Journal of Mathematical Research with Applications
Mar., 2017, Vol. 37, No. 2, pp. 194–198
DOI:10.3770/j.issn:2095-2651.2017.02.007
Http://jmre.dlut.edu.cn

On Semiclean Group Rings

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Abstract A ring R with unity is called semiclean, if each of its elements is the sum of a unit and a periodic. Every clean ring is semiclean. It is not easy to characterize a semiclean group ring in general. Our purpose is to consider the following question: If G is a locally finite group or a cyclic group of order 3, then when is RG semiclean? Some known results on clean group rings are generalized.

Keywords clean ring; semiclean ring; group ring; locally finite group

MR(2010) Subject Classification 16E50

1. Introduction

Throughout this paper, all rings are associative rings with identity. Let R be a ring and G a group. We will denote by RG the group ring of G over R. We use the symbol $U(R)$, $J(R)$ to denote the set of units and the Jacobson radical of R, respectively.

An element of a ring is called clean if it is the sum of an idempotent and a unit, and a ring R is called clean if each of its elements is clean. This notion was first introduced by Nicholson in 1977 (see [1]). A ring whose idempotents are central is called abelian. Usually, we write C_n for the cyclic group of order n. A group G is called locally finite if every finitely generated subgroup of G is finite. Let p be a prime number. A group G is called a p-group if the order of each element of G is a power of p. A group G is said to be an elementary p-group if all non-identity elements of G are of order of p. It is well known that a finite abelian elementary p-group is a direct product of finitely many copies of C_p.

When is a group ring RG clean? This question was first considered by Han and Nicholson [2]. In general, the question when RG is clean seems to be difficult to answer. It is still unanswered when RC_2 is clean. If G is a locally finite group and R is semiperfect or unit-regular or strongly π-regular or abelian clean ring, whether is RG clean? These questions were considered by Zhou [3]. Semiclean ring was first defined by Ye [4]. The author in [4] also proved that the group ring $\mathbb{Z}_pG$ with G a cyclic group of order 3 is semiclean. When is RG semiclean if G is a locally finite group or a cyclic group of order 3? In this paper, this question was mainly considered, and some important results have been obtained.

Received December 7, 2015; Accepted December 2, 2016
Supported by the National Natural Science Foundation of China (Grant No. 11401009) and the Natural Science Foundation of Ahui Province (Grant No. 1408085QA01).
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For a group ring RG, the ring homomorphism $\varepsilon : RG \to R$ such that $\varepsilon(\Sigma_{g\in G} r_g g) = \Sigma_{g\in G} r_g$ is called the augmentation mapping of RG. Its kernel is $\Delta(RG) = \{\sum_{g\in G} a_g(g-1) : 1 \neq g \in G, a_g \in R\}$ and $RG/\Delta(RG) \cong R$. If H is a normal subgroup of G, then $\Delta(RH) = \{\sum_{h\in H} a_h(h-1) : 1 \neq h \in H, a_h \in R\}$, denoting the kernel of $\varepsilon|_{RH}$, is an ideal of RG and $RG/\Delta(RH) \cong R(G/H)$. Let IG denote the elements of RG with coefficients in an ideal I, then IG is an ideal and $RG/IG \cong (R/I)G$. We refer to [5] for further details on group rings. More recent studies on clean rings and semiclean rings can be found in [6–8] and the references therein.

Recall some notion from [4]. An element x of a ring R is called semiclean if $x = u + f$, where f is a periodic, i.e., $f^k = f^l, f \in R$ for some positive integers k and l $(k \neq l)$ and u is a unit in R. A ring R is semiclean if each of its element is semiclean. Let I be an ideal of a ring R. We say that periodics in R can be lifted modulo I, if for any $a \in R$ with $a^k - a^l \in I$, there exists $b \in R$ such that $b^k = b^l \in R$ and $a - b \in I$.

2. Main results

Proposition 2.1 *If R is a semiclean ring, G is a locally finite qroup, and $\Delta(RG) \subseteq J(RG)$, then $RG/J(RG)$ is a semiclean ring.*

Proof Since G is a locally finite group, it implies $J(R) \subseteq J(RG)$, and $J(RG)/\Delta(RG) \cong J(R)$ by [5, Proposition 9]. Then $RG/J(RG) \cong \frac{RG/\Delta(RG)}{J(RG)/\Delta(RG)} \cong R/J(R)$. Note that $R/J(R)$ is a semiclean ring, hence $RG/J(RG)$ is a semiclean ring. □

Lemma 2.2 ([3, Lemma 2]) *Let p be a prime with $p \in J(R)$. If G is a locally finite group, then $\Delta(RG) \subseteq J(RG)$.*

Proposition 2.3 *Let R be a ring, p a prime number with $p \in J(R)$ and G a locally finite group with $G = NH$ where N is a normal p-subgroup of G and H is a subgroup of G. If RH is semiclean, then $RG/J(RG)$ is semiclean.*

Proof By assumption $G = NH$, for $g \in G$, there exists $n \in N$, $h \in H$ such that $g = nh = (n-1)h + h \in \Delta(RN) + RH$, so $RG = \Delta(RN) + RH$. Lemma 4.1 in [9] yields $J(RN) \subseteq J(RG)$ and Lemma 2.2 shows that $\Delta(RN) \subseteq J(RN)$. Hence $RG = J(RG) + RH$. We now prove $J(RH) = RH \bigcap J(RG)$. One obtains $RH \bigcap J(RG) \subseteq J(RH)$ by [5, Proposition 9]. From $J(RG/J(RG)) = 0$, we conclude $RH/[RH \bigcap J(RG)] \cong RG/J(RG)$ is semiprimitive, and so $J(RH) \subseteq RH \bigcap J(RG)$ by [10, Corollary 15.6]. Therefore $RH/J(RH) \cong RG/J(RG)$. We obtain $RG/J(RG)$ is semiclean from RH semiclean. □

Proposition 2.4 *Let R be a ring with $2 \in U(R)$ and G is an abelian elementary 2-group. Then RG is semiclean if and only if R is semiclean.*

Proof We may assume that G is a finite group. Then G is a direct product of n copies of C_2 for some $n \geqslant 1$. Since $2 \in U(R)$, $RC_2 \cong R \bigoplus R$. As 2 is a unit of RC_2, we have $R(C_2 \times C_2) \cong (RC_2)(C_2) \cong RC_2 \bigoplus RC_2 \cong R \bigoplus R \bigoplus R \bigoplus R$. A similar argument shows that

RG is isomorphic to the direct sum of $2n$ copies of R. Therefore RG is semiclean if and only if R is semiclean. $\square$

Theorem 2.5 *For a ring R and a locally finite group G, RG is semiclean if and only if SG is semiclean for every indecomposable image S of R.*

Proof ($\Leftarrow$) If I is an ideal of R and $a_i \in R$ and $g_i \in G$ $(i = 1, \ldots, n)$, we denote $\overline{a}_i = (a_i + I) \in R/I$, so

$$\sum \overline{a}_i g_i = \sum (a_i + I) g_i \in (R/I)G.$$

Suppose that RG is not semiclean. Then there exists a finite subset F of G such that $\sum_{g\in F} a_g g$ is not semiclean in RG, where each $a_g \in R$. Thus, $M = \{I \lhd R | \sum_{g\in F} \overline{a}_i g$ is not semiclean in $(R/I)G\}$ is not empty. For a chain $\{I_\lambda\}$ of elements of M, let $I = \bigcup_\lambda I_\lambda$, then I is an ideal of R. Assume that $\sum_{g\in F} \overline{a}_i g$ is semiclean in $(R/I)G$. Because G is a locally finite group, there exists a finite subgroup H of G with $F \subseteq H$ such that

$$\sum_{g\in H} \overline{a}_g g = \sum_{g\in H} \overline{f}_g g + \sum_{g\in H} \overline{u}_g g, \tag{2.1}$$

where $a_g = 0$ for all $g \in H \backslash F$, $\sum_{g\in H} \overline{f}_g g$ is a periodic in $(R/I)H$ and $\sum_{g\in H} \overline{u}_g g$ is a unit in $(R/I)H$ with inverse $\sum_{g\in H} \overline{v}_g g$. Write $H = \{1 = g_1, g_2, \ldots, g_n\}$. Thus, the following (2.2)–(2.4) hold in R/I for $m = 1, \ldots, n$. By (2.1) we have

$$\overline{a}_{g_m} = \overline{f}_{g_m} + \overline{u}_{g_m}. \tag{2.2}$$

Since $\sum_{g\in H} \overline{f}_g g$ is a periodic in $(R/I)H$, it follows $(\overline{f}_{g_1} g_1 + \overline{f}_{g_2} g_2 + \cdots + \overline{f}_{g_n} g_n)^k = (\overline{f}_{g_1} g_1 + \overline{f}_{g_2} g_2 + \cdots + \overline{f}_{g_n} g_n)^l$ for some positive integers k and l $(k \neq l)$. Comparing the coefficients of the two sides of equal, then we have

$$\sum_{g_{i_1} g_{i_2} \cdots g_{i_k} = g_m} \overline{f}_{g_{i_1}} \overline{f}_{g_{i_2}} \cdots \overline{f}_{g_{i_k}} = \sum_{g_{i_1} g_{i_2} \cdots g_{i_l} = g_m} \overline{f}_{g_{i_1}} \overline{f}_{g_{i_2}} \cdots \overline{f}_{g_{i_l}}. \tag{2.3}$$

Since $\sum_{g\in H} \overline{u}_g g$ is a unit in $(R/I)H$, we have $(\overline{u}_{g_1} g_1 + \overline{u}_{g_2} g_2 + \cdots + \overline{u}_{g_n} g_n)(\overline{v}_{g_1} g_1 + \overline{v}_{g_2} g_2 + \cdots + \overline{v}_{g_n} g_n) = \overline{1}$. Comparing the coefficients of the two sides of equal, then we have

$$\sum_{g_i g_j = g_m} \overline{u}_{g_i} \overline{v}_{g_j} = \delta_{1m} \overline{1} = \sum_{g_i g_j = g_m} \overline{v}_{g_i} \overline{u}_{g_j}, \tag{2.4}$$

where $\delta_{11} = 1$ and $\delta_{1m} = 0$ for $m \neq 1$. It follows that all the following elements (for $m = 1, \ldots, n$) are in I: $a_{g_m} - f_{g_m} - u_{g_m} \in I$, $\delta_{1m} - \sum_{g_i g_j = g_m} u_{gi} v_{gj} \in I$, $\delta_{1m} - \sum_{g_i g_j = g_m} v_{gi} u_{gj} \in I$, $\sum_{g_{i_1} g_{i_2} \cdots g_{i_k} = g_m} f_{g_{i_1}} f_{g_{i_2}} \cdots f_{g_{i_k}} - \sum_{g_{i_1} g_{i_2} \cdots g_{i_l} = g_m} f_{g_{i_1}} f_{g_{i_2}} \cdots f_{g_{i_l}} \in I$. Because $\{I_\lambda\}$ is a chain, there exists some I_λ such that all these elements are in I_λ. Hence (2.2)–(2.4) hold in R/I_λ and (2.1) holds in $(R/I_\lambda)G$. So $\sum_{g\in F} a_g g$ is semiclean in $(R/I_\lambda)G$. This contradiction shows that I is in M. By Zorn's Lemma, M contains a maximal element, say I. It now suffices to show that R/I is indecomposable.

Assume that R/I is decomposable, then there exists ideals $K_j (j = 1, 2)$ of R and $I \subseteq K_j$ such that

$$R/I \cong R/K_1 \bigoplus R/K_2, \text{ via } r + I \mapsto (r + K_1, r + K_2).$$

Accordingly, $(R/I)G \cong (R/K_1 \bigoplus R/K_2)G \cong (R/K_1)G \bigoplus (R/K_2)G$, where the composition of the two isomorphisms is $\sum(r_g+I)g \mapsto (\sum(r_g+K_1)g, \sum(r_g+K_2)g)$. By the maximality of I in M, $(\sum_{g\in F}(a_g+K_j)g$ is semiclean in $(R/K_j)G$ for $j = 1, 2$. Hence $(\sum_{g\in F}(a_g+K_1)g, \sum_{g\in F}(a_g+K_2)g)$ is a semiclean element of $(R/K_1)G \bigoplus (R/K_2)G$; so $\sum_{g\in F} \overline{a}_g g$ is semiclean in $(R/I)G$. This is a contradiction.

$\Rightarrow$. For an image S of R, SG is an image of RG. So SG is semiclean when RG is semiclean.

Lemma 2.6 ([11, Proposition 9]) *Let R be a commutative ring and let C_n be a cyclic group of order n generated by g. Then an element $x = \sum_{i=0}^{n-1} k_i g^i \in RC_n$ is invertible if and only if $\det A \in R$ is invertible, where $k_i \in R$ and* $A = \begin{pmatrix} k_0 & k_{n-1} & \cdots & k_1 \\ k_1 & k_0 & \cdots & k_2 \\ & & \ddots & \\ k_{n-1} & k_{n-2} & \cdots & k_0 \end{pmatrix}$.

Theorem 2.7 *Let R be a commutative local ring with $2 \in U(R)$ and let $G = \{1, a, a^2\}$ be a cyclic group of order 3 generated by a. Then RG is a semiclean ring.*

Proof Let $x = k + la + ma^2 \in RG$, where $k, l, m \in R$. Let us look at the following ways to express $x = k + la + ma^2$:

$$\begin{aligned} k + la + ma^2 &= 1 + [(k-1) + la + ma^2] = a + [k + (l-1)a + ma^2] \\ &= a^2 + [k + la + (m-1)a^2] = -1 + [(k+1) + la + ma^2] \\ &= -a + [k + (l+1)a + ma^2] = -a^2 + [k + la + (m+1)a^2]. \end{aligned}$$

We first consider the elements in the first column on the right of the equal sign. We can see: $1^2 = 1$, $a^4 = a$, $(a^2)^4 = a^2$, $(-1)^3 = (-1)$, $(-a)^7 = -a$, $(-a^2)^7 = -a^2$, so those elements are periodic. In order to show that x is semiclean, we need to show that at least one of the elements in the second column on the right of equal sign is a unit in RG. By Lemma 2.6, we only need to show that at least one of the following six elements is a unit in R:

$$(k-1)^3 + l^3 + m^3 - 3(k-1)lm, \tag{2.5}$$

$$k^3 + (l-1)^3 + m^3 - 3k(l-1)m, \tag{2.6}$$

$$k^3 + l^3 + (m-1)^3 - 3kl(m-1), \tag{2.7}$$

$$(k+1)^3 + l^3 + m^3 - 3(k+1)lm, \tag{2.8}$$

$$k^3 + (l+1)^3 + m^3 - 3k(l+1)m, \tag{2.9}$$

$$k^3 + l^3 + (m+1)^3 - 3kl(m+1). \tag{2.10}$$

Suppose it is not true. Since R is a commutative local ring, all (2.5)–(2.10) belong to $J(R)$. By (2.5) and (2.8), we have $[(k+1)^3 + l^3 + m^3 - 3(k+1)lm] - [(k-1)^3 + l^3 + m^3 - 3(k-1)lm] = 2(3k^2 - 3lm + 1) \in J(R)$. Since 2 is a unit in R, we have

$$3k^2 - 3lm + 1 \in J(R). \tag{2.11}$$

If $3 \in J(R)$, then $1 \in J(R)$, this is a contradiction, so 3 is a unit in R. We have $3k^3 - 3klm + k = k(3k^2 - 3lm + 1) \in J(R)$. Similarly, $3l^3 - 3klm + l \in J(R)$, $3m^3 - 3klm + m \in J(R)$. Thus, we obtain $3(k^3 + l^3 + m^3 - 3klm) + (k + l + m) \in J(R)$. Since 3 is a unit in R,

$$(k^3 + l^3 + m^3 - 3klm) + 3^{-1}(k + l + m) \in J(R). \tag{2.12}$$

By (2.5)+(2.8)−(2.12)×2, $[2k^3+2l^3+2m^3+6k-6klm]-2[(k^3+l^3+m^3-3klm)+3^{-1}(k+l+m)] = 2[3k-3^{-1}(k+l+m)] \in J(R)$. Since $2 \in U(R)$, we have $3k-3^{-1}(k+l+m) \in J(R)$. Similarly, we have $3l-3^{-1}(k+l+m) \in J(R)$, $3m-3^{-1}(k+l+m) \in J(R)$. $3(k+l+m)-3(3^{-1}(k+l+m)) = 3(k+l+m)-(k+l+m) = 2(k+l+m) \in J(R)$. Since 2 is a unit in R, it follows $(k+l+m) \in J(R)$. Therefore, $3k \in J(R)$, which means $k \in J(R)$. Similarly, $l \in J(R)$, $m \in J(R)$. By (2.11), $1 \in J(R)$, a contradiction. Thus, x is a semiclean element.

Corollary 2.8 *Let R be a commutative semiperfect ring with $2 \in U(R)$ and let G be a cyclic group of order 3. Then RG is a semiclean ring.*

Proof Since R is semiperfect, there exists orthogonal local idempotents $\{e_1, e_2, \ldots, e_n\}$ such that $1 = e_1 + e_2 + \cdots + e_n$ by [10, Theorem 27.6]. So $R = e_1Re_1 \times e_2Re_2 \times \cdots \times e_nRe_n$ is a direct product of commutative local rings. Therefore, $RG \cong e_1Re_1G \times e_2Re_2G \times \cdots \times e_nRe_nG$, thus RG is semiclean by Theorem 2.7. □

Remark 2.9 As we all know, the ring $\mathbb{Z}_p = \{m/n | m, n \in \mathbb{Z}, \gcd(p, n) = 1\}$, where $p \neq 2$ is a prime number, is a commutative local ring and $2 \in U(R)$. Let G be a cyclic group of order 3. Then $\mathbb{Z}_pG$ is a semiclean ring [4, Theorem 3.1]. We obtain this result immediately by Theorem 2.7.

Acknowledgements The authors would like to thank the anonymous referees for their careful reading of the manuscript and helpful comments.

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