Journal of Mathematical Research with Applications
Mar., 2017, Vol. 37, No. 2, pp. 223–232
DOI:10.3770/j.issn:2095-2651.2017.02.011
Http://jmre.dlut.edu.cn

Multiple Positive Solutions for Multi-Point Boundary Value Problem of Fractional Differential Equation

Xiping LIU[1], Mei JIA[1,*], Ming NIU[2], Xiufen XIANG[3]

1. *College of Science, University of Shanghai for Science and Technology, Shanghai* 200093*, P. R. China;*
2. *Department of Mechanics and Electronics, Shijiazhuang Vocational Technology Institute, Hebei* 050081*, P. R. China;*
3. *Department of Mathematics and Physics, Chengde Petroleum College, Hebei* 067000*, P. R. China*

Abstract In this paper, we study the multiplicity of positive solutions for multi-point boundary value problem of Riemann-Liouville fractional differential equation with multi-terms fractional derivative in the boundary conditions. By using the properties of the Green function and a generalization of the Leggett-Williams fixed point theorem due to the work of Bai and Ge, the sufficient conditions to guarantee the existence of at least three positive solutions are established. In the end of this paper, we have also given out the example to illustrate the wide range of potential application of our main results.

Keywords fractional differential equation; multi-point boundary value problem; positive solution; Green function; fixed point theorem

MR(2010) Subject Classification 34B10; 34B18; 26A33

1. Introduction

In this paper, we consider the following m-point boundary value problem of fractional differential equation

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t),D^{\beta}u(t))=0,\ t\in(0,1),\\ u(0)=u'(0)=0,\\ D^{\beta}u(1)=\sum\limits_{i=1}^{m-2}\lambda_i D^{\eta_i}u(t_i), \end{cases} \tag{1.1}$$

where D^{α}, D^{β} and D^{η_i} are the Riemann-Liouville fractional derivative operators of order α, β and η_i, respectively, $1<\beta<2<\alpha<3$ with $\alpha-\beta>1$, $0\le\eta_i\le1$, $\lambda_i>0$, $0<t_i<1$, $i=1,2,\dots,m-2$, $f\in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$.

In the past decades, the fractional differential equation theory has gained considerable popularity and importance due to its demonstrated applications in numerous widespread fields of

Received April 25, 2016; Accepted November 2, 2016
Supported by the National Natural Science Foundation of China (Grant No. 11171220) and the Hujiang Foundation of China (Grant No. B14005).
* Corresponding author
E-mail address: xipingliu@usst.edu.cn (Xiping LIU); jiamei@usst.edu.cn (Mei JIA)

science and engineering [1–4]. Driven by the wide range of the applications, the boundary value problems for fractional order differential equations have been studied by more and more researchers [5–15].

On the other hand, although the boundary value problems of fractional differential equations have been investigated by many authors, the multi-point boundary value problem for fractional differential equation involving the fractional derivative in both the nonlinearities and boundary conditions are seldom studied and only limited research papers have been considered [16–18].

The purpose of this paper is to establish the multiplicity of positive solutions of boundary value problem (1.1). Our paper is organized as follows. In Section 2, we give out some basic definitions and lemmas to prove our main results. In Section 3, by using a generalization of the Leggett-Williams fixed point theorem due to the work of Bai and Ge, we establish the existence of at least three positive solutions of boundary value problem (1.1). In Section 4, as applications, some examples are presented to illustrate our main results. Finally, in Section 5, we give out the conclusion of this paper.

2. Preliminary

For the convenience of reading, in this section, we provide the background knowledge on the fractional calculus and fractional differential equations.

Definition 2.1 ([1,2]) *The standard Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : [0, +\infty) \to \mathbb{R}$ is given by*

$$D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{\mathrm{d}}{\mathrm{d}t})^{n}\int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s,$$

where n is an integer with $n-1 < \alpha < n$, provided the right integral converges. And the Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\mathrm{d}s,$$

provided the right integral converges.

Lemma 2.2 ([1,2]) *Suppose $\alpha > 0$, $u \in C(0,1) \cap L[0,1]$ and $D^{\alpha}u \in C(0,1) \cap L[0,1]$. Then*

$$I^{\alpha}D^{\alpha}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n},$$

where $c_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$, n is the smallest integer greater than or equal to α.

Denote by χ_{J_i} the characteristic function of the set $J_i = [0, t_i]$, for $i = 1, 2, \ldots, m-2$, that is

$$\chi_{J_i}(s) = \begin{cases} 1, & s \in J_i, \\ 0, & s \notin J_i, \end{cases}$$

and denote

$$\gamma = \Gamma(\alpha-\beta)\sum_{i=1}^{m-2}\frac{\lambda_i t_i^{\alpha-\eta_i-1}}{\Gamma(\alpha-\eta_i)}.$$

It is clear that $\gamma > 0$.

Lemma 2.3 *Suppose $\gamma \neq 1$ and $h \in C[0,1]$. Then the following problem*

$$\begin{cases} D^\alpha u(t) + h(t) = 0, \ t \in [0,1], \\ u(0) = u'(0) = 0, \\ D^\beta u(1) = \sum_{i=1}^{m-2} \lambda_i D^{\eta_i} u(t_i) \end{cases} \tag{2.1}$$

has the unique solution $u(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s$, where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{2.2}$$

and

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \end{cases}$$

$$G_2(t,s) = \frac{t^{\alpha-1}}{(1-\gamma)\Gamma(\alpha)} \big(\gamma(1-s)^{\alpha-\beta-1} - \Gamma(\alpha-\beta) \sum_{i=1}^{m-2} \frac{\lambda_i}{\Gamma(\alpha-\eta_i)} (t_i - s)^{\alpha-\eta_i-1} \chi_{J_i}(s)\big).$$

Furthermore,

$$D^\beta u(t) = \int_0^1 H(t,s)h(s)\mathrm{d}s,$$

where the function H is given by

$$H(t,s) = H_1(t,s) + H_2(t,s), \tag{2.3}$$

and

$$H_1(t,s) = \frac{1}{\Gamma(\alpha-\beta)} \begin{cases} t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}, & 0 \le t \le s \le 1, \end{cases}$$

$$H_2(t,s) = \frac{t^{\alpha-\beta-1}}{(1-\gamma)\Gamma(\alpha-\beta)} \big(\gamma(1-s)^{\alpha-\beta-1} - \Gamma(\alpha-\beta) \sum_{i=1}^{m-2} \frac{\lambda_i}{\Gamma(\alpha-\eta_i)} (t_i - s)^{\alpha-\eta_i-1} \chi_{J_i}(s)\big).$$

Proof It follows from Lemma 2.2 and $D^\alpha u(t) + h(t) = 0$, we can get

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\mathrm{d}s + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3},$$

where $c_j \in \mathbb{R}$, for $j = 1,2,3$.

The boundary condition $u(0) = u'(0) = 0$ implies that $c_2 = c_3 = 0$. Thus,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\mathrm{d}s + c_1 t^{\alpha-1}. \tag{2.4}$$

We can obtain

$$D^\beta u(t) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} h(s)\mathrm{d}s + c_1 \frac{\Gamma(\alpha) t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}, \tag{2.5}$$

and

$$D^{\eta_i} u(t) = -\frac{1}{\Gamma(\alpha-\eta_i)} \int_0^t (t-s)^{\alpha-\eta_i-1} h(s)\mathrm{d}s + c_1 \frac{\Gamma(\alpha) t^{\alpha-\eta_i-1}}{\Gamma(\alpha-\eta_i)}.$$

By the boundary condition $D^\beta u(1) = \sum_{i=1}^{m-2} \lambda_i D^{\eta_i} u(t_i)$, we can get

$$c_1 = \frac{1}{(1-\gamma)\Gamma(\alpha)} \Big(\int_0^1 (1-s)^{\alpha-\beta-1} h(s)\mathrm{d}s - \Gamma(\alpha-\beta) \sum_{i=1}^{m-2} \frac{\lambda_i}{\Gamma(\alpha-\eta_i)} \int_0^1 (t_i - s)^{\alpha-\eta_i-1} h(s) \chi_{J_i}(s)\mathrm{d}s \Big).$$

Substituting c_1 into (2.4), we can obtain that $u(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s$ and substituting c_1 into (2.5), we can obtain $D^\beta u(t) = \int_0^1 H(t,s)h(s)\mathrm{d}s$. $\square$

Lemma 2.4 *Assume that $\gamma < 1$. Then the functions $G(t,s)$ and $H(t,s)$ are continuous on $[0,1]\times[0,1]$, and satisfy the following properties:*

(i) $0 \le t^{\alpha-1}G(1,s) \le G(t,s) \le G(1,s)$, *for* $(t,s)\in[0,1]\times[0,1]$;

(ii) $0 \le t^{\alpha-\beta-1}H(1,s) \le H(t,s) \le \frac{t^{\alpha-\beta-1}}{(1-\gamma)\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1} \le \frac{1}{(1-\gamma)\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}$, *for* $(t,s)\in[0,1]\times[0,1]$.

Proof (i) It is obvious that $G_1(t,s)$ is increasing on $t\in[0,s]$, so $G_1(s,s)\ge G_1(t,s)\ge 0$, for $0\le t\le s\le 1$.

For $0\le s<t\le 1$,

$$\frac{\partial G_1}{\partial t} = \frac{\alpha-1}{\Gamma(\alpha)}[t^{\alpha-2}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-2}] \ge \frac{\alpha-1}{\Gamma(\alpha)}[t^{\alpha-2}(1-s)^{\alpha-\beta-1}-(t-ts)^{\alpha-2}]\ge 0.$$

Then $G_1(t,s)$ is increasing on $t\in[s,1]$, which implies that $G_1(1,s)\ge G_1(t,s)\ge G_1(s,s)\ge 0$, for $0\le s\le t\le 1$.

Hence, $G_1(1,s)\ge G_1(t,s)\ge 0,\quad (t,s)\in[0,1]\times[0,1]$.

On the other hand, for $0\le s\le t\le 1$,

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)}(t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}) \ge \frac{t^{\alpha-1}}{\Gamma(\alpha)}((1-s)^{\alpha-\beta-1}-(1-s)^{\alpha-1}) = t^{\alpha-1}G_1(1,s).$$

Clearly, for $0\le t\le s\le 1$, $G_1(t,s) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}(1-s)^{\alpha-\beta-1} \ge t^{\alpha-1}G_1(1,s)$.

Thus, $0\le t^{\alpha-1}G_1(1,s)\le G_1(t,s)\le G_1(1,s)$, for $(t,s)\in[0,1]\times[0,1]$.

It follows from a direct application of the definition of $G_2(t,s)$ and the fact that

$$G_2(t,s) = \frac{\Gamma(\alpha-\beta)t^{\alpha-1}}{(1-\gamma)\Gamma(\alpha)}\sum_{i=1}^{m-2}\frac{\lambda_i t_i^{\alpha-\eta_i-1}}{\Gamma(\alpha-\eta_i)}\big((1-s)^{\alpha-\beta-1}-(1-\frac{s}{t_i})^{\alpha-\eta_i-1}\chi_{J_i}(s)\big)\ge 0$$

and $G_2(t,s) = t^{\alpha-1}G_2(1,s)$ that

$0\le t^{\alpha-\beta-1}G(1,s)\le G(t,s)\le G(1,s)$, for $(t,s)\in[0,1]\times[0,1]$, that is (i).

(ii) It is easy to show that

$$H_1(t,s)\ge 0 \text{ for } (t,s)\in[0,1]\times[0,1], \text{ and } H_1(1,s)=0 \text{ for } s\in[0,1].$$

Then, for $(t,s)\in[0,1]\times[0,1]$, we have

$$\begin{aligned} H(t,s) &= H_1(t,s)+H_2(t,s)\ge H_2(t,s) = t^{\alpha-\beta-1}H(1,s) = t^{\alpha-\beta-1}H_2(1,s)\\ &= \frac{t^{\alpha-\beta-1}}{1-\gamma}\sum_{i=1}^{m-2}\frac{\lambda_i}{\Gamma(\alpha-\eta_i)}\big(t_i^{\alpha-\eta_i-1}-(t_i-s)^{\alpha-\eta_i-1}\chi_{J_i}(s)\big)\ge 0.\end{aligned}$$

It is obvious that $H_1(t,s)\le \frac{1}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}$ and $H_2(t,s)\le \frac{\gamma t^{\alpha-\beta-1}}{(1-\gamma)\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}$.

As a result,

$$\begin{aligned} H(t,s) =& H_1(t,s)+H_2(t,s)\le \frac{1}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}+\frac{\gamma t^{\alpha-\beta-1}}{(1-\gamma)\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}\\ =&\frac{t^{\alpha-\beta-1}}{(1-\gamma)\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}\le \frac{1}{(1-\gamma)\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}.\ \square\end{aligned}$$

In order to prove our main results, we need the following fixed point theory of cones in ordered Banach spaces [19,20].

Let $r > a > 0, L > 0$ be constants, ψ be a nonnegative continuous concave functional and P, φ, θ be nonnegative continuous convex functionals on the cone P. Define convex sets

$$P(\varphi^r, \theta^L) = \{x \in P : \varphi(x) < r, \theta(x) < L\}, \ \overline{P}(\varphi^r, \theta^L) = \{x \in P : \varphi(x) \le r, \theta(x) \le L\},$$

$$P(\varphi^r, \theta^L, \psi_a) = \{x \in P : \varphi(x) < r, \ \theta(x) < L, \ \psi(x) > a\},$$

$$\overline{P}(\varphi^r, \theta^L, \psi_a) = \{x \in P : \varphi(x) \le r, \ \theta(x) \le L, \ \psi(x) \ge a\}.$$

The following assumptions about the nonnegative continuous convex functionals φ, θ will be used:

(B1) There exists $M > 0$ such that $\|x\| \le M \max\{\varphi(x), \theta(x)\}$, for all $x \in P$;

(B2) $P(\varphi^r, \theta^L) \ne \emptyset$, for any $r > 0, L > 0$.

Lemma 2.5 (The fixed point theorem [19,20]) *Let P be a cone in the real Banach space E and constants $0 < r_1 < b < d \le r_2$, $0 < L_1 \le L_2$. Assume that φ, θ are nonnegative continuous convex functionals satisfying (B1) and (B2), ψ is a nonnegative continuous concave functional on P such that $\psi(x) \le \varphi(x)$ for all $x \in \overline{P}(\varphi^{r_2}, \theta^{L_2})$, and $T : \overline{P}(\varphi^{r_2}, \theta^{L_2}) \to \overline{P}(\varphi^{r_2}, \theta^{L_2})$ is a completely continuous operator. Suppose*

(C1) $\{x \in \overline{P}(\varphi^d, \theta^{L_2}, \psi_b) : \psi(x) > b\} \ne \emptyset$, *$\psi(Tx) > b$ for $x \in \overline{P}(\varphi^d, \theta^{L_2}, \psi_b)$;*

(C2) *$\varphi(Tx) < r_1$, $\theta(Tx) < L_1$ for all $x \in \overline{P}(\varphi^{r_1}, \theta^{L_1})$;*

(C3) *$\psi(Tx) > b$ for all $x \in \overline{P}(\varphi^{r_2}, \theta^{L_2}, \psi_b)$ with $\varphi(Tx) > d$.*

Then, T has at least three fixed points $x_1, x_2, x_3 \in \overline{P}(\varphi^{r_2}, \theta^{L_2})$ with $x_1 \in P(\varphi^{r_1}, \theta^{L_1})$, $x_2 \in \{\overline{P}(\varphi^{r_2}, \theta^{L_2}, \psi_b) : \psi(x) > b\}$ and $x_3 \in \overline{P}(\varphi^{r_2}, \theta^{L_2}) \setminus (\overline{P}(\varphi^{r_2}, \theta^{L_2}, \psi_b) \cup \overline{P}(\varphi^{r_1}, \theta^{L_1}))$.

By Lemma 2.4, we can easily obtain following lemma.

Lemma 2.6 *If $h(t) \ge 0$, and $u = u(t)$ is the solution of boundary value problem (2.1). Then $u(t) \ge 0$, and $D^\beta u(t) \ge 0$ for $t \in [0, 1]$.*

Let $E := \{u : u \in C[0, 1], \ D^\beta u \in C[0, 1], u(0) = u'(0) = 0\}$ be endowed with the norm $\|u\|_\beta = \|u\|_\infty + \|D^\beta u\|_\infty$, where $\|u\|_\infty = \max_{0 \le t \le 1} |u(t)|$ and $\|D^\beta u\|_\infty = \max_{0 \le t \le 1} |D^\beta u(t)|$. Then $(E, \|\cdot\|_\beta)$ is a Banach space. Let

$$P = \{u \in E : t^{\alpha-1}\|u\|_\infty \le u(t), \ 0 \le D^\beta u(t), \ t \in [0, 1]\}.$$

Then $P \subset E$ is a cone on E. We define a operator $\Phi : P \to E$ by

$$(\Phi u)(t) = \int_0^1 G(t, s) f(s, u(s), D^\beta u(s)) \mathrm{d}s. \tag{2.6}$$

In view of Lemma 2.3, we can get that

$$D^\beta(\Phi u)(t) = \int_0^1 H(t, s) f(s, u(s), D^\beta u(s)) \mathrm{d}s. \tag{2.7}$$

Obviously, the function u is a positive solution of boundary value problem (1.1) if and only if u is a fixed point of the operator Φ in P.

Lemma 2.7 *The operator $\Phi : P \to P$ is completely continuous.*

Proof For a given $u \in P$, by (2.6) and (i) of Lemma 2.4,

$$0 \le t^{\alpha-1}\int_0^1 G(1,s)f(s,u(s),D^\beta u(s))\mathrm{d}s \le (\Phi u)(t) \le \int_0^1 G(1,s)f(s,u(s),D^\beta u(s))\mathrm{d}s.$$

We have

$$t^{\alpha-1}\|\Phi u\|_\infty \le (\Phi u)(t), \text{ for } t \in [0,1].$$

By (2.7) and (ii) of Lemma 2.4,

$$D^\beta(\Phi u)(t) \ge t^{\alpha-\beta-1}\int_0^1 H(1,s)f(s,u(s),D^\beta u(s))\mathrm{d}s \ge 0.$$

Hence, $\Phi u \in P$, which implies that $\Phi : P \to P$.

Let $\{u_j\} \subset P$ and $\lim_{j\to\infty} u_j = u \in P$. Since $f \in C([0,1]\times[0,+\infty)\times[0,+\infty),[0,+\infty))$, then

$$\lim_{j\to\infty} f(t,u_j(t),D^\beta u_j(t)) = f(t,u(t),D^\beta u(t)) \text{ for } t \in [0,1].$$

By Lemma 2.4, we can get

$$|(\Phi u_j)(t) - (\Phi u)(t)| \le \int_0^1 G(1,s)|f(s,u_j(s),D^\beta u_j(s)) - f(s,u(s),D^\beta u(s))|\mathrm{d}s$$

and

$$\begin{aligned}&|D^\beta(\Phi u_j)(t) - D^\beta(\Phi u)(t)|\\ &\le \frac{1}{(1-\gamma)\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}|f(s,u_j(s),D^\beta u_j(s)) - f(s,u(s),D^\beta u(s))|\mathrm{d}s.\end{aligned}$$

It follows from the Lebesgue dominated convergence theorem $\lim_{j\to\infty}(\Phi u_j)(t) = (\Phi u)(t)$ uniformly on $[0,1]$. Hence, Φ is continuous.

Assume $A \subset P$ is any bounded set, then there exists a constant M_1 such that $\|u\|_\beta \le M_1$ for each $u \in A$. So $|u(t)| \le M_1$ and $|D^\beta u(t)| \le M_1$ for $t \in [0,1]$. Because f is continuous, there exists $M_0 > 0$ such that $0 \le f(t,u(t),D^\beta(t)) \le M_0$ for $t \in [0,1]$. Then

$$0 \le (\Phi u)(t) = \int_0^1 G(t,s)f(s,u(s),D^\beta u(s))\mathrm{d}s \le M_0\int_0^1 G(1,s)\mathrm{d}s,$$

and

$$0 \le D^\beta(\Phi u)(t) = \int_0^1 H(t,s)f(s,u(s),D^\beta u(s))\mathrm{d}s \le \frac{M_0}{(1-\gamma)\Gamma(\alpha-\beta+1)},$$

which implies that $\Phi(A)$ is uniformly bounded in P.

Because $G(t,s)$ and $H(t,s)$ are continuous on $[0,1]\times[0,1]$, we can obtain that they are uniformly continuous. Hence, for any $\varepsilon > 0$, there exists $\delta > 0$, whenever $t_1,\ t_2 \in [0,1]$, $s \in [0,1]$ and $|t_2 - t_1| < \delta$,

$$|G(t_2,s) - G(t_1,s)| < \frac{\varepsilon}{2M_0+1} \text{ and } |H(t_2,s) - H(t_1,s)| < \frac{\varepsilon}{2M_0+1}.$$

For any $u \in P$, we have

$$|(\Phi u)(t_2) - (\Phi u)(t_1)| \le \int_0^1 |G(t_2,s) - G(t_1,s)|\cdot|f(s,u(s),D^\beta u(s))|\mathrm{d}s$$

$$\leq M_0 \int_0^1 |G(t_2,s) - G(t_1,s)|\mathrm{d}s < \frac{\varepsilon}{2},$$

and

$$|D^\beta(\Phi u)(t_2) - D^\beta(\Phi u)(t_1)| \leq \int_0^1 |H(t_2,s) - H(t_1,s)| \cdot |f(s,u(s),D^\beta u(s))|\mathrm{d}s$$
$$\leq M_0 \int_0^1 |H(t_2,s) - H(t_1,s)|\mathrm{d}s < \frac{\varepsilon}{2}.$$

Thus, $\Phi(A)$ is equicontinuous.

By Arzela-Ascoli theorem, we can show that Φ is relatively compact.

Therefore, Φ is completely continuous. □

3. The existence of at least three positive solutions

In this section, we are going to prove the existence of multiple positive solution for the m-point boundary value problem (1.1).

Theorem 3.1 *Assume $\delta \in (0,\frac{1}{2})$, there exist constants r_1, r_2, L_1, L_2, b with $0 < r_1 < \delta^{\alpha-\beta-1}b < b < \frac{b}{\delta^\beta} < \frac{b}{\delta^{\alpha-1}} := d < \frac{\int_0^1 G(1,s)\mathrm{d}s}{\delta^{\alpha-1}\int_\delta^{1-\delta} G(1,s)\mathrm{d}s} b < r_2$, $0 < L_1 \leq L_2$, and $b < L_2\delta^{\alpha-1}(1-\gamma)\Gamma(\alpha-\beta+1)\int_\delta^{1-\delta} G(1,s)\mathrm{d}s$, such that the following assumptions hold:*

(A1) $f(t,x,y) \leq \min\{\frac{r_2}{\int_0^1 G(1,s)\mathrm{d}s}, (1-\gamma)\Gamma(\alpha-\beta+1)L_2\}$ *for* $(t,x,y) \in [0,1]\times[0,r_2]\times[0,L_2]$;

(A2) $f(t,x,y) \geq \frac{b}{\delta^{\alpha-1}\int_\delta^{1-\delta} G(1,s)\mathrm{d}s}$ *for* $(t,x,y) \in [\delta,1-\delta]\times[b,d]\times[0,L_2]$;

(A3) $f(t,x,y) \leq \min\{\frac{r_1}{\int_0^1 G(1,s)\mathrm{d}s}, (1-\gamma)\Gamma(\alpha-\beta+1)L_1\}$ *for* $(t,x,y) \in [0,1]\times[0,r_1]\times[0,L_1]$.

Then boundary value problem (1.1) has at least three positive solutions u_1, u_2, u_3 with

$$0 \leq \max_{t\in[0,1]} u_1(t) \leq r_1 \text{ and } 0 \leq \max_{t\in[0,1]} D^\beta u_1(t) \leq L_1; \tag{3.1}$$

$$b < \min_{t\in[\delta,1-\delta]} u_2(t) \leq \max_{t\in[0,1]} u_2(t) \leq r_2 \text{ and } 0 \leq \max_{t\in[0,1]} D^\beta u_2(t) \leq L_2; \tag{3.2}$$

and

$$\min_{t\in[\delta,1-\delta]} u_3(t) \leq b,\ 0 \leq \max_{t\in[0,1]} u_3(t) \leq r_2 \text{ and } 0 \leq \max_{t\in[0,1]} D^\beta u_3(t) \leq L_2. \tag{3.3}$$

Proof In order to prove that boundary value problem (1.1) has at least three positive solutions by using Lemma 2.5, we define three functionals as follows

$$\varphi(u) = \max_{t\in[0,1]} u(t),\ \theta(u) = \max_{t\in[0,1]} D^\beta u(t) \text{ and } \psi(u) = \min_{t\in[\delta,1-\delta]} u(t).$$

Then φ, θ are nonnegative continuous convex functionals, ψ is a nonnegative continuous concave functional on P, and $\psi(u) \leq \varphi(u)$ for all $u \in P$.

Let $M = 2$. Then, for all $u \in P$,

$$\|u\|_\beta = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |D^\beta u(t)| = \varphi(u) + \theta(u) \leq M\max\{\varphi(u),\theta(u)\},$$

which implies that the condition (B1) is satisfied.

For any $r>0$ and $L>0$, let $x=k_0t^\beta$, where $0<k_0<\min\{r,\frac{L}{\Gamma(\beta+1)}\}$. Then

$$\varphi(x)=\max_{t\in[0,1]}k_0t^\beta=k_0<r,\ \theta(x)=\max_{t\in[0,1]}D^\beta k_0t^\beta=k_0\Gamma(\beta+1)<L.$$

So $x=k_0t^\beta\in P(\varphi^r,\theta^L)\neq\emptyset$, which implies that the condition (B2) is satisfied.

Next, we prove that operator Φ satisfies all conditions in Lemma 2.5.

In view of Lemma 2.7, $\Phi:P\to P$ is a completely continuous operator.

Let $d=\frac{b}{\delta^{\alpha-1}}$. Then $b<d$.

For every $u\in\overline{P}(\varphi^{r_2},\theta^{L_2})$, $\varphi(u)=\max_{t\in[0,1]}u(t)\le r_2$ and $\theta(u)=\max_{t\in[0,1]}D^\beta u(t)\le L_2$. By condition (A1),

$$f(t,u(t),D^\beta u(t))\le\ \min\{\frac{r_2}{\int_0^1G(1,s)\mathrm{d}s},\ (1-\gamma)\Gamma(\alpha-\beta+1)L_2\}.$$

And by Lemma 2.4, we get

$$\varphi(\Phi u)=\max_{t\in[0,1]}(\Phi u)(t)\le\int_0^1G(1,s)f(s,u(s),D^\beta u(s))\mathrm{d}s\le r_2,$$

and

$$\theta(\Phi u)=\max_{t\in[0,1]}D^\beta(\Phi u)(t)\le\int_0^1\frac{1}{(1-\gamma)\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}f(s,u(s),D^\beta u(s))\mathrm{d}s\le L_2,$$

which imply that $\Phi:\overline{P}(\varphi^{r_2},\theta^{L_2})\to\overline{P}(\varphi^{r_2},\theta^{L_2})$.

From condition (A3), in the same way, we can prove that $\Phi:\overline{P}(\varphi^{r_1},\theta^{L_1})\to P(\varphi^{r_1},\theta^{L_1})$. Then the condition (C2) of Lemma 2.5 is satisfied.

Let $y=k_1t^\beta$, where $\frac{b}{\delta^\beta}<k_1<\min\{d,\frac{L_2}{\Gamma(\beta+1)}\}$. Then $\varphi(y)=\max_{t\in[0,1]}k_1t^\beta=k_1<d$, $\theta(y)=\max_{t\in[0,1]}D^\beta k_1t^\beta=k_1\Gamma(\beta+1)<L_2$, and $\psi(y)=\min_{t\in[\delta,1-\delta]}k_1t^\beta=k_1\delta^\beta>b$. So $y=k_1t^\beta\in\{x\in\overline{P}(\varphi^d,\theta^{L_2},\psi_b):\psi(x)>b\}\neq\emptyset$.

From (A2) and Lemma 2.4, for $u\in\overline{P}(\varphi^d,\theta^{L_2},\psi_b)$

$$\begin{aligned}\psi(\Phi u)=&\min_{t\in[\delta,1-\delta]}(\Phi u)(t)\ge\min_{t\in[\delta,1-\delta]}\int_\delta^{1-\delta}G(t,s)f(s,u(s),D^\beta u(s))\mathrm{d}s\\ \ge&\delta^{\alpha-1}\int_\delta^{1-\delta}G(1,s)f(s,u(s),D^\beta u(s))\mathrm{d}s>b.\end{aligned}$$

Hence the condition (C1) in Lemma 2.5 is satisfied.

By (A1), in the same way, we can show that $\psi(\Phi u)>b$, for all $x\in\overline{P}(\varphi^{r_2},\theta^{L_2},\psi_b)$ with $\varphi(\Phi x)>d$. Then the condition (C3) in Lemma 2.5 is satisfied.

Since all the conditions of Lemma 2.5 are satisfied, then Φ will have at least three fixed points, which implies that boundary value problem (1.1) has at least three positive solutions $u_1,u_2,u_3\in\overline{P}(\varphi^{r_2},\theta^{L_2})$. Furthermore,

$$u_1\in P(\varphi^{r_1},\theta^{L_1}),\ u_2\in\{\overline{P}(\varphi^{r_2},\theta^{L_2},\psi_b):\ \psi(x)>b\},$$

and

$$x_3\in\overline{P}(\varphi^{r_2},\theta^{L_2})\setminus(\overline{P}(\varphi^{r_2},\theta^{L_2},\psi_b)\cup\overline{P}(\varphi^{r_1},\theta^{L_1})),$$

which imply that the inequalities (3.1), (3.2) and (3.3) hold. $\Box$

4. Illustration

To illustrate our main results, we present an example in this section.

We examine the existence of solutions for the following three-point boundary value problem of fractional differential equation

$$\begin{cases} D^{\frac{5}{2}}u(t) + 80e^{-\frac{200}{1+300u^3(t)}} + \dfrac{\sin t}{100}e^{\frac{D^{\frac{4}{3}}u(t)}{100}} = 0, \ t \in (0,1), \\ u(0) = u'(0) = 0, \\ D^{\frac{4}{3}}u(1) = \dfrac{1}{20}D^{\frac{1}{2}}u(\dfrac{1}{2}), \end{cases} \tag{4.1}$$

where $\alpha = \frac{5}{2}$, $\beta = \frac{4}{3}$, $\eta_1 = \frac{1}{2}$, $t_1 = \frac{1}{2}$, $\lambda_1 = \frac{1}{20}$, $\delta = \frac{1}{4}$, $f(t,x,y) = 80e^{-\frac{200}{1+300x^3}} + \frac{\sin t}{100}e^{\frac{y}{100}}$. Then $\alpha - \beta = \frac{7}{6} > 1$, and

$$\gamma = \Gamma(\alpha-\beta)\sum_{i=1}^{m-2}\frac{\lambda_i t_i^{\alpha-\eta_i-1}}{\Gamma(\alpha-\eta_i)} = \Gamma(\alpha-\beta)\frac{\lambda_1 t_1^{\alpha-\eta_1-1}}{\Gamma(\alpha-\eta_1)} = \frac{1}{40}\Gamma(\frac{7}{6}) \approx 0.023193 < 1.$$

It is easy to check that $f(t,x,y)$ is increasing with respect to t, x and y, respectively.

Let $r_1 = \frac{1}{2}$, $b = 1$, $r_2 = 30$, $L_1 = 2$, $L_2 = 80$. Then $0 < r_1 = \frac{1}{2} < \delta^{\alpha-\beta-1}b = \frac{1}{\sqrt[3]{2}} < b = 1 < \frac{b}{\delta^\beta} < \frac{\int_0^1 G(1,s)\mathrm{d}s}{\delta^{\alpha-1}\int_\delta^{1-\delta} G(1,s)\mathrm{d}s}b \approx 14.0056 < r_2 = 15$, $0 < L_1 \leq L_2$, and $b = 1 < \delta^{\alpha-1}(1-\gamma)\Gamma(\alpha-\beta+1)\int_\delta^{1-\delta} G(1,s)\mathrm{d}sL_2 \approx 2.1422$, and $d = \frac{b}{\delta^{\alpha-1}} = 8$.

We can easily show that

$0 < f(0,0,0) \leq f(t,x,y) \leq f(1,15,80) \approx 80.0029 \leq \min\{\frac{r_2}{\int_0^1 G(1,s)\mathrm{d}s}, (1-\gamma)\Gamma(\alpha-\beta+1)L_2\} \approx \min\{84.5711, 84.5789\} = 84.5711$ on $(t,x,y) \in [0,1]\times[0,r_2]\times[0,L_2] = [0,1]\times[0,30]\times[0,80]$;

$f(\frac{3}{4},8,80) \approx 79.9111 \geq f(t,x,y) \geq f(\frac{1}{4},1,0) \approx 41.1669 \geq \frac{b}{\delta^{\alpha-1}\int_\delta^{1-\delta} G(1,s)\mathrm{d}s} \approx 39.4823$ for $(t,x,y) \in [\delta, 1-\delta]\times[b,d]\times[0,L_2] = [\frac{1}{4},\frac{3}{4}]\times[1,8]\times[0,80]$;

$$\begin{aligned} f(t,x,y) \leq f(1,\frac{1}{2},2) \approx 0.452208 &\leq \min\{\frac{r_1}{\int_0^1 G(1,s)\mathrm{d}s}, (1-\gamma)\Gamma(\alpha-\beta+1)L_1\} \\ &\approx \min\{1.40952, 2.11447\} \text{ for } (t,x,y) \in [0,1]\times[0,r_1]\times[0,L_1] = [0,1]\times[0,\frac{1}{2}]\times[0,2]. \end{aligned}$$

Thus, all the conditions of Theorem 3.1 are satisfied. By using Theorem 3.1, boundary value problem (4.1) has at least three positive solutions u_1, u_2, u_3, and

$$0 \leq \max_{t\in[0,1]} u_1(t) \leq r_1 = \frac{1}{2}, \text{ and } 0 \leq \max_{t\in[0,1]} D^\beta u_1(t) \leq L_1 = 2;$$

$$1 = b < \min_{t\in[\frac{1}{4},\frac{3}{4}]} u_2(t) \leq \max_{t\in[0,1]} u_2(t) \leq r_2 = 30 \text{ and } 0 \leq \max_{t\in[0,1]} D^\beta u_2(t) \leq L_2 = 80;$$

and

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} u_3(t) \leq b = 1, \ 0 \leq \max_{t\in[0,1]} u_3(t) \leq r_2 = 30, \text{ and } 0 \leq \max_{t\in[0,1]} D^\beta u_3(t) \leq L_2 = 80.$$

5. Conclusions

In this paper, we have studied a class of the fractional differential equation multi-point boundary value problem. The aim of our study is to provide an analytical approach which can be used to determine the multiplicity of positive solutions of this boundary value problem. By using the fixed point theorem which is due to the work of Bai and Ge, we have presented the sufficient conditions such that this class of fractional differential equation with multi-point boundary conditions has at least three positive solutions. In the end of this paper, an example is presented to illustrate the main results.

Acknowledgements We thank the referees for their time and comments.

References

[1] K. DIETHELM. *The Analysis of Fractional Differential Equations*. Springer-Verlag, Berlin, 2010.

[2] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO. *Theory and applications of fractional differential equations*. Elsevier Science B.V., Amsterdam, 2006.

[3] I. PODLUBNY. *Fractional Differential Equations*. Mathematics in Science and Engineering 198. Academic Press, New York, 1999.

[4] Yong ZHOU. *Basic Theory of Fractional Differential Equations*. World Scientific, Singapore, 2014.

[5] A. BASHIR, S. K. NTOUYAS. *Existence results for a coupled system of Caputo type sequential fractional differential equations with nonlocal integral boundary conditions*. Appl. Math. Comput., 2015, **266**: 615–622.

[6] Fudong GE, Chunhai KOU. *Stability analysis by Krasnoselskii's fixed point theorem for nonlinear fractional differential equations*. Appl. Math. Comput., 2015, **257**: 308–316.

[7] Jianxin HE, Xinguang ZHANG, Yonghong WU. *Existence of positive solution for a fractional order nonlinear differential system involving a changing sign perturbation*. J. Nonlinear Sci. Appl., 2016, **9**(5): 2076–2085.

[8] Mei JIA, Xiping LIU. *Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions*. Appl. Math. Comput., 2014, **232**: 313–323.

[9] Xiping LIU, Mei JIA, Weigao GE. *Multiple solutions of a p-Laplacian model involving a fractional derivative*. Adv. Differ. Equ., 2013, **126**: 1–12.

[10] Xiping LIU, Mei JIA, Weigao GE. *The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator*. Appl. Math. Lett., 2017, **65**: 56–62.

[11] Xiping LIU, Legang LIN, Haiqin FANG. *Existence of positive solutions for nonlocal boundary value problem of fractional differential equation*. Cent. Eur. J. Phy., 2013, **11**: 1423–1432.

[12] Yuji LIU. *Existence and non-existence of positive solutions of BVPs for fractional order elastic beam equations with a non-Caratheodory nonlinearity*. Appl. Math. Model., 2014, **38**(2): 620–640.

[13] S. K. NYOUYAS, J. TARIBOON, P. THIRAMANUS. *Mixed problems of fractional coupled systems of Riemann-Liouville differential equations and Hadamard integral conditions*. J. Comput. Anal. Appl., 2016, **21**(5): 813–828.

[14] Jinong WANG, Yong ZHOU, Wei WEI. *A class of fractional delay nonlinear integrodifferential controlled systems in Banach spaces*. Commun. Nonlinear Sci. Numer. Simul., 2011, **16**(10): 4049–4059.

[15] Liu YANG, Chunfang SHEN, Dapeng XIE. *Multiple positive solutions for nonlinear boundary value problem of fractional order differential equation with the Riemann-Liouville derivative*. Adv. Differ. Equ., 2014, **284**: 1–12.

[16] M. EL-SHAHED, J. J. NIETO. *Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order*. Comput. Math. Appl., 2010, **59**(11): 3438–3443.

[17] H. A. H. SALEM. *On the fractional order m-point boundary value problem in reflexive Banach spaces and weak topologies*. J. Comput. Appl. Math., 2009, **224**(2): 565–572.

[18] M. REHMAN, R. KHAN. *Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations*. Appl. Math. Lett., 2010, **23**(9): 1038–1044.

[19] Zhanbing BAI, Weigao GE. *Existence of three positive solutions for some second-order boundary value problems*. Comput. Math. Appl., 2004, **48**(5-6): 699–707.

[20] Zhanbing BAI, Weigao GE, Yifu WANG. *Multiplicity results for some second-order four-point boundary-value problems*. Nonlinear Anal., 2005, **60**(3): 491–500.