Journal of Mathematical Research with Applications
Mar., 2017, Vol. 37, No. 2, pp. 209–213
DOI:10.3770/j.issn:2095-2651.2017.02.009
Http://jmre.dlut.edu.cn

Finite p-Groups with Large or Small Normal Closures of Non-Normal Cyclic Subgroups

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Abstract A p-group G is called a JC-group if the normal closure H^G of every cyclic subgroup H satisfies $|G : H^G| \leq p$ or $|H^G : H| \leq p$. In this paper, we classify the non-Dedekindian JC-groups for $p > 2$.

Keywords JC-group; non-Dedekindian p-group; regular p-group

MR(2010) Subject Classification 20D15

1. Introduction

All groups considered in this paper are finite.

It is well-known that the normality of subgroups plays an important role in the research of group theory. Thus it is very interesting to investigate the structure of groups by using normal closures of subgroups. For example, [1] investigated p-groups in which the normal closures of the non-normal cyclic subgroups have small index. Herzog, Longobardi, Maj and Mann investigated groups G in which $\langle a, a^g\rangle = \langle a\rangle^G$ for every non-normal cyclic subgroup $\langle a\rangle$ and every $g \in G \setminus N_G(\langle a\rangle)$, which are called J-groups [2]. Then in [3], Guo and first author continued to investigate the J-groups of prime power order. And Lv, Zhou and Yu [4] proved that when G is of odd prime power order $|\langle a\rangle^G : \langle a\rangle| \leq p$ for every $a \in G$ if and only if G is a J-group.

On the other hand, in Section 62 of [5], Janko obtained the structure of p-groups G in which $|G : \langle a\rangle^G| \leq p$ for every non-normal cyclic subgroup $\langle a\rangle$ in G. Moreover, Zhao and Zhang in [6] investigated finite p-groups with large normal closures of non-normal cyclic subgroups. Zhang et al. in [7] and [8] classified finite p-groups G with G/H^G being cyclic (abelian) for every non-normal subgroup H, respectively. Wang and Qu in [9] classified finite groups in which the normal closures of non-normal subgroups have the same order. What we are interested in this paper is the structure of p-groups G in which the normal closure H^G of every cyclic subgroup H satisfies $|G : H^G| \leq p$ or $|H^G : H| \leq p$, for convenience, we call such groups JC-groups. In this paper, we classify non-Dedekindian JC-groups for $p > 2$.

Received February 20, 2016; Accepted October 12, 2016
Supported by the National Natural Science Foundation of China (Grant Nos. 11526114; 11601245), the Natural Science Foundation of Guangdong Province (Grant No. 2015A030313791), the Innovative Team Project of Guangdong Province (CHINA) (Grant No. 2014KTSCX196), Guangdong Province Innovation Talent Project for Youths (Grant No. 2015KQNCX107) and the Appropriative Researching Fund for Doctors, Guangdong University of Education (Grant No. 2013ARF07).
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The terminology and the notation in this paper are standard. The Frattini subgroup, the commutator subgroup and the center of a group G will be denoted by $\Phi(G)$, G' and $Z(G)$, respectively, and the normal closure and the normalizer of a subgroup H in G are denoted by H^G and $N_G(H)$. For an element a in G, $o(a)$ denotes the order of $\langle a\rangle$. If G is a p-group, then $\Omega_1(G)=\langle g\in G\mid g^p=1\rangle$ and $\mho_1(G)=\langle g^p\mid g\in G\rangle$. We use $c(G)$ and G_i to denote the nilpotent class and the ith term of the lower central series of a group G, respectively.

2. Preliminaries

In this section, we give the definition of L-series and some basic facts which are useful for the later use.

Definition 2.1 ([10, Definition 5.5.3]) *Let G be a regular p-group. Set $W_s(G)=\mho_1(G)\Omega_1(G)$, for $0\le s\le e$, we get*

$$G=W_e(G)\ge W_{e-1}(G)\ge\cdots\ge W_0(G)=\mho_1(G).$$

Delete the repeated terms and refine it to a chief series of G to $\mho_1(G)$,

$$G=L_0(G)>L_1(G)>\cdots>L_w(G)=\mho_1(G).$$

We call this series an L-series of G.

Lemma 2.2 ([10, Theorem 5.5.5]) *Let G be a regular p-group and*

$$G=L_0(G)>L_1(G)>\cdots>L_w(G)=\mho_1(G)$$

be a L-series of G. For $1\le s\le w$, choose $b_s\in L_{s-1}(G)-L_s(G)$ such that $o(b_s)$ is as small as possible. Then $(b_1,\dots,b_w)$ is a uniqueness basis of G.

Lemma 2.3 ([11]) *Let G be a minimal non-abelian p-group. Then G is isomorphic to one of the following p-groups:*

(1) Q_8;

(2) $M_p(n,m)=\langle a,b|a^{p^n}=b^{p^m}=1,a^b=a^{1+p^{n-1}}\rangle$ with $n\ge2$ (metacyclic);

(3) $M_p(n,m,1)=\langle a,b,c|a^{p^n}=b^{p^m}=c^p=1,[a,b]=c,[c,a]=[c,b]=1\rangle$ with $n\ge m$, and $n+m\ge3$ when $p=2$ (non-metacyclic).

Lemma 2.4 ([2]) *Let G be a non-abelian J-group of prime power order. If $p>3$, then $c(G)=2$ and $\exp(G')=p$.*

Lemma 2.5 ([2]) *Let G be a non-abelian J-group of 3 power order. Then $c(G)\le3$.*

(1) If $c(G)=2$, then $\exp(G')=3$.

(2) If $c(G)=3$, then $G=AB$ is a product of two subgroups A and B satisfying the following properties:

(i) $A=\langle a,b,c|a^9=b^9=c^3=1,a^3=b^{-3},[a,b]=c,[c,a]=a^{-3},[c,b]=1\rangle$;

(ii) $c(B)\le2,\exp(B)=3,B'\le\langle a^3\rangle$, $[B,a]=1$ and $[B,b]\le\langle a^3\rangle$.

3. Main results

Hereinafter we always assume that G is a finite p-group and $p>2$. By Lemmas 2.3–2.5, it is easy to see that the following Lemma 3.1 holds:

Lemma 3.1 *Let G be a non-abelian J-group of prime power order. If $d(G)=2$ and $|G|\geq p^5$, then G is isomorphic to $M_p(n_1,m_1)$ or $M_p(n,m,1)$, where $n_1\geq 2$, $n_1+m_1\geq 5$ and $n+m\geq 4$.*

Lemma 3.2 *Let G be a non-abelian JC-group. If $c(G)=2$, then G is a J-group.*

Proof If G is not a J-group, then there exists an element x such that $|\langle x\rangle^G:\langle x\rangle|>p$ and $|G:\langle x\rangle^G|=p$. Now we can assume that $G=\langle\langle x\rangle^G,a\rangle$. It follows from $c(G)=2$ that $\langle x\rangle^G=\langle x\rangle\langle[x,a]\rangle$ is abelian. Hence $[a^p,x]=1$. On the other hand, $|\langle x\rangle^G:\langle x\rangle|=|\langle x\rangle\langle[x,a]\rangle:\langle x\rangle|=|\langle[x,a]\rangle:\langle[x,a]\rangle\cap\langle x\rangle|>p$. Then $[x,a]>p$ and therefore $[a^p,x]=[a,x]^p\neq 1$, a contradiction. □

Lemma 3.3 *Let G be a non-abelian JC-group. If G' is cyclic, then $|G'|\leq p^2$.*

Proof Since G' is cyclic, G is regular and then there exist elements a and b such that $[a,b]=c$, $\langle a\rangle\cap\langle c\rangle=1$ and $G'=\langle c\rangle$. If $o(a)=p$, then, it follows from

$$1=[a^p,b]=[a,b]^p[a,b,a]^{\binom{p}{2}}[a,b,a,a]^{\binom{p}{3}}\cdots=c^{p(1+kp)}$$

that $c^p=1$. If $o(a)\geq p^2$ and $o(c)>p^2$, then $\langle a^p\rangle$ is not normal in G and $\langle a^p\rangle^G=\langle a^p,[a^p,G]\rangle\leq\langle a^p,c^p\rangle=\langle a^p\rangle\Phi(G')$. Thus $|\langle a^p\rangle^G:\langle a^p\rangle|>p$ and therefore $|G:\langle a^p\rangle^G|=p$. Hence $G'=G'\cap\langle a^p\rangle^G\leq(G'\cap\langle a^p\rangle)\Phi(G')$, and so $G'\leq\langle a^p\rangle$, in contradiction to the fact that $\langle a^p\rangle$ is not normal in G. Therefore $|G'|\leq p^2$. □

Lemma 3.4 *Let G be a non-abelian JC-group. If G is not a J-group and G' is cyclic, then G is $\langle a,b\mid a^{p^3}=b^{p^m}=1,[b,a]=a^p\rangle$, where $2\leq m\leq 3$.*

Proof It is easy to see $d(G)=2$ and $|G'|=p^2$. Since G' is cyclic, G is regular. Assume that $G=L_0>L_1>\cdots>\mho_1(G)$ is a L-series. It follows from $d(G)=2$ that $\Phi(G)=L_2$. Take $a\in L_0-L_1, b\in L_1-L_2$ of minimal order, then, by Lemma 2.2, $G=\langle a,b\rangle$ and $\langle a\rangle\cap\langle b\rangle=1$. Also we see $|G/\mho_1(G)|\leq p^4$, $w(G)=\log_p(|G/\mho_1(G)|)\leq 4$.

If $w(G)=2$, then G is meta-cyclic by [10, Theorem 2.4.4]. We may assume that $G=\langle a,b\mid a^{p^n}=b^{p^m}=1,[a,b]=a^{p^{n-2}}\rangle$. Since $|G:\langle b\rangle^G|=|G:\langle b,a^{p^{n-2}}\rangle|=p^2$, we see $|\langle b,a^{p^{n-2}}\rangle:\langle b\rangle|=p$ and therefore $n=3$. If $m>3$, then $\langle ab^{p^{m-3}}\rangle$ is of order p^3 and its normal closure is of order p^5. By the hypothesis, $|G|\leq p^6$ and $m\leq 3$. Thus G is the group in theorem.

If $w(G)\neq 2$, then $[a,b]=c, c^{p^2}=1$ and $[c,a],[c,b]\in\langle c^p\rangle$. Noticing that $\langle b\rangle$ has index p^2 in its normal closure, we see $|G:\langle b\rangle^G|=p$ if $G'\cap\mho_1(G)=1$. Thus $a^p=1$ and $1=[a^p,b]=c^{p(1+kp)}\neq 1$, a contradiction. If $G'\cap\mho_1(G)\neq 1$, then we may assume that $G=\langle a,b\mid a^{p^n}=b^{p^m}=1,[a,b]=c,c^p=a^{p^{n-1}},[c,a],[c,b]\in\langle c^p\rangle\rangle$. Since $|\langle b\rangle^G:\langle b\rangle|=p^2$, $|G:\langle b\rangle^G|=p$ and then $a^{p^2}=1$. Now we see that ac^{-1} is of order p and $[(ac^{-1})^p,b]=1$, however, $\langle[(ac^{-1})^p,b]\rangle=\langle c^p\rangle\neq 1$, a contradiction. □

Lemma 3.5 *There exists no JC-group G which satisfies the following properties:*

(1) $|G|\geq p^6$, *where* $p>2$;

(2) G/N is isomorphic to $M_p(n,m,1)$, where N is of order p and $N \leq G_3 \cap Z(G)$.

Proof If there exists a JC-group G satisfying the above properties, then we assume that G/N is isomorphic to $M_p(n,m,1) = \langle \bar{a}, \bar{b} | \bar{a}^{p^n} = \bar{b}^{p^m} = \bar{c}^p = 1, [\bar{a}, \bar{b}] = \bar{c}, [\bar{c}, \bar{a}] = [\bar{c}, \bar{b}] = 1 \rangle$ and $G = \langle a, b \rangle$, where $N = \langle x \rangle$ is of order p and $N \leq G_3 \cap Z(G)$. We consider following two cases according to $a^{p^n} = 1$ or not.

Case 1 $a^{p^n} = 1$. We see $G' \leq \langle a \rangle^G$ by $G = \langle a, b \rangle$. Then $|\langle a \rangle^G : \langle a \rangle| = |\langle a, G' \rangle : \langle a \rangle| = p^2$ and therefore $|G : \langle a \rangle^G| = p$. Hence $m = 1$. If $b^p = 1$, then $|\langle b \rangle^G : \langle b \rangle| = p^2$. Thus we see $|G : \langle b \rangle^G| = p$ and $|G| = p^4$, which contradicts property (1). If $b^p \neq 1$, then, by $|G| \geq p^6$, we see $n \geq 3$. Now we consider $\langle ba^p \rangle$, which is of order p^{n-1}. It is easy to see $|\langle ba^p \rangle^G| = |\langle ba^p \rangle||G'| = p^{n+1}$ and $|\langle ba^p \rangle^G : \langle ba^p \rangle| = |G : \langle ba^p \rangle^G| = p^2$, a contradiction.

Case 2 $a^{p^n} \neq 1$. If $b^{p^m} = 1$, then we may get contradictions using the same discussions as in the Case 1. So we may assume that $b^{p^m} \neq 1$ and $m \geq n$. Then there exists ab^i of order p^n such that $\langle ab^i \rangle \cap G' = 1$. Hence $|\langle ab^i \rangle^G : \langle ab^i \rangle| = p^2$ and then $|G : \langle ab^i \rangle^G| = p = p^m$. Thus $m = n = 1$ and $|G| = p^4$, a contradiction. □

Theorem 3.6 *Let G be a JC-group. If G is not a J-group and $|G| \geq p^5$, then G is one of the following groups:*

$p \geq 5$

(1) $\langle a, b | a^{p^2} = b^{p^2} = c^p = 1, [b, a] = c, [c, a] = b^{pv} \rangle$, where $v = 1$ or ν, ν is a quadratic non-residue modulo p;

(2) $\langle a, b | a^{p^2} = b^{p^2} = c^p = 1, [b, a] = c, [c, a] = [c, b] = b^p \rangle$;

(3) $\langle a, b | a^{p^3} = b^{p^m} = 1, [b, a] = a^p \rangle$, where $m = 2$ or 3;

$p = 3$

(4) SmallGroup$(3^5, 14)$;

(5) SmallGroup$(3^5, 15)$;

(6) SmallGroup$(3^5, 18)$;

(7) SmallGroup$(3^5, 22)$;

(8) SmallGroup$(3^5, 25)$;

(9) SmallGroup$(3^5, 26)$;

(10) SmallGroup$(3^5, 27)$;

(11) SmallGroup$(3^5, 28)$;

(12) SmallGroup$(3^5, 29)$;

(13) SmallGroup$(3^5, 30)$;

(14) SmallGroup$(3^6, 22)$.

Proof It is easy to see that $d(G) = 2$ and $c(G) > 2$. Firstly, we consider 3-groups. Checking the groups of order $3^5, 3^6$ and 3^7 in the Database of Small Groups of Magma [12], we know that G is isomorphic to one of the groups listed in the theorem. And there is no group of order 3^7 satisfying the hypothesis of theorem. If $|G| \geq 3^8$, then we consider G/N which is also a JC-group, where N is of order p and $N \leq G_3 \cap Z(G)$. Thus by the Lemmas 3.1, 3.4 and 3.5 we see our conclusion in the theorem.

Secondly, we discuss the case that $p > 3$. If $|G| = p^5$, then G is regular. Checking the groups listed in [13, Theorem 5.1], we see G is isomorphic to one of the groups listed in the theorem. If $|G| = p^6$, then we consider G/N, where $N = \langle x\rangle$ is of order p and $N \leq G_3 \cap Z(G)$. If G/N is meta-cyclic, then G is meta-cyclic by [10, Theorem 2.4.1]. Hence G is type (3) group by Lemma 3.4. If G/N is not meta-cyclic, then we see that G/N is not a J-group by Lemmas 3.1 and 3.5. Then G/N is the type (1) or type (2) group in the theorem. Now we proceed in the following two cases:

Case 1 $G/N = \langle \bar{a}, \bar{b} | \bar{a}^{p^2} = \bar{b}^{p^2} = 1, [\bar{b}, \bar{a}] = \bar{c}, [\bar{c}, \bar{a}] = \bar{b}^{pv}, [\bar{c}, \bar{b}] = 1\rangle$ and $G = \langle a, b\rangle$. It is easy to see $\langle b\rangle^G = \langle b, G'\rangle$ is of order p^4. If $b^{p^2} = 1$, then $|G : \langle b\rangle^G| = |\langle b\rangle^G : \langle b\rangle| = p^2$, a contradiction. If $b^{p^2} \neq 1$ and $c^p = 1$, then $\langle c\rangle^G = \langle c, [c, a]\rangle = \langle c, b^p\rangle$ is of order p^3, a contradiction. If $b^{p^2} \neq 1$ and $c^p \neq 1$, then there exists $\langle b^{pi}c^{-1}\rangle$ of order p. Since $\langle b^{pi}c^{-1}\rangle^G = \langle b^{pi}c^{-1}, [b^{pi}c^{-1}, a]\rangle = \langle b^{pi}c^{-1}, b^p\rangle = G'$, $|\langle b^{pi}c^{-1}\rangle^G| = p^3$, a contradiction.

Case 2 If G/N is type (2) group, then by the same arguments in Case 1, we may see the contradictions.

If $|G| \geq p^7$, then we consider G/N, where N is of order p and $N \leq G_3 \cap Z(G)$. Then by the Lemmas 3.1, 3.4 and 3.5 we see there is no group satisfying the conditions of theorem. The proof is completed. $\Box$

Acknowledgements The authors would like to thank referees for their valuable suggestions and useful comments contributed to this paper.

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