Journal of Mathematical Research with Applications
Mar., 2017, Vol. 37, No. 2, pp. 214–222
DOI:10.3770/j.issn:2095-2651.2017.02.010
Http://jmre.dlut.edu.cn

On the Growth Properties of Solutions for a Generalized Bi-Axially Symmetric Schrödinger Equation

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Abstract In this paper, we have considered the generalized bi-axially symmetric Schrödinger equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{2\nu}{x}\frac{\partial\varphi}{\partial x}+\frac{2\mu}{y}\frac{\partial\varphi}{\partial y}+\{K^2-V(r)\}\varphi=0,$$

where $\mu,\nu\geq 0$, and $rV(r)$ is an entire function of $r=+(x^2+y^2)^{1/2}$ corresponding to a scattering potential $V(r)$. Growth parameters of entire function solutions in terms of their expansion coefficients, which are analogous to the formulas for order and type occurring in classical function theory, have been obtained. Our results are applicable for the scattering of particles in quantum mechanics.

Keywords Schroödinger equation; scattering potential; Jacobi polynomials; order and type

MR(2010) Subject Classification 30E10; 30E15

1. Introduction

In this paper we shall study the growth properties of solutions of the partial differential equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{2\nu}{x}\frac{\partial\varphi}{\partial x}+\frac{2\mu}{y}\frac{\partial\varphi}{\partial y}+\{K^2-V(r)\}\varphi=0,\tag{1.1}$$

where $\mu,\nu\geq 0$ and $rV(r)$ is an entire function of $r=+(x^2+y^2)^{1/2}$.

For the purposes of motivation of the work, we should like to mention, that this study has a bearing on the study of the scattering of particles in quantum mechanics. For the case, when $x=x_1, y=(x_2^2+x_3^2)^{1/2}$ and $\mu=1,\nu=0$, (1.1) becomes the ordinary axially symmetric, Schrödinger equation corresponding to a scattering potential $V(r)$. Solutions to this equation, which satisfy a suitable radiation condition, correspond to scattered waves and their singularities are related to the quantum states of the scattered particles.

Received October 30, 2015; Accepted June 8, 2016
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Our results are the extension of the work done by Gilbert and Howard [1] and Kumar and Singh [2] for the equation of generalized axially symmetric helmholtz equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{2\mu}{y}\frac{\partial\varphi}{\partial y}+K^2\varphi=0,\quad \mu>0. \tag{1.2}$$

Gilbert [3–7] considered the partial differential equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{2\mu}{y}\frac{\partial\varphi}{\partial y}=0 \tag{1.3}$$

and studied the various properties of solutions of (1.3). The function theoretic approaches for (1.3) have been used by Erdélyi [8], Henrici [9–12] and Mackie [13] (Heins and MacCamy [14,15] discussed this equation for $\mu=\frac{1}{2}$).

Gilbert and Howard [1] have also obtained results for the generalized, bi-axially symmetric helmholtz equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{2\nu}{x}\frac{\partial\varphi}{\partial x}+\frac{2\mu}{y}\frac{\partial\varphi}{\partial y}+K^2\varphi=0, \tag{1.4}$$

which were extensions of the results studied by Gilbert [6] for the generated, bi-axially symmetric potential equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{2\nu}{x}\frac{\partial\varphi}{\partial x}+\frac{2\mu}{y}\frac{\partial\varphi}{\partial y}=0. \tag{1.5}$$

Equations (1.4) and (1.5) have also been investigated by Henrici [9–12] using function-theoretic methods. Fryant [16] studied equation (1.5) by direct method.

Recently, the author [17] studied the growth estimates for entire function solutions of the equation (1.4), in terms of their Jacobi-Bessel coefficients and ratio of these coefficients. Some results also have been obtained for order and type (analogous to the formulas in classical function theory) in terms of Taylor and Neumann coefficients.

These results represent an extension of the results obtained by Gilbert and Howard [1,18], McCoy [19] and Kumar [20].

The Euler-Poisson-Darboux equation, arising in gas dynamics, is viewed in terms of equation (1.1) for $K^2-V(r)=0$ after a transformation [21, p.223] and has a variety of physical interpretations. Although, bi-axially symmetric potential theory is a well developed subject with many applications to the physical sciences it is, perhaps, not fully appreciated that certain biological problems suggest the use of this theory. The problem of steady-state differential flow through a cylindrical structure arises frequently. Not surprisingly, the physiological situation may provide motivation for solving problems and seeking techniques that are different from those arising from purely mathematical or physical considerations. Also, these potentials play an important role in many aspects of mathematical physics, in particular to an understanding of compressible flow in the transonic region. In this paper our study is applicable for the scattering of particles in quantum mechanics.

The transformation of equation (1.1) in polar coordinates is, namely,

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{(2[\mu+\nu]+1)}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\Big\{\frac{2\nu\cot\theta}{r^2}-\frac{2\mu\tan\theta}{r^2}\Big\}\frac{\partial\varphi}{\partial\theta}+\{K^2-V(r)\}\varphi=0. \tag{1.6}$$

In order to find the solutions of (1.6) by separation of variables, we consider the solutions of the form $\varphi(r,\theta) = R(r)\Theta(\theta)$ where $R(r)$ and $\Theta(\theta)$ must satisfy, respectively, the ordinary differential equations

$$r^2\frac{\partial^2 R}{\partial r^2} + r(2[\mu+\nu]+1)\frac{\partial R}{\partial r} + \{[K^2 - V(r)]r^2 - \Delta\}R = 0 \tag{1.7}$$

and

$$\frac{\partial^2 \Theta}{\partial \theta^2} + [2\nu\cot\theta - 2\mu\tan\theta]\frac{\partial \Theta}{\partial \theta} + \Delta\Theta = 0. \tag{1.8}$$

If we introduce the new independent variable $\xi = \cos 2\theta$ in (1.8) and set $X(\xi) = \Theta(\theta)$, then (1.8) becomes

$$(1-\xi^2)\frac{\partial^2 X}{\partial \xi^2} + [\mu - \nu - \xi(1+\mu+\nu)]\frac{\partial X}{\partial \xi} + \frac{\Delta}{4}X = 0. \tag{1.9}$$

Corresponding to the separation constant $\Delta = 4n(n+\mu+\nu)$, the regular solution of (1.9) is $X(\xi) = P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)$, which has the value $\binom{n+\mu-\frac{1}{2}}{n}$ at $\xi = 1$, here $P_n^{(\alpha,\beta)}(\xi)$ are Jacobi polynomials which may be defined uniquely by their generating function expansion

$$\sum_{n=0}^{\infty} \Im^n P_n^{(\alpha,\beta)}(\xi) = 2^{\alpha+\beta}T^{-1}(1-\Im+T)^{-\alpha}(1+\Im+T)^{-\beta}, \tag{1.10}$$

$|\Im| < 1$, $Tr(1-2\xi\Im+\Im^2)^{1/2} = 1$ when $\Im = 0$ (see [22, Vol.II, p.169]).

In this case equation (1.7) leads us to consider the function $R_n(r)$:

$$r^2\frac{\partial^2 R_n}{\partial r^2} + r(2[\mu+\nu]+1)\frac{\partial R_n}{\partial r} + \{r^2[K^2 - V(r)] - 4n(n+\mu+\nu)\}R_n = 0. \tag{1.11}$$

Set $W_n(r) = r^{\mu+\nu}R_n(r)$, in above equation we obtain

$$r^2\frac{\partial^2 W_n(r)}{\partial r^2} + r\frac{\partial W_n(r)}{\partial r} + \{r^2[K^2 + V(r)] - (2n+\mu+\nu)^2\}W_n(r) = 0. \tag{1.12}$$

For the case $V(r) \equiv 0$, this equation is a Bessel's equation. If $rV(r)$ is not equivalent to 0, it is an entire function of the form

$$V(r) = \frac{\upsilon_0}{r} + \sum_{\nu=1}^{\infty} \upsilon_\nu r^{\nu-1}, \quad \text{with} \quad \upsilon_0 > 0, \tag{1.13}$$

then the indicial equation for (1.12) is the same as in the case of Bessel's equation, namely,

$$F(\alpha) \equiv \alpha(\alpha-1) + \alpha - (2n+\mu+\nu)^2 = 0, \quad \alpha = \pm(2n+\mu+\nu).$$

If $2\mu + 2\nu \neq$ an integer, then the two independent solutions of (1.12) are of the form

$$W_{n,1}(r) = r^{+(2n+\mu+\nu)}\sum_{m=0}^{\infty} a_m r^m, \quad W_{n,2}(r) = r^{-(2n+\mu+\nu)}\sum_{m=0}^{\infty} b_m r^m,$$

where the coefficients $C_m = \{a_m, b_m\}$ may be computed by the relations

$$C_m = \frac{1}{F(\alpha+m)}\Big\{\sum_{p=0}^{m-1} C_p\widetilde{\upsilon}_{m-p}\Big\}, \ \widetilde{\upsilon}_m = \upsilon_m \ (m \neq 1), \ \widetilde{\upsilon}_1 = -K^2 + \upsilon_1,$$

$\alpha = +(2n+\mu+\nu), -(2n+\mu+\nu)$ when $C_m = a_m, b_m$, respectively. The regular and irregular solutions of (1.11) about $r = 0$ are locally $R_{n,1} \approx r^{2n}, R_{n,2} \approx r^{-2(n+\mu+\nu)}$; furthermore, if $rV(r)$

is entire then $R_{n,1}(r)$ is also entire [23]. The local behavior of $R_{n,1}(r)$ and $R_{n,2}(r)$, implies that as $n \to \infty$

$$R_{n,1}(r) \approx R_n^{(1)}(r) \approx g_n r^{2n}. \tag{1.14}$$

Let us rewrite equation (1.1) in the form

$$\frac{\partial}{\partial x}(x^{2\nu}y^{2\mu}\frac{\partial\varphi}{\partial x}) + \frac{\partial}{\partial y}(x^{2\nu}y^{2\mu}\frac{\partial\varphi}{\partial y}) - x^{2\nu}y^{2\mu}(V(r) - K^2)\varphi = 0. \tag{1.15}$$

Then, if either μ or $\nu > 1$, the coefficient $P(x, y) \equiv x^{2\nu}y^{2\mu}(V(r) - K^2)$ of φ is continuous in a neighborhood of $x = y = 0$ in R_2. If $V(r)$ has the form (1.13) with $v_0 > 0$, then $P(x, y) \geq 0$ in the first quadrant of a sufficiently small disk about the origin, if r_0 is the smallest zero of $V(r_0) = K^2$, then our quarter-circle is given to be

$$Q_c(r_0) \equiv D(r_0) \cap \{(x, y)|x \geq 0, y \geq 0\},$$

where

$$D(r_0) \equiv \{(x, y)|x^2 + y^2 \leq r_0^2\}.$$

Using uniqueness theorem for elliptic partial differential equation with continuous coefficients (Courant-Hilbert [24, Vol.II, p.321]) in this case, we see that there exists at most one solution which is twice continuously differentiable in $Q_c(r_0)$ and takes on prescribed values on $\partial Q_c(r_0)$. Since the family of functions $\{R_{n,1}(r)\ P_n^{(\mu-1/2,\nu-1/2)}(\xi)\}_{n=0}^{\infty}$ are symmetric with respect to the origin, and reflections through the coordinate axes, the uniqueness theorem may be formulated to hold for the region $D(r_0)$ with boundary data prescribed on $\partial D(r_0)$ if we also require $\frac{\partial\varphi}{\partial x} = 0$ on $x = 0$, and $\frac{\partial\varphi}{\partial y} = 0$ on $y = 0$.

It should be mentioned here that we are interested in studying the growth of solutions $\varphi(r, \theta)$ in the region $r > r_0$, in the case of scattering.

The decomposition of $\varphi(r, \theta)$ into an everywhere-regular solution and a solution satisfying the Sommerfield radiation condition

$$\lim_{r\to\infty} \{r^{\mu+\nu+1/2}(\frac{\partial\varphi}{\partial r} - ik\varphi)\} \to 0 \tag{1.16}$$

is unique [25, p.107]. Moreover, our study of the growth properties of the class $\mathsf{C}^{(2)}[D]$ has bearing on the study of the growth properties of the scattered solutions satisfying the condition (1.16).

2. Auxiliary results

Theorem 2.1 *The series*

$$\varphi(r, \theta) = \sum_{n=0}^{\infty} a_n R_n(r) P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\cos 2\theta) \tag{2.1}$$

converges absolutely and uniformly on compact subsets of the open disk of the convergence $|z| < \rho$, where

$$\frac{1}{\rho} = \limsup_{n\to\infty} |a_n|^{\frac{1}{2n}}. \tag{2.2}$$

Further, such convergence cannot be obtained on any longer disk.

Proof Let ρ_0 denote the radius of the largest disk centered at the origin in which the series (2.1) will converge uniformly on compact subsets. Recalling the orthogonality relation for the Jacobi polynomials [22]

$$\int_{-1}^{+1}(1-\xi)^{\nu-\frac{1}{2}}(1+\xi)^{\mu-\frac{1}{2}}P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\xi)P_m^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\xi)\mathrm{d}\xi$$
$$=\delta_{nm}\frac{2^{\mu+\nu}\Gamma(n+\nu+\frac{1}{2})\Gamma(n+\mu+\frac{1}{2})}{(2n+\mu+\nu)n!\Gamma(n+\mu+\nu)},$$

and the bound $|P_n^{(\alpha,\beta)}(\xi)|\le\binom{n+q-1}{n}$, where $q=\max(\alpha,\beta)$ (see [22, Vol.2, p.206]), in the relation [23]

$$f(\Im)=\int_{\vartheta-1}^{+1}\Big[\frac{(1-\xi)^{\nu-\frac{1}{2}}(1+\xi)^{\mu-\frac{1}{2}}}{2^{\mu+\nu}}\sum_{n=0}^{\infty}\frac{(2n+\mu+\nu)n!\Gamma(n+\mu+\nu)}{\Gamma(n+\nu+\frac{1}{2})\Gamma(n+\mu+\frac{1}{2})}$$
$$\frac{\Im^n}{R_n(r)}P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\xi)\Big]\varphi(r,\xi)\mathrm{d}\xi,$$

where $\vartheta\equiv\{\xi|-1\le\xi\le+1|\}$, $f(\Im)=\sum_{n=0}^{\infty}a_n\Im^n$ and $\frac{\Im^n}{R_n(r)}\approx\Im^n r^{-2n}$, then by termwise integration we get

$$\frac{2^{\mu+\nu}\Gamma(n+\nu+\frac{1}{2})\Gamma(n+\mu+\frac{1}{2})}{(2n+\mu+\nu)\Gamma(n+1)\Gamma(n+\mu+\nu)}a_nR_n(r)$$
$$=\int_{-1}^{1}(1-\xi)^{\nu-\frac{1}{2}}(1+\xi)^{\mu+\frac{1}{2}}P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\xi)\varphi(r,\varphi)\mathrm{d}\xi$$
$$=2\int_{0}^{\pi/2}(1-\cos2\theta)^{\nu-\frac{1}{2}}(1+\cos2\theta)^{\mu+\frac{1}{2}}P_n^{(\nu-\frac{1}{2},\mu-\frac{1}{2})}(\cos2\theta)\varphi(r,\varphi)\sin2\theta\mathrm{d}\theta.$$

Thus, using the Schwartz inequality yields

$$|a_n|\le\Big[\frac{(2n+\mu+\nu)\Gamma(\nu+\frac{1}{2})\Gamma(\mu+\frac{1}{2})\Gamma(n+1)\Gamma(n+\mu+\nu)}{\Gamma(\mu+\nu+1)\Gamma(n+\nu+\frac{1}{2})\Gamma(n+\mu+\frac{1}{2})}\Big]^{1/2}\cdot\frac{M(r,\varphi)}{r^{2n}},\tag{2.3}$$

where $M(r,\varphi)=\max_{x^2+y^2=\rho^2}|\varphi(x,y)|, r^2\le\rho(\frac{m-1}{m})$.

Now for all sufficiently large n, we have

$$\frac{\Gamma(n+1)\Gamma(n+\mu+\nu)}{\Gamma(n+\nu+\frac{1}{2})}\approx\big(\frac{n+\mu}{n}\big)^{\nu-\frac{1}{2}}\approx1,$$

this yields $\limsup_{n\to\infty}|a_n|^{1/2n}\le\frac{1}{r}$, and since the choice of $r<\rho_0$ was arbitrary,

$$\limsup_{n\to\infty}|a_n|^{\frac{1}{2n}}\le\frac{1}{\rho_0}.$$

On the other hand, suppose

$$\limsup_{n\to\infty}|a_n|^{\frac{1}{2n}}=\frac{1}{\rho},\tag{2.4}$$

then the series (2.1) is dominated by

$$\sum_{n=0}^{\infty}|a_n||R_n(r)|\binom{n+q-1}{n}$$

Thus (2.4) implies the series (2.1) converges absolutely and uniformly on compact subsets of the disk centered at the origin of radius ρ and hence the theorem follows.

In particular, the solution $\varphi(r,\theta)$ is entire if and only if its series representation (2.1) has an infinite radius if convergence, if and only if

$$\rho = \big(\limsup_{n\to\infty} |a_n|^{1/2n}\big)^{-1} \to \infty.$$

Now, we introduce the idea of the growth parameters order ρ^* and type T^* of an entire function solution $\varphi(r,\theta)$ following the usual function theoretic definitions [26, p.3]:

$$\rho^* = \limsup_{r\to\infty} \frac{\log\log M(r,\varphi)}{\log \rho} \tag{2.5}$$

and

$$T^* = \limsup_{r\to\infty} \frac{\log M(r,\varphi)}{\rho^{\rho^*}} \tag{2.6}$$

3. Main results

In this section we characterize ρ^* and T^* in terms of coefficients $\{a_n\}$ occurring in series development (2.1) of $\varphi(r,\theta)$.

Theorem 3.1 *Let $\varphi(r,\theta)$ be a solution of (1.1) with a series development (2.1). Furthermore, let $\varphi(r,\theta)$ be an entire function solution of order ρ^*. Then*

$$\rho^*(\varphi) = \limsup_{n\to\infty} \frac{n\log n}{-\log|a_n|}, \tag{3.1}$$

where $\{a_n\}$ are the coefficients occurring in series development (2.1) of $\varphi(r,\theta)$.

Proof We have [23, p.69]:

$$\varphi(r,\theta) = \frac{1}{2\pi i}\int_{\vartheta} f(\Im)\Big(\sum_{n=0}^{\infty} \Im^{-n} R_n(r) P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)\Big)\frac{d\Im}{\Im} \tag{3.2}$$

where $|r^2\Im|<1, f(\Im)=\sum_{n=0}^{\infty} a_n\Im^n$ and $\vartheta \equiv \{|\Im|| \Im| = (r+\varepsilon)^2, \varepsilon>0$ arbitrary small$\}$.

Now first we estimate

$$\begin{aligned}&\sum_{n=0}^{\infty} \Im^{-n} R_n(r) P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)\\ &= \sum_{n=0}^{N} \Im^{-n}(R_n(r)-r^{2n}) P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\xi) + \big(1+0(\frac{1}{N})\big)\\ &\quad \big\{2^{\mu+\nu-1}T^{-1}\big(1-\frac{r^2}{\Im}+T\big)^{-\mu+\frac{1}{2}}\big(1+\frac{r^2}{\Im}+T\big)^{-\nu+\frac{1}{2}}\big\},\end{aligned}$$

for n sufficiently large, where $T=(1-2\xi(r^2/\Im)+r^4/\Im^2)^{1/2}$, and $T=1$ when $\frac{r^2}{\Im}=0$.

Suppose, since the $R_n(r)$ are entire functions of complex r, that

$$M_1(r,\varepsilon') = \max_{\substack{|\Im|=r^2+\varepsilon'\\ -1\le\xi\le+1}} \Big\{\Big|\sum_{n=0}^{\infty} \Im^{-n} R_n(r) P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)\Big|\Big\},$$

and

$$M_2(r,\varepsilon') = \max_{\substack{|\Im|=r^2+\varepsilon'\\ -1\le\xi\le+1}} \Big\{\frac{1}{|T|}\Big\},$$

then

$$\Big|\sum_{n=0}^{\infty}\Im^{-n}R_n(r)P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)\Big| \le M_1(r,\varepsilon') + 2^{\mu+\nu}(M_2(r,\varepsilon'))^{\mu+\nu}, \frac{r^2}{|\Im|} < 1.$$

We see that $M_1(r,\varepsilon') < \infty$ for $N < \infty$, $|\Im| = r^2 + \varepsilon'$, $-1 \le \xi \le +1$.

In order to estimate $M_2(r,\varepsilon')$ we set $\Im = (r^2+\varepsilon')e^{-i\theta}$ in T and consider the extrema of $|T|$. We conclude that $|T|$ assumes its minimum over the domain $0 \le \theta \le 2\pi, -1 \le \xi \le +1$ for $\cos\theta = 1$ and for $\xi = \frac{r^2}{(r^2+\varepsilon')}$. Hence, the minimum value of T is greater than $(\varepsilon')^2(r^2+\varepsilon')^{-2}$. Consequently, for ε' sufficiently small, we have

$$\begin{aligned}\Big|\sum_{n=0}^{\infty}\Im^{-n}R_n(r)P_n^{(\mu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)\Big| &\le M_1(r,\varepsilon') + [2(\varepsilon')^{-2}(r^2+\varepsilon')^2]^{\mu+\nu}\\ &\le 2M_1(r,0) + [2(\varepsilon')^{-2}(r^2+\varepsilon')^2]^{\mu+\nu}. \qquad (3.3)\end{aligned}$$

Using (3.3) in (3.2), we get

$$|\varphi(r,\theta)| \le \{2M_1(r,0) + [2(\varepsilon')^{-2}(r^2+\varepsilon')^2]^{\mu+\nu}\}|f(\Im)|, \text{ for } |\Im| \le r_0^2 + \varepsilon'. \qquad (3.4)$$

Recalling that associate of $\varphi(r,\theta)$ must also be entire and has the same coefficients $\{a_n\}$. The formula expressing the order of an entire function of a single complex variable in terms of its Taylor coefficients [26, p.4] yields

$$\rho^*(\varphi) \le \limsup_{n\to\infty}\frac{n\log n}{\log|a_n|^{-1}} = \rho^*(f). \qquad (3.5)$$

In view of (2.3) and the definition (2.5) of order, we have for sufficiently large ρ and finite r

$$|a_n| \le \Big[\frac{(2n+\mu+\nu)\Gamma(\nu+\frac{1}{2})\Gamma(\mu+\frac{1}{2})\Gamma(n+1)\Gamma(n+\mu+\nu)}{\Gamma(\mu|\nu+1)\Gamma(n+\nu+\frac{1}{2})\Gamma(n+\mu+\frac{1}{2})}\Big]^{1/2}\frac{\exp(\rho^{(\rho^*+\varepsilon)})}{r^{2n}}.$$

Since $r^2 \le \rho[\frac{(m-1)}{m}]$, the minimum value of $r^{-2n}\exp(\rho^{(\rho^*+\varepsilon)})$ is attained when

$$\rho = \Big(\frac{n}{\rho^*(\varphi)+\varepsilon}\Big)^{\frac{1}{(\rho^*(\varphi)+\varepsilon)}}.$$

Thus, for n sufficiently large,

$$\begin{aligned}|a_n| \le &\Big[\frac{(2n+\mu+\nu)\Gamma(\nu+\frac{1}{2})\Gamma(\mu+\frac{1}{2})\Gamma(n+1)\Gamma(n+\mu+\nu)}{\Gamma(\mu+\nu+1)\Gamma(n+\nu+\frac{1}{2})\Gamma(n+\mu+\frac{1}{2})}\Big]^{1/2}\times\\ &\Big[\frac{(\rho^*(\varphi)+\varepsilon)e}{n(\frac{m-1}{m})}\Big]^{\frac{n}{(\rho^*(\varphi)+\varepsilon)}},\end{aligned}$$

now one easily computes

$$\rho^*(\varphi)+\varepsilon \ge \frac{\log n}{\log|a_n|^{-1/n}} = \rho^*(f). \qquad (3.6)$$

Combining (3.5) and (3.6), the proof is completed. □

Theorem 3.2 *Let $\varphi(r,\theta)$ be a solution of (1.1) with a series development (2.1). Furthermore, let $\varphi(r,\theta)$ be an entire function solution of order ρ^* and type T^*. Then*

$$T^*(\varphi) = \frac{1}{e\rho^*}\limsup_{n\to\infty} n|a_n|^{\rho^*/n},$$

where $\{a_n\}$ are the coefficients occurring in series development (2.1) of $\varphi(r,\theta)$.

Proof By the result of Theorem 3.1 and the expression of the order of an entire function in terms of its Taylor coefficients, it follows that the order of $\varphi(r,\theta)$ equals the order of $f(\Im)$. Thus inequality (3.4) implies $T^*(\varphi) \le T^*(f)$. Using the expression giving for the type of an entire function in terms of its Taylor coefficients [26, p.4], we have

$$T^*(\varphi) \le T^*(f) = \frac{1}{e\rho^*(f)} \limsup n|a_n|^{\frac{\rho^*(f)}{n}}. \tag{3.7}$$

Using (2.3) and the definition (2.6) of type, we have for sufficiently large ρ and finite r

$$|a_n| \le \Big[\frac{(2n+\mu+\nu)\Gamma(\nu+\frac{1}{2})\Gamma(\mu+\frac{1}{2})\Gamma(n+1)\Gamma(n+\mu+\nu)}{\Gamma(\mu+\nu+1)\Gamma(n+\nu+\frac{1}{2})\Gamma(n+\mu+\frac{1}{2})}\Big]^{1/2}\times \frac{\exp[(T^*(\varphi)+\varepsilon)\rho^{\rho^*(\varphi)}]}{r^{2n}}.$$

The minimum value of $r^{-2n}\exp[(T^*+\varepsilon)\rho^{\rho^*(\varphi)}]$ attains at

$$\rho = \Big[\frac{n}{(T^*(\varphi)+\varepsilon)\rho^*(\varphi)}\Big]^{\frac{1}{\rho^*(\varphi)}}.$$

For sufficiently large n,

$$|a_n| \le \Big[\frac{(2n+\mu+\nu)\Gamma(\nu+\frac{1}{2})\Gamma(\mu+\frac{1}{2})\Gamma(n+1)\Gamma(n+\mu+\nu)}{\Gamma(\mu+\nu+1)\Gamma(n+\nu+\frac{1}{2})\Gamma(n+\mu+\frac{1}{2})}\Big]^{1/2}\times \Big[\frac{(T^*(\varphi)+\varepsilon)e\rho^*(\varphi)}{n(\frac{m-1}{m})}\Big]^{\frac{n}{\rho^*(\varphi)}},$$

it gives

$$\frac{1}{e\rho^*(\varphi)} \limsup_{n\to\infty} n|a_n|^{\frac{\rho^*(\varphi)}{n}} \le T^*(\varphi). \tag{3.8}$$

Since $\rho^*(f) = \rho^*(\varphi)$, (3.7) and (3.8) together completes the proof. □

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