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A Generalization of VNL-Rings and PP-Rings

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Abstract Let R be a ring. An element a of R is called a left PP-element if Ra is projective. The ring R is said to be a left almost PP-ring provided that for any element a of R, either a or $1-a$ is left PP. We develop, in this paper, left almost PP-rings as a generalization of von Neumann local (VNL) rings and left PP-rings. Some properties of left almost PP-rings are studied and some examples are also constructed.

Keywords VNL-rings; left PP-rings; left almost PP-rings

MR(2010) Subject Classification 16E50; 16L99

1. Introduction

As a common generalization of von Neumann regular rings and local rings, Contessa in [1] called a commutative ring R von Neumann local (VNL) if for each $a \in R$, either a or $1-a$ is von Neumann regular (An element $a \in R$ is von Neumann regular provided that there exists an element $x \in R$ such that $a = axa$). VNL-rings are also exchange rings. Some properties of VNL-rings and SVNL-rings were investigated in [2]. Later Chen and Tong in [3] defined a noncommutative ring to be a VNL-ring. Some results on commutative VNL-rings were extended. Moreover, Grover and Khurana in [4] characterized VNL-rings in the sense of relating them to some familiar classes of rings. On the other hand, we recall that a ring R is said to be left PP (see [5]) (or left Rickart) provided that every principal left ideal is projective, or equivalently the left annihilator of any element of R is a summand of R_R. A ring is called a PP-ring if it is both left and right PP-ring. Examples include von Neumann regular rings and domains. The PP-rings and their generalizations have been extensively studied by many authors [5–15].

We say that, in this paper, an element a of R is left PP in R if Ra is projective, or equivalently, if $l_R(a) = Re$ for some $e^2 = e \in R$. Obviously, R is a left PP-ring if and only if every element of R is left PP. A ring R is said to be a left almost PP-ring provided that for any element a of R, either a or $1-a$ is left PP. left almost PP-rings are introduced as the generalization of left PP-rings and VNL-rings. Some examples turn out to show that this generalization is non-trivial. In Section 2, we investigate the properties of left almost PP-rings. Extensions of left PP-rings are considered in Section 3. Some results on left PP-rings are

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extended onto left almost PP-rings. Section 4 focuses on semiperfect, left almost PP-rings. We give the structure of this class rings.

Throughout R is an associative ring with identity and all modules are unitary. $J(R)$ will denote the Jacobson radical of R. $\mathbb{Z}_n$ stands for the ring of integers mod n. $M_n(R)$ denotes the ring of all $n\times n$ matrices over a ring R with an identity I_n. If X is a subset of R, the left (resp., right) annihilator of X in R is denoted by $l_R(X)$ (resp., $r_R(X)$). If $X=\{a\}$, we usually abbreviate it to $l_R(a)$ (resp., $r_R(a)$). For the usual notations we refer the reader to [1], [7] and [16].

2. Left almost PP-rings

We start this section with the definition.

Definition 2.1 Let R be a ring and $a\in R$. a is called a left PP-element in R if Ra is projective, or equivalently, $l_R(a)=Re$ for some $e^2=e\in R$. The ring R is said to be a left almost PP-ring provided that for any element a of R, either a or $1-a$ is left PP. Similarly, right almost PP-rings can be defined. A ring R is called almost PP if it is left and right almost PP.

Remark 2.2 (1) Obviously, left PP-rings are left almost PP-rings.

(2) Every VNL-ring is a left and right almost PP-ring.

(3) Clearly, $a\in R$ is left PP if and only if au is left PP for every unit element $u\in R$.

Example 2.3 (1) The ring $\mathbb{Z}$ of integers is an almost PP-ring but not a VNL-ring.

(2) The ring $\mathbb{Z}_4$ of integers mod 4 is an almost PP-ring but not a PP-ring.

(3) Let $R=\{\left(\begin{smallmatrix} a & b\\ 0 & a\end{smallmatrix}\right)|a,b\in\mathbb{Z}_2\}$. Then $R=\{\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right),\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right),\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right),\left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right)\}$.

If $c=\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right)$, let $e=\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)$; If $c=\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right)$, let $e=\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right)$; If $c=\left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right)$, consider $1-c=\left(\begin{smallmatrix} 1 & -1\\ 0 & 1\end{smallmatrix}\right)$, let $e=\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right)$. In either case, we have $l_R(c)=Re$ or $l_R(1-c)=Re$. So R is a left almost PP-ring. choose $c=\left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right)\in R$, Rc is not projective since $l_R(c)=J(R)$ cannot be generated by an idempotent, then R is not a left PP-ring.

Example 2.4 If R is a left PP-ring, S is local and let ${}_RM_S$ be bimodule, then $\left(\begin{smallmatrix} R & M\\ 0 & S\end{smallmatrix}\right)$ is a left almost PP-ring.

Proof Let $T=\left(\begin{smallmatrix} R & M\\ 0 & S\end{smallmatrix}\right)$. For any $\alpha=\left(\begin{smallmatrix} a & m\\ 0 & b\end{smallmatrix}\right)\in T$. Since S is local, b or 1_S-b is invertible. Assume that b is invertible. Note that a is a left PP-element in R, so there exists $e^2=e\in R$ such that $l_R(a)=Re$. Then

$$\begin{pmatrix} e & -emb^{-1}\\ 0 & 0\end{pmatrix}\begin{pmatrix} a & m\\ 0 & b\end{pmatrix}=\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}.$$

Let $\beta=\left(\begin{smallmatrix} e & -emb^{-1}\\ 0 & 0\end{smallmatrix}\right)$. Then $\beta^2=\beta\in T$ and $T\beta\subseteq l_T(\alpha)$. Now for any $\left(\begin{smallmatrix} a_1 & m_1\\ 0 & b_1\end{smallmatrix}\right)\in l_T(\alpha)$, $\left(\begin{smallmatrix} a_1 & m_1\\ 0 & b_1\end{smallmatrix}\right)\left(\begin{smallmatrix} a & m\\ 0 & b\end{smallmatrix}\right)=\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right)$, we have $a_1\in l_R(a)=Re$, $b_1=0$ and $m_1=-a_1mb^{-1}$. So $\left(\begin{smallmatrix} a_1 & m_1\\ 0 & b_1\end{smallmatrix}\right)=\left(\begin{smallmatrix} r & 0\\ 0 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} e & -emb^{-1}\\ 0 & 0\end{smallmatrix}\right)\in T\beta$. This implies that α is a left PP-element in T.

Assume that 1_S-b is invertible. As 1_R-a is a left PP-element in R, there exists $f^2=f\in R$

such that $l_R(1_R - a) = Rf$. Similarly, $1_T - \alpha = \begin{pmatrix} 1_R & 0 \\ 0 & 1_S \end{pmatrix} - \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1_R - a & -m \\ 0 & 1_S - b \end{pmatrix}$ is a left PP-element in T. Therefore, we complete the proof. □

Now we elaborate some properties of left almost PP-rings.

Proposition 2.5 *Let R be a left almost PP-ring. Then the following results hold.*

(1) The center of R is an almost PP-ring.

(2) For every $e^2 = e \in R$, the corner ring eRe is a left almost PP-ring.

Proof (1) Let $C(R)$ be the center and $x \in C(R)$. Since R is left almost PP, x or $1 - x$ is a left PP-element in R. If x is a left PP-element, then $l_R(x) = Re$ for some $e = e^2 \in R$. Note that $l_R(x) = Re$ is an ideal and $l_R(x) = r_R(x)$ because $x \in C(R)$. It follows that for every $r \in R$, $er = ere = re$, and hence $e \in C(R)$. We now prove that $l_{C(R)}(x) = C(R)e$. Clearly, $l_{C(R)}(x) = l_R(x) \cap C(R)$ and $C(R)e \subseteq l_{C(R)}(x)$. Let $a \in l_{C(R)}(x)$, then $a \in l_R(x)$ and so $a = ae \cap C(R)e$. Thus, $l_{C(R)}(x) \in C(R)e$. Consequently $l_{C(R)}(x) = C(R)e$. Note that $1 - x$ is also in $C(R)$, if $1 - x$ is a left PP-element in R, then $1 - x$ is a left PP-element in $C(R)$ by using the similar method above. Therefore, $C(R)$ is also an almost PP-ring.

(2) For any $0 \neq a \in eRe$, a or $1 - a$ is left PP in R by hypothesis. Assume that a is left PP, then $l_R(a) = Rf$ for some $f^2 = f \in R$. Note that $l_{eRe}(a) = l_R(a) \cap eRe$ and $1 - e \in l_R(a)$, so $1 - e = (1 - e)f$ and $fe = efe$. Write $fe = g$, then $g^2 = g \in eRe$. So $ga = fea = fa = 0$. On the other hand, for any $b \in l_{eRe}(a)$, $b = be = bef = befe = bg \in eReg$. It implies $l_{eRe}(a) = eReg$. If $1 - a$ is a left PP-element in R, then $e - a$ is a left PP-element in eRe by using the similar method. Thus eRe is a left almost PP-ring. □

An elementary argument using condition in Definition 2.1 shows that a direct product of rings is left PP if and only if each factor is left PP. However, for left almost PP-rings we have the next result.

Theorem 2.6 *Let $R = \prod_{\alpha \in I} R_\alpha$. Then R is a left almost PP-ring if and only if there exists $\alpha_0 \in I$, such that R_{α_0} is a left almost PP-ring and for each $\alpha \in I - \alpha_0$, R_α is a left PP-ring.*

Proof $\Leftarrow$. Let $x = (x_\alpha) \in R$, $\alpha \in I$. By hypothesis, x_{α_0} or $1_{R_{\alpha_0}} - x_{\alpha_0}$ is left PP. If x_{α_0} is a left PP-element in R_{α_0}, then x is a left PP-element in R. If $1_{R_{\alpha_0}} - x_{\alpha_0}$ is a left PP-element in R_{α_0}, then $1 - x$ is a left PP-element in R. Thus, the result follows.

$\Rightarrow$. Assume that R is a left almost PP-ring. Then every R_α is also a left almost PP-ring. Write $R = R_{\alpha_0} \times S$, where $S = \prod R_\alpha$, $\alpha \in I - \alpha_0$. If neither R_{α_0} nor S is left PP, then we can find non-left PP-elements $r \in R_{\alpha_0}$ and $s \in S$. Choose $a = (1_{R_{\alpha_0}} - r, s)$. Then neither a nor $1 - a = (r, 1_S - s)$ is left PP in R, a contradiction. Hence, either R_{α_0} or S is a left PP-ring. If S is a left PP-ring, the result follows. If S is a left almost PP-ring, by iteration of this process, we complete the proof. □

Remark 2.7 (1) Note that the direct product of left almost PP-rings may not be a left almost PP-ring. Clearly, $\mathbb{Z}_4$ and $\mathbb{Z}_9$ are almost PP-rings. But we claim that $\mathbb{Z}_4 \times \mathbb{Z}_9$ is not an almost PP-ring. Choose $a = (\overline{2}, \overline{4})$. Then neither a nor $1 - a$ is PP in $\mathbb{Z}_4 \times \mathbb{Z}_9$, and we are done. The example also shows that the homomorphic image of a left almost PP-ring need not to be a left almost PP-ring.

(2) By the theorem above, if $R\times S$ is a left almost PP-ring, then either R or S is left PP. So, in general, the ring $\mathbb{Z}_n$ of integers mod n is an almost PP-ring if and only if $(pq)^2$ does not divide n, where p and q are distinct primes. It is easy to see that $n=36$ is the least positive integer such that $\mathbb{Z}_n$ is not an almost PP-ring.

Let D be a ring and C a subring of D with $1_D\in C$. We set

$$R[D,C]=\{(d_1,\ldots,d_n,c,c\ldots):d_i\in D,c\in C,n\geq 1\}$$

with addition and multiplication defined componentwise. Since Nicholson used $R[D,C]$ to construct rings which are semiregular but not regular, more and more algebraists use this structure to construct various counterexamples in ring theory.

Theorem 2.8 *$R[D,C]$ is a left almost PP-ring if and only if the following hold:*

(1) D is a left PP-ring.

(2) For any $c\in C$, there exists an $e^2=e\in C$ such that $l_C(c)=Ce$, $l_D(c)=De$ or $l_C(1-c)=Ce$, $l_D(1-c)=De$.

Proof $\Rightarrow$. For convenience, let $S=R[D,C]$. Assume that D is not a left PP-ring. Then there exists a non-left PP-element $x\in D$. Choose $a=(x,1-x,1,1,\ldots)\in S$. By hypothesis, either a or $1-a$ is left PP in S. If a is a left PP-element in S, then x is left PP in D, a contradiction. If $1-a$ is left PP in S, then x is also left PP in D, a contradiction. Thus, D is a left PP-ring.

To prove condition (2), let $c\in C$ and $\overline{c}=(c,c,\ldots)\in S$. Since S is a left almost PP-ring, either $\overline{c}$ or $\overline{1}-\overline{c}$ is left PP in S. Assume that $\overline{c}$ is left PP, then $l_S(\overline{c})=S\overline{e}$, where $\overline{e}=(e_1,\ldots,e_m,e,e,\ldots)$ and $e_i\in D,e\in C$ are also idempotents. Thus $Ce\subseteq l_C(c)$ and $De\subseteq l_D(c)$.

If $x\in l_C(c)$, let $\overline{x}=(x,x,\ldots)$. Then $\overline{x}\in l_S(\overline{c})=S\overline{e}$, and $\overline{x}=(a_1e_1,\ldots,a_me_m,be,be,\ldots)$. Thus, by computing the $(m+1)$th component of $\overline{x}$, we have $x=be\in Ce$, thus $l_C(c)=Ce$.

If $s\in l_D(c)$, let $\overline{s}=(d_1,d_2,\ldots,d_{m+1},0,\ldots)$, where $d_i=s$ for $i=1,\ldots,m+1$. Then $\overline{s}\in l_S(\overline{c})=S\overline{e}$, showing that $s\in De$, thus $l_D(c)=De$.

Assume that $\overline{1}-\overline{c}$ is left PP, then we have $l_C(1-c)=Ce$, $l_D(1-c)=De$ by the similar argument.

$\Leftarrow$. Let $\overline{a}=(a_1,\ldots,a_n,c,c,\ldots)\in S$. For any $\overline{x}=(x_1,\ldots,x_n,\ldots,x_m,x,x,\ldots)\in l_S(\overline{a})$, we have $x_ia_i=0$ $(i=1,\ldots,m)$, where $a_{n+1}=\cdots=a_m=c$, and $xc=0$. Note that $l_D(a_i)=De_i(i=1,\ldots,n)$. If $l_C(c)=Ce$ and $l_D(c)=De$, where $e^2=e\in C$ are also idempotent. So $x_i=d_ie_i$ $(i=1,\ldots,n)$, $x_i=d_ie$ $(i=n+1,\ldots,m)$, $x=c'e$ with all $d_i\in D$, $c'\in C$. Thus

$$\overline{x}=(d_1,\ldots,d_n,d_{n+1},\ldots,d_m,c',c',\ldots)(e_1,\ldots,e_n,e,\ldots,e,e,e,\ldots)\in S\overline{e}.$$

On the other hand, for any $\overline{y}=(y_1,\ldots,y_m,y,y,\ldots)\in S\overline{e}$, we have $y_i\in De_i$ $(i=1,\ldots,n)$, $y_i\in De$ $(i=n+1,\ldots,m)$ and $y\in Ce$. Then $y_ia_i=0$ $(i=1,\ldots,n)$, $y_ic=0$ $(i=n+1,\ldots,m)$ and $yc=0$. It implies that $\overline{y}\,\overline{a}=0$, and hence $\overline{y}\in l_S(\overline{a})$. Therefore, $\overline{a}$ is left PP in S.

If $l_C(1-c)=Ce$, $l_D(1-c)=De$, using the similar argument above, we can prove $\overline{1}-\overline{a}$ is left PP in S.

Therefore, S is a left almost PP-ring. □

By Theorem 2.8, we have the next corollaries immediately.

Corollary 2.9 *$R[D,D]$ is a left almost PP-ring if and only if D is a left PP-ring.*

Corollary 2.10 *$R[D,C]$ is a left PP-ring if and only if D and C are left PP-rings and for any $c\in C$, there exists an $e^2=e\in C$ such that $l_D(c)=De$.*

Example 2.11 Let $S=R[D,C]$, where $D=\mathbb{Q}$ and $C=\mathbb{Z}$. Then S is an almost PP-ring by Theorem 2.8. But S is not a VNL-ring in view of the argument of [4, Example 2.5].

Example 2.12 Let $S=R[D,C]$, where $D=M_2(\mathbb{Z}_2)$ and $C=\{\left(\begin{smallmatrix} a & b\\ 0 & a\end{smallmatrix}\right)|a,b\in\mathbb{Z}_2\}$. Then S is an almost PP-ring which is not left PP, not local.

Proof Obviously, $D=M_2(\mathbb{Z}_2)$ is a PP-ring. $C=\{\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right),\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right),\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right),\left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right)\}$.

If $c=\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right)$, let $e=\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)$; If $c=\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right)$, let $e=\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right)$; If $c=\left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right)$, consider $1-c=\left(\begin{smallmatrix} 1 & -1\\ 0 & 1\end{smallmatrix}\right)$, let $e=\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right)$.

In either case, we have $l_C(c)=Ce$, $l_D(c)=De$ or $l_C(1-c)=Ce$, $l_D(1-c)=De$. By Theorem 2.8, S is a left almost PP-ring. Similarly, we can prove that S is a right almost PP-ring.

Choose $c=\left(\begin{smallmatrix} 0 & 1\\ 0 & 0\end{smallmatrix}\right)\in C$, Rc is not projective since $l_C(c)=J(C)$ cannot be generated by an idempotent, then C is not a left PP-ring. Thus S is not a left PP-ring by Corollary 2.10. Note that $J(S)=R[J(D),J(D)\cap J(C)]=0$, then S is not local, otherwise, S is regular, a contradiction. □

3. Matrix extensions

Matrix constructions will provide new sources of examples of left almost PP-rings. In this section, we will develop results which allows us to study when full matrices and triangular matrices are left almost PP-rings.

Lemma 3.1 *Let R be a ring and $a\in R$. Then the following are equivalent:*

(1) $a\in R$ is a left PP-element.

(2) $\alpha=\left(\begin{smallmatrix} a & 0\\ 0 & 1\end{smallmatrix}\right)\in M_2(R)$ is a left PP-element.

(3) $\beta=\left(\begin{smallmatrix} 1 & 0\\ 0 & a\end{smallmatrix}\right)\in M_2(R)$ is a left PP-element.

Proof Write $S=M_2(R)$.

(1)⇒(2). If $a\in R$ is left PP, there exists an idempotent $e^2=e\in R$ such that $l_R(a)=Re$. Hence $\left(\begin{smallmatrix} e & 0\\ 0 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix} a & 0\\ 0 & 1\end{smallmatrix}\right)=\left(\begin{smallmatrix} 0 & 0\\ 0 & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} e & 0\\ 0 & 0\end{smallmatrix}\right)^2=\left(\begin{smallmatrix} e & 0\\ 0 & 0\end{smallmatrix}\right)\in l_S(\alpha)$. If $\left(\begin{smallmatrix} b & c\\ m & n\end{smallmatrix}\right)\in l_S(\alpha)$,

$$\begin{pmatrix} b & c\\ m & n\end{pmatrix}\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix} ba & c\\ ma & n\end{pmatrix}=\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}.$$

It implies that $b,m\in l_R(a)=Re$, $c=n=0$. Then $b=r_1e, m=r_2e$, and so

$$\begin{pmatrix} b & c \\ m & n \end{pmatrix}=\begin{pmatrix} r_1 & 0 \\ r_2 & 0 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}\in S\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}.$$

We prove that $l_S(\alpha)=S\left(\begin{smallmatrix} e & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

(2)⇒(1). Assume that $\alpha=\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\in S$ is left PP, there exists an idempotent $E=\left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right)\in S$ such that $l_S(\alpha)=SE$. So

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and hence $a_2=a_4=0$, $a_1=a_1^2$, $a_3a_1=a_3$, $a_1,a_3\in l_R(a)$. Conversely, if $x\in l_R(a)$, then

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and hence $\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)\in l_S(\alpha)=SE$. Thus

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ a_3 & 0 \end{pmatrix}=\begin{pmatrix} (r_1+r_2a_3)a_1 & 0 \\ r_3a_1+r_4a_3 & 0 \end{pmatrix}.$$

It implies that $x=(r_1+r_2a_3)a_1\in Ra_1$. Therefore, $l_R(a)=Ra_1$, where $a_1^2=a_1\in R$.

(1)⇔(3) is similar to the proof of (1)⇔(2). □

Now we are in a position to prove when a matrix ring is a left almost PP-ring. The following result is a generalization of [16, Proposition 7.63].

Theorem 3.2 *Let R be a ring. Then the following are equivalent:*

(1) R is a left semihereditary ring;

(2) $M_n(R)$ is a left PP-ring for every $n\geq 1$;

(3) $M_n(R)$ is a left almost PP-ring for every $n\geq 1$.

Proof (1)⇔(2) is dual to [16, Proposition 7.63]. (2)⇒(3) is trivial.

(3)⇒(2). It is enough to show that if $M_2(R)$ is left almost PP, then R is left PP. For any $a\in R$. Choose $A=\left(\begin{smallmatrix} a & a \\ -a & 1-a \end{smallmatrix}\right)\in M_2(R)$. By hypothesis, either $A\in M_2(R)$ or $I_2-A\in M_2(R)$ is left PP. Suppose that $A\in M_2(R)$ is left PP. Note that

$$A=\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

by Remark 2.2(3), $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)\in M_2(R)$ is left PP. So $a\in R$ is left PP by Lemma 3.1.

If $I_2-A\in M_2(R)$ is left PP, noting that

$$I_2-A=\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

by Remark 2.2(3), we have $\left(\begin{smallmatrix} 0 & a \\ 1 & 0 \end{smallmatrix}\right)\in M_2(R)$ is left PP. So $a\in R$ is left PP by Lemma 3.1 again.
□

By the theorem above and Example 2.3, being a left almost PP-ring is not Morita invariant. The next example shows that the definition of almost PP-rings is not left-right symmetric.

Example 3.3 Let S be a von Neumann regular ring with an ideal I such that, as a submodule of S, I is not a direct summand. Let $R = S/I$ and $T = \left(\begin{smallmatrix} R & R \\ 0 & S \end{smallmatrix}\right)$. By the augment of [16, Example 2.34], T is left semihereditary but not right semihereditary. Then there exists some n $(n \geq 2)$ such that the matrix ring $M_n(T)$ is a left almost PP ring but not right almost PP.

A ring R is said to be right Kasch if every simple right R-module embeds in R_R.

Proposition 3.4 *If R is a right Kasch and left almost PP ring, then it is a right almost PP-ring.*

Proof For any $a \in R$, a or $1 - a$ is left PP in R. Assume that a is a left PP-element in R. There exists $e^2 = e \in R$ such that $l(a) = R(1 - e)$. Then $a = ea$, and hence $aR \subseteq eR$. Now we prove that $aR = eR$. Otherwise, $aR \subseteq M$, where M is a maximal submodule of eR. Since R is right Kasch, there exists a monomorphism $f : eR/M \to R$ by $f(e + M) = b$. Then $eb = b$ and $ba = 0$. So $b \in l(a) = R(1 - e)$, and hence $b = be = 0$. Since f is a monomorphism, $e \in M$, contradicting with the maximality of M. So $aR = eR$ is projective. It implies that a is a right PP-element. Assume that $1 - a$ is a left PP-element. We can prove $1 - a$ is a right PP-element by the similar method. □

Let R and S be rings and ${}_RM_S$ a bimodule. We write the generalized triangular matrix as $T = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$. Following [13], a left module is a PP-module if every principal submodule is projective. Now we consider the necessary and sufficient conditions of what a generalized triangular matrix ring is left almost PP.

Proposition 3.5 *Let R and S be rings and ${}_RM_S$ a bimodule. If the following hold:*

(1) R is left PP and S is left almost PP;

(2) If $b \in S$ is a left PP-element, then $l_M(b) = Ml_S(b)$ and M/Mb is a left PP-module. If $b \in S$ is not a left PP-element, then $l_M(1-b) = Ml_S(1-b)$ and $M/M(1-b)$ is a left PP-module. then $T = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ is a left almost PP-ring.

Proof For any $\alpha = \left(\begin{smallmatrix} a & m \\ 0 & b \end{smallmatrix}\right) \in T$. If b is left PP in S, then $l_S(b) = Sf$ with $f^2 = f \in S$. Note that $l_R(a) = Re_1$, $l_R(m + Mb) = Re_2$, where $e_i^2 = e_i \in R$, $i = 1, 2$. By [13, Lemma 1], $Re_1 \cap Re_2 = Re$ for $e^2 = e \in R$. So $e \in Re_1, Re_2$, we can let $em = m_1b$ for some $m_1 \in M$. Write $m_2 = em_1(1-f)$, then $m_2b = em$, and hence $\beta = \left(\begin{smallmatrix} e & -m_2 \\ 0 & f \end{smallmatrix}\right) \in l_T(\alpha)$. Conversely, if $\left(\begin{smallmatrix} x & y \\ 0 & s \end{smallmatrix}\right) \in l_T(\alpha)$, $xa = 0, sb = 0, xm = -yb$, and so $xe = x, sf = s$ and $(y + xm_2)b = yb + xm_2b = -xm + xem = -xm + xm = 0$. Hence $y + xm_2 \in l_M(b) = Ml_S(b)$, and we have $y + xm_2 = (y + xm_2)f$. It follows that $\left(\begin{smallmatrix} x & y \\ 0 & s \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & y+xm_2 \\ 0 & s \end{smallmatrix}\right)\left(\begin{smallmatrix} e & -m_2 \\ 0 & f \end{smallmatrix}\right) \in T\beta$. Thus α is a left PP-element in T.

If $b \in S$ is not left PP-element, then $1 - b$ is left PP because S is a left almost PP-ring. Using the similar method above, we can prove that $1 - \alpha = \left(\begin{smallmatrix} 1-a & -m \\ 0 & 1-b \end{smallmatrix}\right)$ is a left PP-element in T.

Therefore, T is left almost PP. □

Proposition 3.6 *Let R and S be rings and ${}_RM_S$ a bimodule. If $T = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ is a left almost PP-ring, then one of R and S is left PP and the other is left almost PP.*

Proof The result follows from Proposition 2.5(2) and Lemma 4.1 below. □

Corollary 3.7 *Let $T_n(R)$ be the rings of upper triangular matrices over R. Then the following are equivalent:*

(1) *R is regular;*

(2) *$T_n(R)$ is a left PP-ring for every $n \geq 1$;*

(3) *$T_n(R)$ is a left almost PP-ring for every $n \geq 1$.*

Proof It follows by [13, Theorem 4] and Proposition 3.6. □

4. Semiperfect left almost PP-rings

Now we consider the structure of semiperfect, left almost PP-rings.

Lemma 4.1 *If R is a left almost PP-ring and $e^2 = e \in R$, then either eRe or $(1-e)R(1-e)$ is a left PP-ring.*

Proof We have the Pierce decomposition

$$R \cong \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}.$$

If $x \in eRe$ and $y \in (1-e)R(1-e)$ are not left PP-elements, then neither $a = \left(\begin{smallmatrix} x & 0 \\ 0 & 1-y \end{smallmatrix}\right)$ nor $1-a = \left(\begin{smallmatrix} 1-x & 0 \\ 0 & y \end{smallmatrix}\right)$ are left PP-elements. □

Recall a ring R is abelian if each idempotent in R is central. An element a of a ring R is called an exchange element if there exists an idempotent $e \in R$ such that $e \in Ra$ and $1-e \in R(1-a)$. The ring R is an exchange ring if and only if every element of R is an exchange element.

Proposition 4.2 *The following are equivalent for an abelian, exchange ring R.*

(1) *R is an almost PP-ring;*

(2) *For every $e^2 = e \in R$, either eRe or $(1-e)R(1-e)$ is a left PP-ring.*

Proof (1)⇒(2). It follows by Lemma 4.1.

(2)⇒(1). For any $a \in R$, as R is an exchange ring, there exists $e^2 = e \in R$ such that $e \in Ra$ and $1-e \in R(1-a)$. So $Ra + R(1-e) = R$ and $R(1-a) + Re = R$. It implies that $Rae = Re$ and $R(1-a)(1-e) = R(1-e)$. Thus ae is left PP-element in Re and $(1-a)(1-e)$ is left PP-element in $R(1-e)$.

Now if $eRe = Re$ is left PP, then $(1-a)e$ is a left PP-element in eRe, and hence $1-a = (1-a)e + (1-a)(1-e)$ is left PP in R. Similarly, if $(1-e)R(1-e) = R(1-e)$ is left PP, then a is left PP in R. Therefore, R is an almost PP-ring. □

Lemma 4.3 *Let R be a local ring. Then R is a left PP-ring if and only if R is a domain.*

Proposition 4.4 *Let R be a semiperfect, left almost PP-ring with $1 = e_1 + e_2$, where e_1, e_2 are orthogonal local idempotents. Then R is isomorphic to one of the following:*

(1) $M_2(D)$ for some domain D;

(2) $\left(\begin{smallmatrix} D_1 & Y \\ X & D_2 \end{smallmatrix}\right)$, where D_1 is a domain, D_2 is a local ring and $XY \subseteq J(D_1), YX \subseteq J(D_2)$.

In particular, if R is also abelian, then $R \cong M_2(D)$ or $R \cong A \times B$, where D, A are domains and B is a local ring.

Proof We use the Pierce decomposition

$$R \cong \begin{pmatrix} e_1Re_1 & e_1Re_2 \\ e_2Re_1 & e_2Re_2 \end{pmatrix}.$$

If $e_1R \cong e_2R$, then $R \cong M_2(e_1Re_1)$, where e_1Re_1 is a local left PP-ring by Lemma 4.1. So e_1Re_1 is a domain. If $e_1R \not\cong e_2R$, then $e_1Re_2 \subseteq J(R)$ and $e_2Re_1 \subseteq J(R)$ by [4, Lemma 4.2]. We assume that e_1Re_1 is a local left PP-ring by Lemma 4.1, and hence e_1Re_1 is a domain. Note $e_1Re_2Re_1 \subseteq e_1Re_1 \cap J(R) = J(e_1Re_1)$ and $e_2Re_1Re_2 \subseteq e_2Re_2 \cap J(R) = J(e_2Re_2)$. So write $D_1 = e_1Re_1$, $D_2 = e_2Re_2$, $X = e_1Re_2$ and $Y = e_2Re_1$, then (2) follows. □

Proposition 4.5 *Let R be a semiperfect, left almost PP-ring with $1 = e_1 + e_2 + e_3$, where e_1, e_2, e_3 are orthogonal local idempotents. Then R is isomorphic to one of the following:*

(1) $M_3(D)$ for some domain D;

(2) $\left(\begin{smallmatrix} D_1 & Y \\ X & D_2 \end{smallmatrix}\right)$, where D_1 is a domain, D_2 is a local ring and $XY \subseteq J(D_2), YX \subseteq J(D_1)$;

(3) $\left(\begin{smallmatrix} D_1 & Y \\ X & D_2 \end{smallmatrix}\right)$, where D_1 is a prime ring, D_2 is a local ring and $XY \subseteq J(D_2), YX \subseteq J(D_1)$;

(4) $\left(\begin{smallmatrix} S & Y \\ X & D \end{smallmatrix}\right)$ with $S \cong \left(\begin{smallmatrix} D_1 & Y_1 \\ X_1 & D_2 \end{smallmatrix}\right)$ and D_1, D_2, D are domains, $X_1Y_1 \subseteq J(D_2), Y_1X_1 \subseteq J(D_1), XY \subseteq J(D), YX \subseteq J(S)$.

Proof Case 1 If $e_iR \cong e_jR$ for $i, j = 1, 2, 3$, then $R \cong M_3(e_1Re_1)$, where e_1Re_1 is a local left PP-ring by Lemma 4.1. So e_1Re_1 is a domain.

We now consider the the Pierce decomposition

$$R \cong \begin{pmatrix} (1-e_1)R(1-e_1) & (1-e_1)Re_1 \\ e_1R(1-e_1) & e_1Re_1 \end{pmatrix}.$$

Case 2 Assume that e_1Re_1 is local but not a left PP-ring by Lemma 4.1, then $(1-e_1)R(1-e_1)$ is a domain, and hence e_2Re_2 and e_3Re_3 are also domains. By [4, Lemma 4.2], e_1Re_2, e_2Re_1, e_1Re_3 and e_3Re_1 are all contained in $J(R)$. So $(1-e_1)Re_1R(1-e_1) \subseteq J(R) \cap (1-e_1)R(1-e_1) = J((1-e_1)R(1-e_1))$ and $e_1R(1-e_1)Re_1 \subseteq J(R) \cap e_1Re_1 = J(e_1Re_1)$. Thus R is isomorphic to the ring in (2).

Case 3 Assume that e_iRe_i is a domain for $i = 1, 2, 3$. If $e_1R \not\cong e_2R$ but $e_2R \cong e_3R$, then $(1-e_1)R(1-e_1) \cong M_2(D)$ for some domain D, and hence $(1-e_1)R(1-e_1)$ is a prime ring. By [4, Lemma 4.2], e_1Re_2, e_2Re_1, e_1Re_3 and e_3Re_1 are all contained in $J(R)$. So $(1-e_1)Re_1R(1-e_1) \subseteq J(R) \cap (1-e_1)R(1-e_1) = J((1-e_1)R(1-e_1))$ and $e_1R(1-e_1)Re_1 \subseteq J(R) \cap e_1Re_1 = J(e_1Re_1)$. Then (3) is done.

Case 4 Assume that e_iRe_i is a domain for $i = 1, 2, 3$ and $e_1R \ncong e_2R \ncong e_3R$. Then

$$(1 - e_1)R(1 - e_1) \cong \begin{pmatrix} e_2Re_2 & e_2Re_3 \\ e_3Re_2 & e_3Re_3 \end{pmatrix},$$

where $e_2Re_3Re_2 \subseteq J(e_2Re_2)$ and $e_3Re_2Re_3 \subseteq J(e_3Re_3)$. Note that $(1 - e_1)Re_1R(1 - e_1) \subseteq J(R) \cap (1-e_1)R(1-e_1) = J((1-e_1)R(1-e_1))$. So write $e_2Re_2 = D_1$, $e_3Re_3 = D_2$, $e_3Re_2 = X_1$, $e_2Re_3 = Y_1$, $e_1Re_1 = D$, $(1 - e_1)Re_1 = X$, $e_1R(1 - e_1) = Y$, then (4) is also done. $\square$

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