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Deductive Systems in Hyper EQ-Algebras

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Abstract In this paper, we introduce and investigate some types of deductive systems in hyper EQ-algebras and discuss relationships among them. Especially, we focus on investigating two types of important deductive systems, namely, (positive) implicative strong deductive systems, respectively. Moreover we give equivalent characterizations of them.

Keywords hyper EQ-algebra; (strong) deductive system; (positive) implicative strong deductive system; $S_{\rightarrow}$-reflexive subset; $S_{\otimes}$-semiclosed subset

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1. Introduction

EQ-algebras is a new class of logic algebra which was proposed by Nov*á*k in [1]. It has three connectives: meet $\wedge$, product $\otimes$ and fuzzy equality $\sim$. One of the motivations was to introduce a special algebra as the correspondence of truth values for high-order fuzzy type theory (FTT). Another motivation is from the equational style of proof in logic. It is well known that filter (or deductive system) theory plays an important role in studying logic systems. From a logical point of view, various filters correspond to various provable formula. Implicative filters and positive implicative filters are two types of important filters and many researchers have studied them [2–4]. In [5], Liu introduced and studied (positive) implicative prefilters (filters) in EQ-algebras. The hyper structure theory was introduced by Marty [6], at the 8th Congress of Scandinavian Mathematicians. In an algebraic hyper structure, the composition of two elements is not an element but a set. Since then hyper structure theory has been intensively researched in [7–11]. Recently, Borzooei has applied the hyper theory to EQ-algebras to introduce the notion of hyper EQ-algebras [13] which is a generalization of EQ-algebras. Now hyper structure theory has applied to many disciplines such as geometry, graphs, automata, cryptography, artificial intelligence, probability theory, dismutation reactions and inheritance, etc. Similar to logic algebras, filter (or deductive system) theory is an important tool in studying hyper structures

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and one can see [7,12]. The above are the motivation of studying deductive systems in hyper EQ-algebras.

This paper is organized as follows: in Section 2, we recall some notions of hyper EQ-algebras. In Section 3, we introduce the notion of (strong) deductive systems in hyper EQ-algebras and discuss the relation between them. In Section 4, we introduce the notion of positive implicative strong deductive systems in hyper EQ-algebras and give a characterization of it. In Section 5, we introduce the notion of implicative strong deductive systems in hyper EQ-algebras and discuss the relations between implicative strong deductive systems and strong deductive systems, positive implicative strong deductive systems, respectively. Moreover we obtain some equivalent characterizations of implicative strong deductive systems in hyper EQ-algebras.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in the following.

Definition 2.1 ([13]) *A hyper EQ-algebra* $(H;\wedge,\otimes,\sim,1)$ *is a nonempty set* H *endowed with a binary operation* $\wedge$, *two binary hyper operations* $\otimes,\sim$ *and a top element* 1 *satisfying the following axioms, for all* $x,y,z,t\in H$:

(HE1) $(H;\wedge,1)$ *is a* $\wedge$*-semilattice with top element* 1;

(HE2) $(E,\otimes,1)$ *is a commutative semihypergroup with* 1 *as an identity and* $\otimes$ *is isotone w.r.t.* $\leq$ *(i.e., if* $x\leq y$, *then* $x\otimes z\ll y\otimes z$*)*;

(HE3) $((x\wedge y)\sim z)\otimes(t\sim x)\ll z\sim(t\wedge y)$;

(HE4) $(x\sim y)\otimes(z\sim t)\ll(x\sim z)\sim(y\sim t)$;

(HE5) $(x\wedge y\wedge z)\sim x\ll(x\wedge y)\sim x$;

(HE6) $(x\wedge y)\sim x\ll(x\wedge y\wedge z)\sim(x\wedge z)$;

(HE7) $x\otimes y\ll x\sim y$,

where $x\leq y$ *if and only if* $x\wedge y=x$, *and* $A\ll B$ *means that for any* $x\in A$, *there exists* $y\in B$ *such that* $x\leq y$. *For all nonempty subsets* A *and* B *of* H, $A\wedge B=\{a\wedge b|a\in A,b\in B\}$ *and* $A\circ B=\bigcup_{a\in A,b\in B}a\circ b$, $\circ\in\{\otimes,\sim\}$.

Example 2.2 ([13]) Let $H=[0,1]$. Define $\wedge,\sim$ and $\otimes$ as follows: $x\sim y=\{x\wedge y,1\}$, $x\wedge y=\min\{x,y\}$ and

$$x\otimes y=\begin{cases}0, & x+y\leq 1,\\ x\wedge y, & \text{otherwise.}\end{cases}$$

Then $(H;\wedge,\otimes,\sim,1)$ is a hyper EQ-algebra.

Example 2.3 ([13]) Let $H=\{0,a,1\}$ with $0<a<1$. Define the operations $\wedge,\otimes$ and $\sim$ on H as follows: $x\wedge y=\min\{x,y\}$ and

$\otimes$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0\}$	$\{0,a\}$
1	$\{0\}$	$\{0,a\}$	$\{1\}$

$\sim$	0	a	1
0	$\{1\}$	$\{0,a\}$	$\{0\}$
a	$\{0,a\}$	$\{1\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

Table 1 Definition of $\otimes$ and $\sim$ in Example 2.3

Then $(H;\otimes,\sim,\wedge,1)$ is a hyper EQ-algebra.

Example 2.4 ([13]) Let $H=\{0,a,b,1\}$ with $0<a<b<1$. Define the operations $\wedge,\otimes$ and $\sim$ on H as follows: $x\wedge y=x\otimes y=\min\{x,y\}$ and

$\sim$	0	a	b	1
0	$\{1\}$	$\{a,b,1\}$	$\{a,1\}$	$\{1\}$
a	$\{a,b,1\}$	$\{a,1\}$	$\{a,b,1\}$	$\{1\}$
b	$\{a,1\}$	$\{a,b,1\}$	$\{b,1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

Table 2 Definition of $\sim$ in Example 2.4

Then $(H;\wedge,\otimes,\sim,1)$ is a hyper EQ-algebra.

Let $(H;\wedge,\otimes,\sim,1)$ be a hyper EQ-algebra. Put the operations $\to$ and $*$ as follows: $x\to y:=(x\wedge y)\sim x$, $x^*:=x\sim 1$ for any $x,y\in H$. If H has a bottom element 0, we call H bounded. In this case, we define $\neg$ by $\neg x:=0\sim x$ for any $x\in H$.

Proposition 2.5 ([13]) *Let $(H;\wedge,\otimes,\sim,1)$ be a hyper EQ-algebra. Then for all $x,y,z,t\in H$:*

(P1) $1\in x\sim x$ *and* $1\in A\sim A$*;*

(P2) $x\le y$ *implies* $1\in x\to y$ *and* $A\ll B$ *implies* $1\in A\to B$*;*

(P3) $x\le y$ *implies* $x\sim y=y\to x$*;*

(P4) $A\ll B$ *and* $B\ll C$ *imply* $A\ll C$*;*

(P5) $x\sim y\ll y\sim x$ *and* $A\sim B\ll B\sim A$*;*

(P6) $x\ll x\sim 1$*;*

(P7) $x\le y$ *implies* $z\to x\ll z\to y$ *and* $y\to z\ll x\to z$*;*

(P8) $y\ll x\to y$ *and* $B\ll A\to B$*;*

(P9) $(x\sim y)\otimes(z\sim t)\ll(x\wedge z)\sim(y\wedge t)$*;*

(P10) $x\to y=x\to x\wedge y$*;*

(P11) $y\to z\ll(x\to y)\to(x\to z)$ *and* $B\to C\ll(A\to B)\to(A\to C)$*;*

(P12) $A\ll B$ *implies* $A\otimes C\ll B\otimes C$*;*

(P13) $(x\to y)\otimes(y\to z)\ll x\to z$ *and* $(A\to B)\otimes(B\to C)\ll A\to C$*.*

Proposition 2.6 *Let $(H;\wedge,\otimes,\sim,1)$ be a hyper EQ-algebra. Then for all $x,y,z\in H$ and $A,B,C\subseteq H$:*

(1) $A\wedge B\ll A,B$ *and* $A\otimes B\ll A\wedge B$*;*

(2) $x \in x \otimes 1$ *and* $A \subseteq A \otimes 1$*;*

(3) $A \subseteq B$ *and* $B \ll C$ *imply* $A \ll C$*;*

(4) $x \to y \ll (x \wedge z) \to (y \wedge z)$ *and* $A \to B \ll (A \wedge C) \to (B \wedge C)$*.*

Proof (1), (2) and (3) are straightforward.

(4) By (P1), (P3), (P5), (P7), (P9), (P12) and Proposition 2.6 (2), $x \to y \subseteq (x \to y) \otimes 1 \subseteq ((x \wedge y) \sim x) \otimes (z \sim z) \ll (x \sim (x \wedge y)) \otimes (z \sim z) \ll (x \wedge z) \sim (x \wedge y \wedge z) = (x \wedge z) \to (x \wedge y \wedge z) \ll (x \wedge z) \to (y \wedge z)$. Combining Proposition 2.6 (4) and (P4), we can get that $x \to y \ll (x \wedge z) \to (y \wedge z)$. $\square$

Definition 2.7 ([13]) *Let* $(H; \wedge, \otimes, \sim, 1)$ *be a hyper EQ-algebra.* H *is called good if* $x \sim 1 = x = 1 \sim x$ *for all* $x \in H$*.*

3. (Strong) deductive systems in hyper EQ-algebras

From now on, unless otherwise stated we assume that H is a hyper EQ-algebra.

Definition 3.1 *A nonempty subset* S *containing* 1 *of* H *is called a subalgebra of* H *if* S *is closed with respect to the operations* $\wedge, \otimes, \sim$*. That is,* $x \wedge y \in S$ *and* $x \circ y \subseteq S$*, where* $\circ \in \{\otimes, \sim\}$ *for all* $x, y \in S$*.*

Example 3.2 Let H be a hyper EQ-algebra defined in Example 2.3. One can calculate that $D = \{1, a\}$ is not a subalgebra of H, since $1 \otimes a = \{0, a\} \nsubseteq D$. But $\{1\}$ is a subalgebra of H.

Example 3.3 (1) In Example 2.2, the subset $S = [0.5, 1]$ is a subalgebra of H.

(2) In Example 2.4, the subset $S = \{1, a\}$ is a subalgebra of H.

Definition 3.4 *Let* H *be a hyper EQ-algebra. A nonempty subset* D *of* H *is called a*

• *deductive system (or briefly, DS) in* H *if* D *satisfies*

(D) $x \in D$ *and* $x \leq y$ *imply* $y \in D$ *for all* $x, y \in H$*.*

(HD) $x \in D$ *and* $D \ll x \to y$ *imply* $y \in D$ *for all* $x, y \in H$*.*

• *strong deductive system (or briefly, SDS) in* H *if* $1 \in D$ *and* D *satisfies*

(SHD) $x \in D$ *and* $x \to y \cap D \neq \emptyset$ *imply* $y \in D$ *for all* $x, y \in H$*.*

Example 3.5 Let $(H; \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra given in Example 2.2. One can calculate that $D = [0.5, 1]$ is not a DS in H, because $0.5 \in D$ and $D \ll 0.5 \to 0 = \{0, 1\}$, while $0 \notin D$. $D = [0.5, 1]$ is either not an SDS in H, because $0.5 \in D$ and $0.5 \to 0 = \{0, 1\} \cap D \neq \emptyset$, while $0 \notin D$.

Example 3.6 Let H be a hyper EQ-algebra defined in Example 2.3. It is easily checked that $D = \{1, a\}$ is a DS and $\{1\}$ is a DS (SDS) in H. But $D = \{1, a\}$ is not an SDS in H, since $a \in D$ and $a \to 0 = \{0, a\} \cap D \neq \emptyset$, while $0 \notin D$.

Example 3.7 Let $H = \{0, a, 1\}$ with $0 < a < 1$. Define the operations $\wedge, \otimes$ and $\sim$ on H as follows: $x \wedge y = \min\{x, y\}$ and

$\otimes$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0, a\}$
1	$\{0\}$	$\{0, a\}$	$\{1\}$

$\sim$	0	a	1
0	$\{1\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{1\}$	$\{1, a\}$
1	$\{0\}$	$\{1, a\}$	$\{1\}$

Table 3 Definition of $\otimes$ and $\sim$ in Example 3.7

Then one can check that $(H; \otimes, \sim, \wedge, 1)$ is a hyper EQ-algebra and $D = \{a, 1\}$ is a DS (SDS) in H.

Proposition 3.8 *In any hyper EQ-algebra H,*

(1) every DS in H contains 1;

(2) every SDS in H satisfies (D);

(3) every SDS in H is a DS.

Proof (1) Evident.

Assume that D is an SDS in H. For any $x, y \in H$.

(2) Let $x \in D$ and $x \leq y$. Then $1 \in x \to y$ by (P2) and hence $x \to y \cap D \neq \emptyset$. Therefore $y \in D$.

(3) Let $x \in D$ and $D \ll x \to y$. Then there exists $b \in x \to y$ such that $x \leq b$. By (2), we have $b \in F$. This implies that $x \to y \cap D \neq \emptyset$ and so $y \in D$. □

Lemma 3.9 *Let D be a nonempty subset satisfying (D) of H. Then for any nonempty subset A, B of H, $A \cap D \neq \emptyset$ and $A \ll B$ imply $B \cap D \neq \emptyset$.*

Proof Since $A \cap D \neq \emptyset$, then there is $a \in A$ such that $a \in F$. For the above $a \in A$, it follows from $A \ll B$ that there exists $b \in B$ such that $a \leq b$. Again since $a \in D$ and D satisfies (D), we get $b \in D$. This shows that $B \cap D \neq \emptyset$. □

Proposition 3.10 *Let H be a hyper EQ-algebra and D be an SDS in H. Then*

(1) $x \in D$ and $x \otimes y \cap D \neq \emptyset$ imply $y \in D$;

(2) $x \in D$ and $x \sim y \cap D \neq \emptyset$ imply $y \in D$;

(3) $x \to y \subseteq D$ and $y \to z \cap D \neq \emptyset$ imply $x \to z \cap D \neq \emptyset$.

Proof (1) Let $x \in D$ and $x \otimes y \cap D \neq \emptyset$. Since $x \otimes y \ll x \to y$, then by Lemma 3.9 $x \to y \cap D \neq \emptyset$. Again since D is an SDS in H and $x \in D$, we obtain $y \in D$.

(2) Similar to (1).

(3) For any $x, y, z \in H$, $y \to z \ll (x \to y) \to (x \to z)$ by (P11). Since $y \to z \cap D \neq \emptyset$, then by Lemma 3.9, $(x \to y) \to (x \to z) \cap D \neq \emptyset$. Hence there exist $a \in x \to y \subseteq D$ and $b \in x \to z$ such that $a \to b \cap D \neq \emptyset$, which implies $b \in D$. Therefore $x \to z \cap D \neq \emptyset$. □

Definition 3.11 *Let H be a hyper EQ-algebra. A nonempty subset S of H is said to be*

$S_{\to}$−reflexive if $x \to y \cap S \neq \emptyset$ implies $x \to y \subseteq S$ for all $x, y \in H$.

Example 3.12 (1) In Example 2.3, $D = \{1, a\}$ is not $D_{\to}$-reflexive, since $a \to 0 = \{0, a\} \cap D \neq \emptyset$, but $a \to 0 \nsubseteq D$.

(2) In Example 3.7, one can easily check that $D = \{1, a\}$ is $D_{\to}$-reflexive.

Proposition 3.13 *Let D be a nonempty $S_{\to}$-reflexive subset satisfying (D). Then*

(1) D is closed with respect to the operation $\to$;

(2) If D is an SDS in H, then D is closed with respect to the operation $\wedge$.

Proof Let $x, y \in D$.

(1) Since $y \ll x \to y$, we have $x \to y \cap D \neq \emptyset$ by Lemma 3.9. Applying the $D_{\to}$-reflexivity of D, $x \to y \subseteq D$.

(2) According to the proof of (1) and (P9) we can get $x \to (x \wedge y) = x \to y \cap D \neq \emptyset$. Since D is an SDS, it follows from $x \in D$ that $x \wedge y \in D$.

Let H be a hyper EQ-algebra and S be a nonempty subset of H. Denote by $[S)$ the least strong deductive system of H containing S, called the strong deductive system generated by S. In particular, if $S = \{a\}$, we write $[\{a\}) = [a)$, called the principal strong deductive system generated by the element a in H. In addition, we use $[D \cup \{x\})$ to denote the strong deductive system generated by D and x, where $x \in H - D$. The following are some results about the generated strong deductive system. □

Theorem 3.14 *Let S be a nonempty subset of a hyper EQ-algebra H. Then $[S) \supseteq \{x \in H : 1 \in a_n \to (\cdots (a_2 \to (a_1 \to x)) \cdots)$ for some $a_1, a_2, \ldots, a_n \in S\}$.*

Proof Similar to the proof of Theorem 3.9 in [12]. □

Theorem 3.15 *Let D be an SDS of a hyper EQ-algebra H and $a \in H - D$. Then $[D \cup \{a\}) \supseteq \{x \in H : a \to x \cap D \neq \emptyset\}$.*

Proof Similar to the proof of Theorem 3.9 in [12]. □

Corollary 3.16 *Let H be a hyper EQ-algebra and $a \in H$. Then $[a) \supseteq \{x \in H : 1 \in a \circ x\}$.*

4. Positive implicative strong deductive systems in hyper EQ-algebras

In this section, we introduce the notion of positive implicative strong deductive systems in hyper EQ-algebras, and give an equivalent characterization for strong deductive systems to be positive implicative strong deductive systems in hyper EQ-algebras.

Definition 4.1 *Let H be a hyper EQ-algebra. A nonempty subset D in H is called a positive implicative strong deductive system (or briefly, PISDS) if it satisfies (D) and $x \to ((z \to y) \to z) \cap D \neq \emptyset$, $x \in D$ imply $z \in D$ for all $x, y, z \in H$.*

Clearly, every PISDS contains 1.

Example 4.2 Let H be a hyper EQ-algebra given in Example 2.3. It is easily calculated that D is not a PISDS in H, since $a \in D$ and $a \to ((0 \to 1) \to 0) \cap D \neq \emptyset$, but $0 \notin D$.

Example 4.3 Let H be a hyper EQ-algebra defined in Example 3.7. Routine calculation shows that $D = \{1, a\}$ is a PISDS in H.

Theorem 4.4 *Let H be a hyper EQ-algebra and D is a PISDS in H. Then D is an SDS in H.*

Proof Let $x \in D$ and $x \to y \cap D \neq \emptyset$ for any $x, y \in H$. Since $y \ll 1 \to y \ll (y \to y) \to y$, then from (P7) $x \to y \ll x \to ((y \to y) \to y)$. According to Lemma 3.9 we get that $x \to ((y \to y) \to y) \cap D \neq \emptyset$. Since D is a PISDS and $x \in D$, then $y \in D$, which implies that D is an SDS in H. □

Example 4.5 In Example 3.7 we can see that $D = \{a, 1\}$ is both an SDS and a PISDS in H.

The converse of Theorem 4.4 is not true in general. That is, an SDS may not be a PISDS in H. See the following example.

Example 4.6 Let $H = \{0, a, 1\}$ with $0 < a < 1$. Define the operations $\wedge, \otimes$ and $\sim$ on H as follows: $x \wedge y = \min\{x, y\}$ and

$\otimes$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0, a\}$	$\{0, a\}$
1	$\{0\}$	$\{0, a\}$	$\{1\}$

$\sim$	0	a	1
0	$\{1\}$	$\{0, a\}$	$\{a\}$
a	$\{0, a\}$	$\{1\}$	$\{a\}$
1	$\{0, a\}$	$\{0, a\}$	$\{1\}$

Table 4 Definition of $\otimes$ and $\sim$ in Example 4.6

Then $(H; \otimes, \sim, \wedge, 1)$ is a hyper EQ-algebra [13]. It is easily verified that $D = \{1\}$ is an SDS in H. But D is not a PISDS in H, since $1 \in D$ and $1 \to ((a \to 0) \to a) \cap D \neq \emptyset$, while $a \notin D$.

In the following, we give a characterization about the PISDS in H.

Theorem 4.7 *Let H be a hyper EQ-algebra and D be an SDS in H. Then the following are equivalent:*

(1) D is a PISDS in H;

(2) $(x \to y) \to x \cap D \neq \emptyset$ implies $x \in D$ for all $x, y \in H$.

Proof (1) $\Rightarrow$ (2). Assume that D is a PISDS in H. Let $(x \to y) \to x \cap D \neq \emptyset$ for any $x, y \in H$. By (P8), $(x \to y) \to x \ll 1 \to ((x \to y) \to x)$. According to Lemma 3.9 we have $1 \to ((x \to y) \to x) \cap D \neq \emptyset$. Considering $1 \in D$, we can obtain $x \in D$.

(2) $\Rightarrow$ (1). Assume that $z \in D$ and $z \to ((x \to y) \to x) \cap D \neq \emptyset$ for any $x, y, z \in H$. Since D is an SDS, then $(x \to y) \to x \cap D \neq \emptyset$. Therefore by hypothesis $x \in D$. This implies that D is a PISDS in H. □

Corollary 4.8 *Let H be a bounded hyper EQ-algebra and D be a PISDS in H. Then $\neg x \to x \cap D \neq \emptyset$ implies $x \in D$ for all $x \in H$.*

5. Implicative strong deductive systems in hyper EQ-algebras

In this section, we introduce the notion of implicative strong deductive systems in hyper EQ-algebras, and we give some characterizations about implicative strong deductive systems in hyper EQ-algebras. In particular, we obtain the representation of the generated strong deductive systems via implicative strong deductive systems.

Definition 5.1 *Let H be a hyper EQ-algebra. A nonempty subset D in H is called an implicative strong deductive system (or briefly, ISDS) if it satisfies (D) and $x \to (y \to z) \cap D \neq \emptyset$, $x \to y \cap D \neq \emptyset$ imply $x \to z \cap D \neq \emptyset$ for all $x, y, z \in H$.*

Clearly, every ISDS in H contains 1.

Example 5.2 Let H be a hyper EQ-algebra given in Example 2.3. It is easily calculated that $D = \{1, a\}$ is not an ISDS in H, since $1 \to (a \to 0) = \{0, a\} \cap D \neq \emptyset$ and $1 \to a \cap D \neq \emptyset$, but $1 \to 0 = \{0\} \cap D = \emptyset$.

Example 5.3 Let H be a hyper EQ-algebra defined in Example 3.7. Routine calculation shows that $D = \{1, a\}$ is an ISDS in H.

Note that an ISDS of H may not be an SDS of H. Indeed, consider Example 2.4, it is clear that $D = \{b, 1\}$ is an ISDS in H. Since $b \in D$ and $b \to a = \{a, b, 1\} \cap D \neq \emptyset$ while $a \notin D$, we know that $D = \{b, 1\}$ is not an SDS in H.

Theorem 5.4 *Let H be a good hyper EQ-algebra and D be an ISDS in H. Then D is an SDS in H.*

Proof Let $x \in D$ and $x \to y \cap D \neq \emptyset$ for any $x, y \in H$. Using $x \ll 1 \to x$ and $x \to y \ll 1 \to (x \to y)$, we have $1 \to x \cap D \neq \emptyset$ and $1 \to (x \to y) \cap D \neq \emptyset$. Since D is an ISDS of H, then $1 \to y \cap D \neq \emptyset$. Again since H is good, we have $y \in D$. This shows that D is an SDS in H. □

Example 5.5 In Example 2.3, H is good and $\{1\}$ is both an ISDS in H and an SDS in H.

The condition that H is good of Theorem 5.5 is not necessary. In Example 3.7, H is not good EQ-algebra, but $D = \{1, a\}$ is an SDS in H. Moreover $D = \{1, a\}$ is an ISDS in H. Note that an SDS need not be an ISDS in H in general. See the following example.

Example 5.6 In Example 4.6, $D = \{1\}$ is an SDS in H. But D is not an ISDS in H, since $a \to 1 = \{0, a\} \cap D \neq \emptyset$ and $a \to (1 \to 0) = \{1\} \cap D \neq \emptyset$, while $a \to 0 = \{0, a\} \cap D = \emptyset$.

Now we give some conditions that an SDS is an ISDS in H.

Theorem 5.7 Let H be a hyper EQ-algebra and D be an SDS in H. If $x \to (y \to z) \cap D \neq \emptyset$ implies $(x \to y) \to (x \to z) \cap D \neq \emptyset$ for all $x, y \in H$, then D is an ISDS in H.

Proof Suppose that $x \to (y \to z) \cap D \neq \emptyset$ and $x \to y \cap D \neq \emptyset$. Since D is an SDS, by hypothesis we get that $x \to z \cap D \neq \emptyset$. Therefore D is an ISDS in H. □

Definition 5.8 *Let H be a hyper EQ-algebra. A nonempty subset S of H is called $S_{\otimes}$-semiclosed* *if $x, y \in S$ implies $x \otimes y \cap S \neq \emptyset$.*

Example 5.9 (1) Let $(H; \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra defined in Example 2.3. Then $D = \{1, a\}$ is not $D_{\otimes}$-semiclosed, because $a \otimes a = \{0\} \cap D = \emptyset$.

(2) Let $(H; \wedge, \otimes, \sim, 1)$ be a hyper EQ-algebra defined in Example 3.7. Then it is clear that $D = \{1, a\}$ is $D_{\otimes}$-semiclosed.

Lemma 5.10 *Let H be a hyper EQ-algebra and D be a nonempty $D_{\otimes}$-semiclosed subset of H. Then for any $A, B \subseteq H$, $A \cap D \neq \emptyset, B \cap D \neq \emptyset$ imply $A \otimes B \cap D \neq \emptyset$. Furthermore if D satisfies (D), we have $A \wedge B \cap D \neq \emptyset$, $A \sim B \cap D \neq \emptyset$ and $A \to B \cap D \neq \emptyset$.*

Proof Suppose that $A \cap D \neq \emptyset$ and $B \cap D \neq \emptyset$. Then there exist $a \in A$ and $b \in B$ such that $a, b \in D$. Since D is $D_{\otimes}$-semiclosed, then $a \otimes b \cap D \neq \emptyset$ and hence $A \otimes B \cap D \neq \emptyset$. Again since $A \otimes B \ll A \wedge B, A \sim B, A \to B$ and D satisfies (D), from Lemma 3.9 we verify that $A \wedge B \cap D \neq \emptyset$, $A \sim B \cap D \neq \emptyset$ and $A \to B \cap D \neq \emptyset$. □

Theorem 5.11 *Let H be a hyper EQ-algebra and D be a $D_{\to}$-reflexive and $D_{\otimes}$-semiclosed SDS in H. If $(x \wedge (x \to y)) \to y \cap D \neq \emptyset$ for all $x, y \in H$, then D is an ISDS in H.*

Proof Let $x \to (y \to z) \cap D \neq \emptyset$ and $x \to y \cap D \neq \emptyset$ for any $x, y \in H$. From Proposition 2.6 (5) and (P10), we have $x \to (y \to z) \ll (x \wedge y) \to (y \wedge (y \to z))$ and $x \to y = x \to (x \wedge y)$. Hence $(x \wedge y) \to (y \wedge (y \to z)) \cap D \neq \emptyset$ and $x \to (x \wedge y) \cap D \neq \emptyset$. Again since $(x \wedge y) \to (y \wedge (y \to z)) \ll (x \to (x \wedge y)) \to (x \to (y \wedge (y \to z)))$ by (P11), then $(x \to (x \wedge y)) \to (x \to (y \wedge (y \to z))) \cap D \neq \emptyset$. Thus there exist $a \in x \to (x \wedge y)$ and $b \in x \to (y \wedge (y \to z))$ such that $a \to b \cap D \neq \emptyset$. Since $x \to (x \wedge y) \cap D \neq \emptyset$ and D is $D_{\to}$-reflexive SDS, we have $a \in x \to (x \wedge y) \subseteq D$ and so $b \in D$. This implies that $x \to (y \wedge (y \to z)) \cap D \neq \emptyset$. Considering Lemma 5.10, $x \to (y \wedge (y \to z)) \cap D \neq \emptyset$ and the hypothesis of $(y \wedge (y \to z)) \to z \cap D \neq \emptyset$, we obtain that $(x \to (y \wedge (y \to z))) \otimes (((y \wedge (y \to z)) \to z) \cap D \neq \emptyset$ as D is $D_{\otimes}$-semiclosed. Again by (P13) $(x \to (y \wedge (y \to z))) \otimes (((y \wedge (y \to z)) \to z) \ll x \to z$, it follows from Lemma 3.9 that $x \to z \cap D \neq \emptyset$. That is, D is an ISDS in H. □

Definition 5.12 *A nonempty subset S of a hyper EQ-algebra H is said to satisfy exchange property, if $x \to (y \to z) \cap S \neq \emptyset$ implies $y \to (x \to z) \cap S \neq \emptyset$ for all $x, y, z \in H$.*

Example 5.13 (1) In Example 4.6 $D = \{1\}$ does not satisfy the exchange property since $a \to (1 \to 0) \cap D \neq \emptyset$, but $1 \to (a \to 0) \cap D \neq \emptyset$.

(2) In Example 3.7 we can easily check that $D = \{a, 1\}$ satisfies the exchange property.

The following theorems give characterizations about an ISDS in H.

Theorem 5.14 *Let H be a hyper EQ-algebra and D be $D_{\otimes}$-semiclosed subset satisfying (D) and the exchange property in H. Then the following are equivalent:*

(1) D is an ISDS in H;

(2) $x \to (x \to y) \cap D \neq \emptyset$ implies $x \to y \cap D \neq \emptyset$ for all $x, y \in H$.

Proof (1) $\Rightarrow$ (2). Let $x \to (x \to y) \cap D \neq \emptyset$ for any $x, y \in H$. Since $x \to x \cap D \neq \emptyset$ and D is an ISDS, we have $x \to y \cap D \neq \emptyset$.

(2) $\Rightarrow$ (1). Assume that $x \to (y \to z) \cap D \neq \emptyset$ and $x \to y \cap D \neq \emptyset$ for all $x, y \in H$. Since D satisfies the exchange property, we have $y \to (x \to z) \cap D \neq \emptyset$. Again since D is $D_{\otimes}$-semiclosed, by Lemma 5.10 we can get that $(x \to y) \otimes (y \to (x \to z)) \cap D \neq \emptyset$. By use of $(x \to y) \otimes (y \to (x \to z)) \ll x \to (x \to z)$ from (P13), it follows from Lemma 3.9 that $x \to (x \to z) \cap D \neq \emptyset$. By (2) $x \to z \cap D \neq \emptyset$ and therefore D is an ISDS in H.

Corollary 5.15 *Let H be a hyper EQ-algebra and D be an ISDS in H. Then $x^{**} \cap D \neq \emptyset$ implies $x^{*} \cap D \neq \emptyset$ for any $x \in H$.*

Theorem 5.16 *Let D be a nonempty subset of a hyper EQ-algebra H. Then D is an ISDS of H if and only if $\{x \in H : a \to x \cap D \neq \emptyset\}$ is an SDS of H for all $a \in H$.*

Proof Similar to the proof of Theorem 4.22 in [12]. □

Corollary 5.17 *Let D be an ISDS in a hyper EQ-algebra H and $a \in H - D$. Then the SDS generated by D and a can be represented as $[D \cup \{a\}) = \{x \in H : a \to x \cap D \neq \emptyset\}$.*

Proof Similar to the proof of Corollary 4.23 in [12]. □

Corollary 5.18 *Let H be a hyper EQ-algebra and let $a \in H$. Then $\{1\}$ is an ISDS in H if and only if $[a) = \{x \in H : 1 \in a \to x\}$.*

The following is a relationship between the ISDS and the PISDS in H.

Theorem 5.19 *Let H be a hyper EQ-algebra and D be a $D_{\to}$-reflexive and $D_{\otimes}$-semiclosed subset of H. If D is a PISDS in H, then D is an ISDS in H.*

Proof Assume that D is a PISDS in H. Then from Theorem 4.4 D is an SDS in H. By Proposition 2.6 $x \wedge (x \to y) \ll x, x \to y$, it follows from (P7) $x \to y \ll (x \wedge (x \to y)) \to y$. Hence $x \wedge (x \to y) \ll (x \wedge (x \to y)) \to y$. Again applying (P7), we can get that $((x \wedge (x \to y)) \to y) \to y \ll (x \wedge (x \to y)) \to y$. This shows that $(((x \wedge (x \to y)) \to y) \to y) \to ((x \wedge (x \to y)) \to y) \cap D \neq \emptyset$. By means of Theorem 4.7 we can get that $(x \wedge (x \to y)) \to y \cap D \neq \emptyset$. Again since D is $D_{\to}$-reflexive and $D_{\otimes}$-semiclosed, using Theorem 5.11 we can verity that D is an ISDS in H. □

Example 5.20 It is easily seen that $D = \{a, 1\}$ satisfies the conditions of Theorem 5.19. Moreover $D = \{a, 1\}$ is both a PISDS in H and an ISDS in H.

Note that the converse of Theorem 5.19 is not true in general. See the following example.

Example 5.21 In Example 4.6, $D = \{a, 1\}$ is an ISDS in H. But D is not a PISDS since $a \in D$ and $a \to ((0 \to 1) \to 0) \cap D \neq \emptyset$, while $0 \notin D$.

In [5], we can see that positive implicative prefilters are implicative prefilters in EQ-algebras, but we do not know whether the above result holds in hyper EQ-algebras. So we have the following question.

Question. Is there a PISDS in H which is not an ISDS in H? In other words, can the condition of Theorem 5.19 be removed?

6. Conclusion and future research

Filter (or deductive system) theory plays an important role in studying the structure of algebras. This paper investigates some types of deductive systems in hyper EQ-algebras and focuses on (positive) implicative strong deductive systems. We find some conditions that a strong deductive system is an implicative strong deductive system. Especially, we give characterizations of (positive) implicative strong deductive systems. In the next work we will construct quotient hyper EQ-algebras via (strong) deductive systems.

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