let $p \geq 3$ be an odd prime. Then for any integers m, s with $(m, p) = 1$ and $(s, p) = 1$, we have the identity*

$$\sum_{n=1}^{p} \sum_{\chi \bmod p} |C(m, s, n, k, t, \chi; p)|^4 = 2p^4 - 7p^3 + 8p^2 - 3p.$$

Theorem 1.2 *Let $p \geq 3$ be an odd prime, and let m, n be integers with $(m, p) = 1$ and $(n, p) = 1$. For $t \mid k$, we have*

$$\sum_{s=1}^{p} \sum_{\chi \bmod p} |C(m, s, n, k, t, \chi; p)|^4 = 2p^4 - 5p^3 + 3p^2 - 2p(p-1)^2(t, p-1) + p(p-1)(t, p-1)^2.$$

Theorem 1.3 *Let p be an odd prime, and let χ denote a Dirichlet character modulo p. Let χ_0 be the principal character modulo p. Write*

$$\chi(a) = \begin{cases} 0, & \text{if } (a, p) > 1, \\ e(\frac{m \operatorname{ind} a}{p-1}), & \text{if } (a, p) = 1. \end{cases}$$

Then for any integer n with $(n, p) = 1$, we have

$$\sum_{m=1}^{p} \sum_{s=1}^{p} |C(m, s, n, 4, 2, \chi; p)|^4 = \begin{cases} 2p^4 - 11p^3 + 16p^2, & \text{if } \chi = \chi_0, \\ 2p^4 - 7p^3 + 12p^2, & \text{if } \chi \neq \chi_0 \text{ and } 2 \mid m, \\ 2p^4 - 3p^3, & \text{if } \chi \neq \chi_0 \text{ and } 2 \nmid m, \end{cases}$$

and

$$\sum_{m=1}^{p} \sum_{s=1}^{p} |C(m, s, n, 6, 2, \chi; p)|^4$$
$$= \begin{cases} 2p^4 - 15p^3 + 36p^2, & \text{if } \chi = \chi_0 \text{ and } p \equiv 1 \,(\bmod 4), \\ 2p^4 - 11p^3 + 16p^2, & \text{if } \chi = \chi_0 \text{ and } p \equiv 3 \,(\bmod 4), \\ 2p^4 - 7p^3 + 28p^2, & \text{if } \chi \neq \chi_0, p \equiv 1 \,(\bmod 4) \text{ and } 4 \mid m, \\ 2p^4 - 7p^3 + 12p^2, & \text{if } \chi \neq \chi_0, p \equiv 1 \,(\bmod 4), 2 \mid m \text{ and } 4 \nmid m, \\ 2p^4 - 7p^3 + 12p^2, & \text{if } \chi \neq \chi_0, p \equiv 3 \,(\bmod 4) \text{ and } 2 \mid m, \\ 2p^4 - 3p^3, & \text{if } \chi \neq \chi_0 \text{ and } 2 \nmid m. \end{cases}$$

Moreover, we have the identities

$$\sum_{m=1}^{p} \sum_{s=1}^{p} |C(m, s, n, 6, 3, \chi; p)|^4$$
$$= \begin{cases} 2p^4 - 17p^3 + 45p^2, & \text{if } \chi = \chi_0 \text{ and } p \equiv 1 \,(\bmod 3), \\ 2p^4 - 5p^3 + 3p^2, & \text{if } \chi = \chi_0 \text{ and } p \equiv 2 \,(\bmod 3), \\ 2p^4 - 8p^3 + 36p^2, & \text{if } \chi \neq \chi_0, \chi^2 = \chi_0 \text{ and } p \equiv 1 \,(\bmod 6), \\ 2p^4 - 4p^3 + 2p^2, & \text{if } \chi \neq \chi_0, \chi^2 = \chi_0 \text{ and } p \equiv 2 \,(\bmod 3), \\ 2p^4 - 8p^3 + 36p^2, & \text{if } \chi^2 \neq \chi_0, p \equiv 1 \,(\bmod 3) \text{ and } 3 \mid m, \\ 2p^4 - 5p^3, & \text{if } \chi^2 \neq \chi_0, p \equiv 1 \,(\bmod 3) \text{ and } 3 \nmid m, \\ 2p^4 - 4p^3 + 2p^2, & \text{if } \chi^2 \neq \chi_0 \text{ and } p \equiv 2 \,(\bmod 3), \end{cases}$$

and

$$\sum_{m=1}^{p}\sum_{s=1}^{p}|C(m,s,n,9,3,\chi;p)|^4$$
$$=\begin{cases} 3p^4-33p^3+108p^2, & \text{if } \chi=\chi_0 \text{ and } p\equiv 1\,(\text{mod } 3),\\ 3p^4-9p^3+6p^2, & \text{if } \chi=\chi_0 \text{ and } p\equiv 2\,(\text{mod } 3),\\ 3p^4-15p^3+90p^2, & \text{if } \chi\neq\chi_0, \chi^2=\chi_0 \text{ and } p\equiv 1\,(\text{mod } 12),\\ 3p^4-15p^3+36p^2, & \text{if } \chi\neq\chi_0, \chi^2=\chi_0 \text{ and } p\equiv 7\,(\text{mod } 12),\\ 3p^4-7p^3+4p^2, & \text{if } \chi\neq\chi_0, \chi^2=\chi_0 \text{ and } p\equiv 5\,(\text{mod } 12),\\ 3p^4-7p^3+2p^2, & \text{if } \chi\neq\chi_0, \chi^2=\chi_0 \text{ and } p\equiv 11\,(\text{mod } 12),\\ 2p^4-14p^3+90p^2, & \text{if } \chi^2\neq\chi_0, p\equiv 1\,(\text{mod } 3) \text{ and } 6\mid m,\\ 2p^4-14p^3+36p^2, & \text{if } \chi^2\neq\chi_0, p\equiv 1\,(\text{mod } 3), 3\mid m \text{ and } 6\nmid m,\\ 2p^4-5p^3, & \text{if } \chi^2\neq\chi_0, p\equiv 1\,(\text{mod } 3) \text{ and } 3\nmid m,\\ 2p^4-6p^3+4p^2, & \text{if } \chi^2\neq\chi_0, p\equiv 2\,(\text{mod } 3) \text{ and } 2\mid m,\\ 2p^4-6p^3+2p^2, & \text{if } \chi^2\neq\chi_0, p\equiv 2\,(\text{mod } 3) \text{ and } 2\nmid m. \end{cases}$$

2. Proof of Theorems 1.1 and 1.2

First we prove Theorem 1.1. By the properties of characters we have

$$\sum_{n=1}^{p}\sum_{\chi \bmod p}|C(m,s,n,k,t,\chi;p)|^4=\sum_{n=1}^{p}\sum_{\chi \bmod p}\Big|\sum_{a=1}^{p-1}\chi(a)\mathrm{e}\Big(\frac{ma^k+sa^t+na}{p}\Big)\Big|^4$$
$$=\sum_{n=1}^{p}\sum_{\chi \bmod p}\Big|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a\bar{b})\mathrm{e}\Big(\frac{m(a^k-b^k)+s(a^t-b^t)+n(a-b)}{p}\Big)\Big|^2$$
$$=\sum_{n=1}^{p}\sum_{\chi \bmod p}\Big|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\mathrm{e}\Big(\frac{mb^k(a^k-1)+sb^t(a^t-1)+nb(a-1)}{p}\Big)\Big|^2$$
$$=\sum_{n=1}^{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\chi \bmod p}\chi(a\bar{c})\mathrm{e}\Big(\frac{m[b^k(a^k-1)-d^k(c^k-1)]}{p}\Big)\times$$
$$\mathrm{e}\Big(\frac{s[b^t(a^t-1)-d^t(c^t-1)]+n[b(a-1)-d(c-1)]}{p}\Big)$$
$$=\sum_{n=1}^{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\chi \bmod p}\chi(a\bar{c})\mathrm{e}\Big(\frac{md^k[b^k(a^k-1)-(c^k-1)]}{p}\Big)\times$$
$$\mathrm{e}\Big(\frac{sd^t[b^t(a^t-1)-(c^t-1)]+nd[b(a-1)-(c-1)]}{p}\Big)$$
$$=p(p-1)\sum_{\substack{a=1\\ a\equiv c\,(\text{mod } p)\\ b(a-1)\equiv c-1\,(\text{mod } p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\mathrm{e}\Big(\frac{md^k[b^k(a^k-1)-(c^k-1)]+sd^t[b^t(a^t-1)-(c^t-1)]}{p}\Big)$$

$$= p(p-1) \sum_{\substack{a=1 \\ (b-1)(a-1)\equiv 0 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} \mathrm{e}\Big(\frac{md^k(b^k-1)(a^k-1) + sd^t(b^t-1)(a^t-1)}{p}\Big)$$

$$= p(p-1) \sum_{\substack{a=1 \\ (b-1)(a-1)\equiv 0 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} 1 = p(p-1)^2 \sum_{\substack{a=1 \\ (a-1)(b-1)\equiv 0 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{b=1}^{p-1} 1$$

$$= 2p^4 - 7p^3 + 8p^2 - 3p.$$

This completes the proof of Theorem 1.1. □

Then we prove Theorem 1.2. Suppose that $t \mid k$, we get

$$\sum_{s=1}^{p} \sum_{\chi \bmod p} |C(m,s,n,k,t,\chi;p)|^4 = \sum_{s=1}^{p} \sum_{\chi \bmod p} \Big|\sum_{a=1}^{p-1} \chi(a) \mathrm{e}\Big(\frac{ma^k + sa^t + na}{p}\Big)\Big|^4$$

$$= \sum_{s=1}^{p} \sum_{\chi \bmod p} \Big|\sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a) \mathrm{e}\Big(\frac{mb^k(a^k-1) + sb^t(a^t-1) + nb(a-1)}{p}\Big)\Big|^2$$

$$= \sum_{s=1}^{p} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{\chi \bmod p} \chi(a\bar{c}) \mathrm{e}\Big(\frac{m[b^k(a^k-1) - d^k(c^k-1)]}{p}\Big) \times$$
$$\mathrm{e}\Big(\frac{s[b^t(a^t-1) - d^t(c^t-1)] + n[b(a-1) - d(c-1)]}{p}\Big)$$

$$= \sum_{s=1}^{p} \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \sum_{\chi \bmod p} \chi(a\bar{c}) \mathrm{e}\Big(\frac{md^k[b^k(a^k-1) - (c^k-1)]}{p}\Big) \times$$
$$\mathrm{e}\Big(\frac{sd^t[b^t(a^t-1) - (c^t-1)] + nd[b(a-1) - (c-1)]}{p}\Big)$$

$$= p(p-1) \sum_{\substack{a=1 \\ a\equiv c \,(\mathrm{mod}\, p) \\ b^t(a^t-1)\equiv (c^t-1) \,(\mathrm{mod}\, p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \mathrm{e}\Big(\frac{md^k[b^k(a^k-1) - (c^k-1)] + nd[b(a-1) - (c-1)]}{p}\Big)$$

$$= p(p-1) \sum_{\substack{a=1 \\ (a^t-1)(b^t-1)\equiv 0 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} \mathrm{e}\Big(\frac{md^k(b^k-1)(a^k-1) + nd(b-1)(a-1)}{p}\Big)$$

$$= p(p-1) \sum_{\substack{a=1 \\ (a^t-1)(b^t-1)\equiv 0 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p-1} \mathrm{e}\Big(\frac{nd(b-1)(a-1)}{p}\Big)$$

$$= p(p-1) \sum_{\substack{a=1 \\ (a^t-1)(b^t-1)\equiv 0 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{d=1}^{p} \mathrm{e}\Big(\frac{nd(a-1)(b-1)}{p}\Big) - p(p-1) \sum_{\substack{a=1 \\ (a^t-1)(b^t-1)\equiv 0 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{b=1}^{p-1} 1$$

$$= p^2(p-1) \sum_{\substack{a=1 \\ (a-1)(b-1)\equiv 0 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{b=1}^{p-1} 1 - p(p-1)\big(2(p-1)(t,p-1) - (t,p-1)^2\big)$$

$$= 2p^4 - 5p^3 + 3p^2 - 2p(p-1)^2(t,p-1) + p(p-1)(t,p-1)^2.$$

This completes the proof of Theorem 1.2. □

3. Express the fourth power mean as two congruence equations

Now we consider Theorem 1.3. By the properties of congruence system we get

$$\begin{aligned}
&\sum_{m=1}^{p}\sum_{s=1}^{p}\Big|\sum_{a=1}^{p-1}\chi(a)\mathrm{e}\big(\frac{ma^k+sa^t+na}{p}\big)\Big|^4\\
&=\frac{1}{p-1}\sum_{m=1}^{p}\sum_{s=1}^{p}\sum_{r=1}^{p-1}\Big|\sum_{a=1}^{p-1}\chi(a)\mathrm{e}\big(\frac{ma^k+sa^t+nra}{p}\big)\Big|^4\\
&=\frac{1}{p-1}\sum_{m=1}^{p}\sum_{s=1}^{p}\sum_{r=1}^{p}\Big|\sum_{a=1}^{p-1}\chi(a)\mathrm{e}\big(\frac{ma^k+sa^t+nra}{p}\big)\Big|^4-\\
&\quad\frac{1}{p-1}\sum_{m=1}^{p}\sum_{s=1}^{p}\Big|\sum_{a=1}^{p-1}\chi(a)\mathrm{e}\big(\frac{ma^k+sa^t}{p}\big)\Big|^4\\
&=\frac{1}{p-1}\sum_{m=1}^{p}\sum_{s=1}^{p}\sum_{r=1}^{p}\Big|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\mathrm{e}\big(\frac{mb^k(a^k-1)+sb^t(a^t-1)+nrb(a-1)}{p}\big)\Big|^2-\\
&\quad\frac{1}{p-1}\sum_{m=1}^{p}\sum_{s=1}^{p}\Big|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\mathrm{e}\big(\frac{mb^k(a^k-1)+sb^t(a^t-1)}{p}\big)\Big|^2\\
&=S_1-S_2.
\end{aligned}$$

As to S_1, from the trigonometric identity and the properties of residue system we have

$$\begin{aligned}
S_1=&\frac{1}{p-1}\sum_{m=1}^{p}\sum_{s=1}^{p}\sum_{r=1}^{p}\Big|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\mathrm{e}\big(\frac{mb^k(a^k-1)+sb^t(a^t-1)+nrb(a-1)}{p}\big)\Big|^2\\
=&\frac{1}{p-1}\sum_{m=1}^{p}\sum_{s=1}^{p}\sum_{r=1}^{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(a\bar{c})\mathrm{e}\big(\frac{m[b^k(a^k-1)-d^k(c^k-1)]}{p}\big)\times\\
&\mathrm{e}\big(\frac{s[b^t(a^t-1)-d^t(c^t-1)]+nr[b(a-1)-d(c-1)]}{p}\big)\\
=&\frac{p^3}{p-1}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ b^k(a^k-1)\equiv d^k(c^k-1)\ (\mathrm{mod}\ p)\\ b^t(a^t-1)\equiv d^t(c^t-1)\ (\mathrm{mod}\ p)\\ b(a-1)\equiv d(c-1)\ (\mathrm{mod}\ p)}}^{p-1}\chi(a\bar{c})\\
=&p^3(p-1)+\frac{p^3}{p-1}\sum_{a=2}^{p-1}\sum_{b=1}^{p-1}\sum_{c=2}^{p-1}\sum_{\substack{d=1\\ b^k(a^k-1)\equiv d^k(c^k-1)\ (\mathrm{mod}\ p)\\ b^t(a^t-1)\equiv d^t(c^t-1)\ (\mathrm{mod}\ p)\\ b(a-1)\equiv d(c-1)\ (\mathrm{mod}\ p)}}^{p-1}\chi(a\bar{c})
\end{aligned}$$

$$=p^3(p-1)+\frac{p^3}{p-1}\sum_{\substack{a=2\\ b^k(c-1)^k(a^k-1)\equiv d^k(a-1)^k(c^k-1)\ (\mathrm{mod}\ p)\\ b^t(c-1)^t(a^t-1)\equiv d^t(a-1)^t(c^t-1)\ (\mathrm{mod}\ p)\\ b(c-1)(a-1)\equiv d(a-1)(c-1)\ (\mathrm{mod}\ p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=2}^{p-1}\sum_{d=1}^{p-1}\chi(a\bar{c})$$

$$=p^3(p-1)+p^3\sum_{\substack{a=2\\ (c-1)^k(a^k-1)\equiv(a-1)^k(c^k-1)\ (\mathrm{mod}\ p)\\ (c-1)^t(a^t-1)\equiv(a-1)^t(c^t-1)\ (\mathrm{mod}\ p)}}^{p-1}\sum_{c=2}^{p-1}\chi(a\bar{c}).$$

Similarly, we also get

$$S_2=\frac{1}{p-1}\sum_{m=1}^{p}\sum_{s=1}^{p}\Big|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\mathrm{e}\Big(\frac{mb^k(a^k-1)+sb^t(a^t-1)}{p}\Big)\Big|^2$$

$$=\frac{1}{p-1}\sum_{m=1}^{p}\sum_{s=1}^{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(a\bar{c})\times$$

$$\mathrm{e}\Big(\frac{m[b^k(a^k-1)-d^k(c^k-1)]+s[b^t(a^t-1)-d^t(c^t-1)]}{p}\Big)$$

$$=\frac{p^2}{p-1}\sum_{\substack{a=1\\ b^k(a^k-1)\equiv d^k(c^k-1)\ (\mathrm{mod}\ p)\\ b^t(a^t-1)\equiv d^t(c^t-1)\ (\mathrm{mod}\ p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(a\bar{c})=p^2\sum_{\substack{a=1\\ b^k(a^k-1)\equiv c^k-1\ (\mathrm{mod}\ p)\\ b^t(a^t-1)\equiv c^t-1\ (\mathrm{mod}\ p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c})$$

$$=p^2\sum_{\substack{a=1\\ b^ka^k-b^k\equiv c^k-1\ (\mathrm{mod}\ p)\\ b^ta^t-b^t\equiv c^t-1\ (\mathrm{mod}\ p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c})=p^2\sum_{\substack{a=1\\ a^k+1\equiv b^k+c^k\ (\mathrm{mod}\ p)\\ a^t+1\equiv b^t+c^t\ (\mathrm{mod}\ p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c}).$$

Therefore

$$\sum_{m=1}^{p}\sum_{s=1}^{p}|C(m,s,n,k,t,\chi;p)|^4$$

$$=p^3(p-1)+p^3\sum_{\substack{a=2\\ (c-1)^k(a^k-1)\equiv(a-1)^k(c^k-1)\ (\mathrm{mod}\ p)\\ (c-1)^t(a^t-1)\equiv(a-1)^t(c^t-1)\ (\mathrm{mod}\ p)}}^{p-1}\sum_{c=2}^{p-1}\chi(a\bar{c})-p^2\sum_{\substack{a=1\\ a^k+1\equiv b^k+c^k\ (\mathrm{mod}\ p)\\ a^t+1\equiv b^t+c^t\ (\mathrm{mod}\ p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c})$$

$$=p^3(p-1)+p^3N_1(k,t,\chi;p)-p^2N_2(k,t,\chi;p). \tag{3.1}$$

To prove Theorem 1.3 we need to compute the congruence equations $N_1(k,t,\chi;p)$ and $N_2(k,t,\chi;p)$.

4. The computation of $N_1(k,t,\chi;p)$

We study $N_1(k,t,\chi;p)$ in certain cases.

Theorem 4.1 *Let p be an odd prime, k be a positive integer, and let χ denote any Dirichlet character* mod p. *Then $N_1(k,2,\chi;p)=p-2$.*

Proof From $(c-1)^2(a^2-1) \equiv (a-1)^2(c^2-1) \pmod p$ we know that $(a-c)(a-1)(c-1) \equiv 0 \pmod p$. Therefore

$$\begin{aligned} N_1(k,2,\chi;p) &= \sum_{\substack{a=2 \\ (c-1)^k(a^k-1)\equiv(a-1)^k(c^k-1) \pmod p \\ (c-1)^2(a^2-1)\equiv(a-1)^2(c^2-1) \pmod p}}^{p-1} \sum_{c=2}^{p-1} \chi(a\bar{c}) \\ &= \sum_{\substack{a=2 \\ (c-1)^k(a^k-1)\equiv(a-1)^k(c^k-1) \pmod p \\ a\equiv c \pmod p}}^{p-1} \sum_{c=2}^{p-1} \chi(a\bar{c}) = \sum_{\substack{a=2 \\ a\equiv c \pmod p}}^{p-1} \sum_{c=2}^{p-1} 1 \\ &= p-2. \end{aligned}$$

This completes the proof of Theorem 4.1. □

Theorem 4.2 *Let p be an odd prime, k be a positive integer, and let χ denote any Dirichlet character* mod p. *Then we have*

$$N_1(k,3,\chi;p) = \begin{cases} \displaystyle\sum_{a=2}^{p-1} \chi(a^2) + p - 3, & \text{if } k \text{ is an odd number;} \\ \displaystyle\sum_{\substack{a=2 \\ a^k\equiv 1 \pmod p}}^{p-1} \chi(a^2) + p - 3, & \text{if } k \text{ is an even number.} \end{cases}$$

Proof It is obvious that Theorem 4.2 holds for $p = 3$. Now we suppose $p > 3$, then

$$\begin{aligned} &(c-1)^3(a^3-1) \equiv (a-1)^3(c^3-1) \pmod p \\ &\Longleftrightarrow (c^2-2c+1)(a^2+a+1) \equiv (a^2-2a+1)(c^2+c+1) \pmod p \\ &\Longleftrightarrow 3ac^2 - 3a^2c - 3c + 3a \equiv 0 \pmod p \\ &\Longleftrightarrow (ac-1)(c-a) \equiv 0 \pmod p. \end{aligned}$$

So we can write

$$\begin{aligned} &\sum_{\substack{a=2 \\ (c-1)^k(a^k-1)\equiv(a-1)^k(c^k-1) \pmod p \\ (c-1)^3(a^3-1)\equiv(a-1)^3(c^3-1) \pmod p}}^{p-1} \sum_{c=2}^{p-1} \chi(a\bar{c}) = \sum_{\substack{a=2 \\ (c-1)^k(a^k-1)\equiv(a-1)^k(c^k-1) \pmod p \\ (ac-1)(c-a)\equiv 0 \pmod p}}^{p-1} \sum_{c=2}^{p-1} \chi(a\bar{c}) \\ &= \sum_{\substack{a=2 \\ (c-1)^k(a^k-1)\equiv(a-1)^k(c^k-1) \pmod p \\ ac\equiv 1 \pmod p}}^{p-1} \sum_{c=2}^{p-1} \chi(a\bar{c}) + \sum_{\substack{a=2 \\ (c-1)^k(a^k-1)\equiv(a-1)^k(c^k-1) \pmod p \\ c\equiv a \pmod p}}^{p-1} \sum_{c=2}^{p-1} \chi(a\bar{c}) - \\ &\quad \sum_{\substack{a=2 \\ (c-1)^k(a^k-1)\equiv(a-1)^k(c^k-1) \pmod p \\ ac\equiv 1 \pmod p \\ c\equiv a \pmod p}}^{p-1} \sum_{c=2}^{p-1} \chi(a\bar{c}) \end{aligned}$$

$$= \sum_{\substack{a=2 \\ (1-a)^k(a^k-1)\equiv(a-1)^k(1-a^k) \pmod p}}^{p-1} \chi(a^2) + p - 3$$

$$= \sum_{\substack{a=2 \\ a^k-1\equiv(-1)^k(1-a^k) \pmod p}}^{p-1} \chi(a^2) + p - 3$$

$$= \begin{cases} \sum_{a=2}^{p-1} \chi(a^2) + p - 3, & \text{if } k \text{ is an odd number,} \\ \sum_{\substack{a=2 \\ a^k\equiv 1 \pmod p}}^{p-1} \chi(a^2) + p - 3, & \text{if } k \text{ is an even number.} \end{cases}$$

This completes the proof of Theorem 4.2. □

Corollary 4.3 *Let p be an odd prime, and let χ denote a Dirichlet character modulo p. Let χ_0 be the principal character modulo p. For $2 \nmid k$, we have*

$$N_1(k, 3, \chi; p) = \begin{cases} 2p-5, & \text{if } \chi^2 = \chi_0; \\ p-4, & \text{if } \chi^2 \neq \chi_0. \end{cases}$$

Write

$$\chi(a) = \begin{cases} 0, & \text{if } (a,p) > 1; \\ e(\frac{m \operatorname{ind} a}{p-1}), & \text{if } (a,p) = 1. \end{cases}$$

We further get

$$N_1(6, 3, \chi; p) = \begin{cases} (6, p-1) + p - 4, & \text{if } \chi^2 = \chi_0; \\ p+2, & \text{if } \chi^2 \neq \chi_0, p \equiv 1 \pmod 3 \text{ and } 3 \mid m; \\ p-4, & \text{if } \chi^2 \neq \chi_0, p \equiv 1 \pmod 3 \text{ and } 3 \nmid m; \\ p-2, & \text{if } \chi^2 \neq \chi_0 \text{ and } p \equiv 2 \pmod 3). \end{cases}$$

Proof It is easy to show that Corollary 4.1 holds for $\chi^2 = \chi_0$. Note that the characters χ modulo 3 satisfy $\chi^2 = \chi_0$, thus we can suppose that $p > 3$ and $\chi^2 \neq \chi_0$.

For $2 \nmid k$, by Theorem 4.2 we have

$$N_1(k, 3, \chi; p) = \sum_{a=1}^{p-1} \chi^2(a) + p - 4 = \begin{cases} 2p-5, & \text{if } \chi^2 = \chi_0; \\ p-4, & \text{if } \chi^2 \neq \chi_0. \end{cases}$$

For $k = 6$, from Theorem 4.2 we get

$$N_1(6, 3, \chi; p) = \sum_{\substack{a=1 \\ a^6\equiv 1 \pmod p}}^{p-1} \chi(a^2) + p - 4 = \frac{1}{p-1} \sum_{a=1}^{p-1} \chi(a^2) \sum_{\psi \bmod p} \psi(a^6) + p - 4$$

$$= \frac{1}{p-1} \sum_{\psi \bmod p} \sum_{a=1}^{p-1} \chi^2(a) \psi^6(a) + p - 4.$$

Write

$$\chi(a)=\begin{cases}0, & \text{if } (a,p)>1;\\ \mathrm{e}(\frac{m\,\mathrm{ind}\,a}{p-1}), & \text{if } (a,p)=1\end{cases} \quad \text{and} \quad \psi(a)=\begin{cases}0, & \text{if } (a,p)>1;\\ \mathrm{e}(\frac{n\,\mathrm{ind}\,a}{p-1}), & \text{if } (a,p)=1.\end{cases}$$

We have

$$\begin{aligned} N_1(6,3,\chi;p) &= \frac{1}{p-1}\sum_{n=0}^{p-2}\sum_{a=1}^{p-1}\mathrm{e}(\frac{(2m+6n)\mathrm{ind}\,a}{p-1})+p-4 \\ &= \sum_{\substack{n=0\\ 2m+6n\equiv 0 \,(\mathrm{mod}\, p-1)}}^{p-2} 1+p-4 = \sum_{\substack{n=0\\ 3n\equiv -m \,(\mathrm{mod}\, \frac{p-1}{2})}}^{p-2} 1+p-4 \\ &= \begin{cases} p+2, & \text{if } p\equiv 1\,(\mathrm{mod}\ 3) \text{ and } 3\mid m;\\ p-4, & \text{if } p\equiv 1\,(\mathrm{mod}\ 3) \text{ and } 3\nmid m;\\ p-2, & \text{if } p\equiv 2\,(\mathrm{mod}\ 3).\end{cases} \end{aligned}$$

This completes the proof of Corollary 4.3. □

5. The computation of $N_2(k,t,\chi;p)$

We study $N_2(k,t,\chi;p)$ in special cases.

Theorem 5.1 *Let p be an odd prime, and let χ denote a Dirichlet character modulo p. Let t be a positive integer. Then we have*

$$N_2(2t,t,\chi;p)=\begin{cases} 2(p-1)(t,p-1)^2-(t,p-1)^3, & \text{if } \chi=\chi_0;\\ ((p-1)(t,p-1)-(t,p-1)^2)\displaystyle\sum_{\substack{a=1\\ a^t\equiv 1\,(\mathrm{mod}\, p)}}^{p-1}\chi(a), & \text{if } \chi\neq\chi_0.\end{cases}$$

Proof It is not hard to show that

$$\begin{aligned} &\begin{cases} a^{2t}+1\equiv b^{2t}+c^{2t}\,(\mathrm{mod}\, p)\\ a^t+1\equiv b^t+c^t\,(\mathrm{mod}\, p)\end{cases} \Longleftrightarrow \begin{cases} (a^t+b^t)(a^t-b^t)\equiv (c^t+1)(c^t-1)\,(\mathrm{mod}\, p)\\ a^t-b^t\equiv c^t-1\,(\mathrm{mod}\, p)\end{cases} \\ &\Longleftrightarrow \begin{cases} (a^t+b^t)(c^t-1)\equiv (c^t+1)(c^t-1)\,(\mathrm{mod}\, p)\\ a^t-b^t\equiv c^t-1\,(\mathrm{mod}\, p)\end{cases} \\ &\Longleftrightarrow \begin{cases} (a^t+b^t-c^t-1)(c^t-1)\equiv 0\,(\mathrm{mod}\, p)\\ a^t-b^t\equiv c^t-1\,(\mathrm{mod}\, p)\end{cases} \\ &\Longleftrightarrow \begin{cases} (b^t-1)(c^t-1)\equiv 0\,(\mathrm{mod}\, p)\\ a^t+1\equiv b^t+c^t\,(\mathrm{mod}\, p)\end{cases} \end{aligned}$$

By the principle of inclusion and exclusion we can get

$$N_2(2t,t,\chi;p)=\sum_{\substack{a=1\\ a^{2t}+1\equiv b^{2t}+c^{2t}\,(\mathrm{mod}\, p)\\ a^t+1\equiv b^t+c^t\,(\mathrm{mod}\, p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c}) = \sum_{\substack{a=1\\ a^t+1\equiv b^t+c^t\,(\mathrm{mod}\, p)\\ (b^t-1)(c^t-1)\equiv 0\,(\mathrm{mod}\, p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c})$$

$$= \sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \,(\mathrm{mod}\, p) \\ b^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi(a\overline{c}) + \sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \,(\mathrm{mod}\, p) \\ c^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi(a\overline{c}) - \sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \,(\mathrm{mod}\, p) \\ b^t\equiv 1 \,(\mathrm{mod}\, p) \\ c^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi(a\overline{c})$$

$$= \sum_{\substack{a=1 \\ a^t\equiv c^t \,(\mathrm{mod}\, p) \\ b^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi(a\overline{c}) + \sum_{\substack{a=1 \\ a^t\equiv b^t \,(\mathrm{mod}\, p) \\ c^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi(a\overline{c}) - \sum_{\substack{a=1 \\ a^t\equiv 1 \,(\mathrm{mod}\, p) \\ b^t\equiv 1 \,(\mathrm{mod}\, p) \\ c^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} \chi(a\overline{c})$$

$$= \sum_{\substack{b=1 \\ b^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{c=1}^{p-1} \sum_{\substack{a=1 \\ a^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a) + \sum_{\substack{a=1 \\ a^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{b=1}^{p-1} \sum_{\substack{c=1 \\ c^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a)\chi(b)\chi(\overline{c}) -$$

$$\sum_{\substack{b=1 \\ b^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{\substack{c=1 \\ c^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \sum_{\substack{a=1 \\ a^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a)$$

$$= \begin{cases} 2(p-1)(t,p-1)^2 - (t,p-1)^3, & \text{if } \chi = \chi_0, \\ \big((p-1)(t,p-1) - (t,p-1)^2\big) \displaystyle\sum_{\substack{a=1 \\ a^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a), & \text{if } \chi \neq \chi_0. \end{cases}$$

This completes the proof of Theorem 5.1. □

Theorem 5.2 *Let p be an odd prime, and let χ denote a Dirichlet character modulo p. Let t be a positive integer. Then we have*

$$N_2(3t,t,\chi;p) = \begin{cases} (p-1-2(t,p-1))((2t,p-1)-(t,p-1))^2 + \\ \qquad (2(p-1)-(t,p-1))(t,p-1)^2, \quad \text{if } \chi = \chi_0, \\ (t,p-1)(p-1-(t,p-1))\Big(\displaystyle\sum_{\substack{a=1 \\ a^t\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a)\Big) - \\ \qquad (t,p-1)((2t,p-1)-(t,p-1))\Big(\displaystyle\sum_{\substack{a=1 \\ a^{2t}\equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a)\Big), \quad \text{if } \chi \neq \chi_0. \end{cases}$$

Proof It is obvious that Theorem 5.2 holds for $p=3$. Now we suppose that $p>3$. Note that

$$\begin{cases} a^{3t}+1 \equiv b^{3t}+c^{3t} \,(\mathrm{mod}\, p) \\ a^t+1 \equiv b^t+c^t \,(\mathrm{mod}\, p) \end{cases} \Longleftrightarrow \begin{cases} a^{3t}-b^{3t} \equiv c^{3t}-1 \,(\mathrm{mod}\, p) \\ a^t-b^t \equiv c^t-1 \,(\mathrm{mod}\, p) \end{cases}$$

$$\Longleftrightarrow \begin{cases} (a^t-b^t)(a^{2t}+a^tb^t+b^{2t}) \equiv (c^t-1)(c^{2t}+c^t+1) \,(\mathrm{mod}\, p) \\ a^t-b^t \equiv c^t-1 \,(\mathrm{mod}\, p) \end{cases}$$

$$\Longleftrightarrow \begin{cases} (c^t-1)(a^{2t}+a^tb^t+b^{2t}) \equiv (c^t-1)(c^{2t}+c^t+1) \,(\mathrm{mod}\, p) \\ a^t-b^t \equiv c^t-1 \,(\mathrm{mod}\, p) \end{cases}$$

$$\Longleftrightarrow \begin{cases} (a^{2t}+a^tb^t+b^{2t}-c^{2t}-c^t-1)(c^t-1) \equiv 0 \,(\mathrm{mod}\, p) \\ a^t-b^t \equiv c^t-1 \,(\mathrm{mod}\, p) \end{cases}$$

$$\Longleftrightarrow \begin{cases} (b^tc^t + b^{2t} - b^t - c^t)(c^t - 1) \equiv 0 \pmod p \\ a^t - b^t \equiv c^t - 1 \pmod p \end{cases}$$

$$\Longleftrightarrow \begin{cases} (b^t + c^t)(b^t - 1)(c^t - 1) \equiv 0 \pmod p \\ a^t + 1 \equiv b^t + c^t \pmod p \end{cases}$$

From the principle of inclusion and exclusion we have

$$N_2(3t, t, \chi; p) = \sum_{\substack{a=1 \\ a^{3t}+1\equiv b^{3t}+c^{3t} \pmod p \\ a^t+1\equiv b^t+c^t \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) = \sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \pmod p \\ (b^t+c^t)(b^t-1)(c^t-1)\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c})$$

$$= \sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \pmod p \\ b^t+c^t\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) + \sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \pmod p \\ b^t-1\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) + \sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \pmod p \\ c^t-1\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) -$$

$$\sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \pmod p \\ b^t+c^t\equiv 0 \pmod p \\ b^t-1\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) - \sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \pmod p \\ b^t+c^t\equiv 0 \pmod p \\ c^t-1\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) -$$

$$\sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \pmod p \\ b^t-1\equiv 0 \pmod p \\ c^t-1\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) + \sum_{\substack{a=1 \\ a^t+1\equiv b^t+c^t \pmod p \\ b^t+c^t\equiv 0 \pmod p \\ b^t-1\equiv 0 \pmod p \\ c^t-1\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c})$$

$$= \sum_{\substack{a=1 \\ a^t\equiv -1 \pmod p \\ b^t+c^t\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) + \sum_{\substack{a=1 \\ a^t\equiv c^t \pmod p \\ b^t\equiv 1 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) + \sum_{\substack{a=1 \\ a^t\equiv b^t \pmod p \\ c^t\equiv 1 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) -$$

$$\sum_{\substack{a=1 \\ a^t\equiv -1 \pmod p \\ b^t\equiv 1 \pmod p \\ c^t\equiv -1 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) - \sum_{\substack{a=1 \\ a^t\equiv -1 \pmod p \\ b^t\equiv -1 \pmod p \\ c^t\equiv 1 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) - \sum_{\substack{a=1 \\ a^t\equiv 1 \pmod p \\ b^t\equiv 1 \pmod p \\ c^t\equiv 1 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}).$$

It is not hard to show that

$$\sum_{\substack{a=1 \\ a^t\equiv -1 \pmod p \\ b^t+c^t\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) = \sum_{\substack{a=1 \\ a^t\equiv -1 \pmod p \\ b^tc^t+c^t\equiv 0 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c}) = \sum_{\substack{a=1 \\ a^t\equiv -1 \pmod p \\ b^t\equiv -1 \pmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \chi(a\overline{c})$$

$$= \begin{cases} (p-1)\left((2t, p-1) - (t, p-1)\right)^2, & \text{if } \chi = \chi_0, \\ 0, & \text{if } \chi \neq \chi_0, \end{cases}$$

$$\sum_{\substack{a=1\\a^t\equiv b^t\,(\mathrm{mod}\,p)\\c^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c}) = \sum_{\substack{a=1\\a^t\equiv a^t b^t\,(\mathrm{mod}\,p)\\c^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c}) = \sum_{\substack{a=1\\b^t\equiv 1\,(\mathrm{mod}\,p)\\c^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c})$$

$$= \begin{cases} (p-1)\,(t,p-1)^2, & \text{if } \chi=\chi_0,\\ 0, & \text{if } \chi\neq\chi_0,\end{cases}$$

$$\sum_{\substack{a=1\\a^t\equiv c^t\,(\mathrm{mod}\,p)\\b^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c}) = \sum_{\substack{a=1\\a^t\equiv 1\,(\mathrm{mod}\,p)\\b^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a) = (p-1)(t,p-1)\Big(\sum_{\substack{a=1\\a^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\chi(a)\Big),$$

$$\sum_{\substack{a=1\\a^t\equiv -1\,(\mathrm{mod}\,p)\\b^t\equiv 1\,(\mathrm{mod}\,p)\\c^t\equiv -1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c}) = \sum_{\substack{a=1\\a^t\equiv 1\,(\mathrm{mod}\,p)\\b^t\equiv 1\,(\mathrm{mod}\,p)\\c^t\equiv -1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a)$$

$$=(t,p-1)((2t,p-1)-(t,p-1))\Big(\sum_{\substack{a=1\\a^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\chi(a)\Big),$$

$$\sum_{\substack{a=1\\a^t\equiv -1\,(\mathrm{mod}\,p)\\b^t\equiv -1\,(\mathrm{mod}\,p)\\c^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c}) = \sum_{\substack{a=1\\a^t\equiv -1\,(\mathrm{mod}\,p)\\b^t\equiv -1\,(\mathrm{mod}\,p)\\c^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a)$$

$$=(t,p-1)((2t,p-1)-(t,p-1))\Big(\sum_{\substack{a=1\\a^t\equiv -1\,(\mathrm{mod}\,p)}}^{p-1}\chi(a)\Big),$$

and

$$\sum_{\substack{a=1\\a^t\equiv 1\,(\mathrm{mod}\,p)\\b^t\equiv 1\,(\mathrm{mod}\,p)\\c^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a\bar{c}) = \sum_{\substack{a=1\\a^t\equiv 1\,(\mathrm{mod}\,p)\\b^t\equiv 1\,(\mathrm{mod}\,p)\\c^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\chi(a) = (t,p-1)^2\Big(\sum_{\substack{a=1\\a^t\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\chi(a)\Big).$$

Combining the above formulas, we have

$$\begin{aligned}N_2(3t,t,\chi_0;p) =&(p-1)((2t,p-1)-(t,p-1))^2-(t,p-1)((2t,p-1)-(t,p-1))^2-\\&(t,p-1)^2((2t,p-1)-(t,p-1))+2(p-1)(t,p-1)^2-(t,p-1)^3.\end{aligned}$$

Noting that $(t,p-1)((2t,p-1)-(t,p-1))^2=(t,p-1)^2((2t,p-1)-(t,p-1))$, thus we get

$$\begin{aligned}N_2(3t,t,\chi_0;p) =&(p-1-2(t,p-1))((2t,p-1)-(t,p-1))^2+\\&(2(p-1)-(t,p-1))(t,p-1)^2.\end{aligned}$$

For $\chi \neq \chi_0$, we also have

$$N_2(3t, t, \chi; p) = (p-1)(t, p-1)\Big(\sum_{\substack{a=1 \\ a^t \equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a)\Big) -$$
$$(t, p-1)\left((2t, p-1) - (t, p-1)\right)\Big(\sum_{\substack{a=1 \\ a^t \equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a)\Big) -$$
$$(t, p-1)\left((2t, p-1) - (t, p-1)\right)\Big(\sum_{\substack{a=1 \\ a^t \equiv -1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a)\Big) -$$
$$(t, p-1)^2\Big(\sum_{\substack{a=1 \\ a^t \equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a)\Big)$$
$$= (t, p-1)\left(p-1-(t, p-1)\right)\Big(\sum_{\substack{a=1 \\ a^t \equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a)\Big) -$$
$$(t, p-1)\left((2t, p-1) - (t, p-1)\right)\Big(\sum_{\substack{a=1 \\ a^{2t} \equiv 1 \,(\mathrm{mod}\, p)}}^{p-1} \chi(a)\Big).$$

This completes the proof of Theorem 5.2. □

Corollary 5.3 *Let p be an odd prime, and let χ denote a Dirichlet character modulo p. Write*

$$\chi(a) = \begin{cases} 0, & \text{if } (a,p) > 1; \\ e\left(\frac{m\,\mathrm{ind}\,a}{p-1}\right), & \text{if } (a,p) = 1. \end{cases}$$

We have

$$N_2(4, 2, \chi; p) = \begin{cases} 8p - 16, & \text{if } \chi = \chi_0; \\ 4p - 12, & \text{if } \chi \neq \chi_0 \text{ and } 2 \mid m; \\ 0, & \text{if } \chi \neq \chi_0 \text{ and } 2 \nmid m, \end{cases}$$

$$N_2(6, 2, \chi; p) = \begin{cases} (p-5)((4,p-1)-2)^2 + 8p - 16, & \text{if } \chi = \chi_0; \\ 4p - 28, & \text{if } \chi \neq \chi_0, p \equiv 1 \,(\mathrm{mod}\, 4) \text{ and } 4 \mid m; \\ 4p - 12, & \text{if } \chi \neq \chi_0, p \equiv 1 \,(\mathrm{mod}\, 4), 2 \mid m \text{ and } 4 \nmid m; \\ 4p - 12, & \text{if } \chi \neq \chi_0, p \equiv 3 \,(\mathrm{mod}\, 4) \text{ and } 2 \mid m; \\ 0, & \text{if } \chi \neq \chi_0 \text{ and } 2 \nmid m, \end{cases}$$

$$N_2(6, 3, \chi; p) = \begin{cases} 2(3,p-1)^2(p-1) - (3,p-1)^3, & \text{if } \chi = \chi_0; \\ 9p - 36, & \text{if } \chi \neq \chi_0, p \equiv 1 \,(\mathrm{mod}\, 3) \text{ and } 3 \mid m; \\ 0, & \text{if } \chi \neq \chi_0, p \equiv 1 \,(\mathrm{mod}\, 3) \text{ and } 3 \nmid m; \\ p - 2, & \text{if } \chi \neq \chi_0 \text{ and } p \equiv 2 \,(\mathrm{mod}\, 3), \end{cases}$$

$$N_2(9,3,\chi;p)=\begin{cases} 3(p-1-(3,p-1))(3,p-1)^2, & \textit{if } \chi=\chi_0;\\ 9p-90, & \textit{if } \chi\neq\chi_0, p\equiv 1\,(\mathrm{mod}\,3) \textit{ and } 6\mid m;\\ 9p-36, & \textit{if } \chi\neq\chi_0, p\equiv 1\,(\mathrm{mod}\,3), 3\mid m \textit{ and } 6\nmid m;\\ 0, & \textit{if } \chi\neq\chi_0, p\equiv 1\,(\mathrm{mod}\,3) \textit{ and } 3\nmid m;\\ p-4, & \textit{if } \chi\neq\chi_0, p\equiv 2\,(\mathrm{mod}\,3) \textit{ and } 2\mid m;\\ p-2, & \textit{if } \chi\neq\chi_0, p\equiv 2\,(\mathrm{mod}\,3) \textit{ and } 2\nmid m.\end{cases}$$

Proof By the properties of character sums we get

$$\sum_{\substack{a=1\\ a^k\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\chi(a)=\frac{1}{p-1}\sum_{a=1}^{p-1}\chi(a)\sum_{\psi \,\mathrm{mod}\, p}\psi(a^k)=\frac{1}{p-1}\sum_{\psi\,\mathrm{mod}\,p}\sum_{a=1}^{p-1}\chi(a)\psi^k(a).$$

Write

$$\chi(a)=\begin{cases}0, & \text{if } (a,p)>1;\\ \mathrm{e}(\frac{m\,\mathrm{ind}\,a}{p-1}), & \text{if } (a,p)=1,\end{cases}\quad \text{and}\quad \psi(a)=\begin{cases}0, & \text{if } (a,p)>1;\\ \mathrm{e}(\frac{n\,\mathrm{ind}\,a}{p-1}), & \text{if } (a,p)=1.\end{cases}$$

We have

$$\sum_{\substack{a=1\\ a^k\equiv 1\,(\mathrm{mod}\,p)}}^{p-1}\chi(a)=\frac{1}{p-1}\sum_{n=0}^{p-2}\sum_{a=1}^{p-1}\mathrm{e}(\frac{(m+kn)\mathrm{ind}\,a}{p-1})=\sum_{\substack{n=0\\ m+kn\equiv 0\,(\mathrm{mod}\,p-1)}}^{p-2}1$$

$$=\begin{cases}(k,p-1), & \text{if } (k,p-1)\mid m;\\ 0, & \text{if } (k,p-1)\nmid m.\end{cases}$$

Then from Theorems 5.1 and 5.2 we can get Corollary 5.3. □

6. Proof of Theorem 1.3

Write

$$\chi(a)=\begin{cases}0, & \text{if } (a,p)>1;\\ \mathrm{e}(\frac{m\,\mathrm{ind}\,a}{p-1}), & \text{if } (a,p)=1.\end{cases}$$

Note that if $\chi\neq\chi_0$ and $\chi^2=\chi_0$, then $m=\frac{p-1}{2}$. Combining (3.1), Theorem 4.1, Corollaries 4.3 and 5.3, we immediately get Theorem 1.3.

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