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The Existence of Nodal Solutions for the Half-Quasilinear p-Laplacian Problems

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Abstract In this paper, we study the existence of nodal solutions for the following problem:

$$\begin{aligned} &-(\varphi_p(x'))' = \alpha(t)\varphi_p(x^+) + \beta(t)\varphi_p(x^-) + ra(t)f(x),\ 0 < t < 1,\\ &x(0) = x(1) = 0, \end{aligned}$$

where $\varphi_p(s) = |s|^{p-2}s$, $a \in C([0,1],(0,\infty))$, $x^+ = \max\{x,0\}, x^- = -\min\{x,0\}$, $\alpha(t), \beta(t) \in C[0,1]$; $f \in C(\mathbb{R},\mathbb{R})$, $sf(s) > 0$ for $s \neq 0$, and $f_0, f_\infty \notin (0,\infty)$, where

$$f_0 = \lim_{|s|\to 0} f(s)/\varphi_p(s), \quad f_\infty = \lim_{|s|\to +\infty} f(s)/\varphi_p(s).$$

We use bifurcation techniques and the approximation of connected components to prove our main results.

Keywords Bifurcation; Half-Quasilinear problems; Nodal solutions; p-Laplacian

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1. Introduction

Let E be a real Banach space with the norm $\|\cdot\|$. Consider the operator equation

$$u = \lambda Bu + H(\lambda, u), \tag{1.1}$$

where B is a compact linear operator and $H : \mathbb{R} \times E \to E$ is compact with $H = o(\|u\|)$ at $u = 0$ uniformly on bounded λ intervals. Krasnoselskii [1] has shown that all characteristic values of B which are of odd multiplicity are bifurcation points. Rabinowitz [2] has extended this result by showing that bifurcation has global consequences. Furthermore, if the characteristic value μ of B has multiplicity 1 and

$$\mathcal{S} = \overline{\{(\lambda,u) : (\lambda,u) \text{ satisfies } (1.1) \text{ and } u \not\equiv 0\}}^{\mathbb{R}\times E}.$$

Dancer [3] has shown that there are two distinct unbounded continua $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$, consisting of the bifurcation branch $\mathscr{C}_\mu$ of $\mathcal{S}$ emanating from $(\mu, 0)$, which satisfy either $\mathscr{C}_\mu^+$ and $\mathscr{C}_\mu^-$ are both unbounded or $\mathscr{C}_\mu^+ \cap \mathscr{C}_\mu^- \neq \{(\mu,0)\}$.

In the past ten years, some authors [4–9] studied the existence of Nodal solutions for the problems by applying Rabinowitz bifurcation techniques [2]. By using bifurcation techniques of

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Dancer [3], Dai [10,11] also considered the p-Laplacian problems. Recently, the problem involving non-differentiable nonlinearities have also been investigated by applying bifurcation techniques, see [12–16] and references therein. Meanwhile, Half-linear or Half-quasilinear problems have attracted the attention of many specialists in different equations because of their interesting applications [12,14–17]. Especially, Dai [16] considered the existence of nodal solutions for the following half-quasilinear eigenvalue problem

$$\begin{aligned} &-(\varphi_p(x'))' = \alpha(t)\varphi_p(x^+) + \beta(t)\varphi_p(x^-) + ra(t)f(x),\ 0 < t < 1, \\ &x(0) = x(1) = 0, \end{aligned} \tag{1.2}$$

where $\varphi_p(s) = |s|^{p-2}s$, $x^+ = \max\{x, 0\}$, $x^- = -\min\{x, 0\}$, $\alpha(t), \beta(t) \in C[0,1]$ and $f \in C(\mathbb{R}, \mathbb{R})$. Under the following assumptions:

(A1) $a \in C([0,1], (0,\infty))$.

(A2) $sf(s) > 0$ for $s \neq 0$.

(A3) $f_0, f_\infty \in (0, +\infty)$,

where

$$f_0 = \lim_{|s|\to 0} \frac{f(s)}{\varphi_p(s)}, \quad f_\infty = \lim_{|s|\to +\infty} \frac{f(s)}{\varphi_p(s)},$$

they obtained the following result:

Theorem 1.1 *Let (A1), (A2), and (A3) hold. For some $k \in \mathbb{N}$ and $\nu = +, -$, either $\frac{\lambda_k^\nu}{f_\infty} < r < \frac{\lambda_k^\nu}{f_0}$ or $\frac{\lambda_k^\nu}{f_0} < r < \frac{\lambda_k^\nu}{f_\infty}$. Then the problem (1.2) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0 and x_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0. Where λ_k^ν be given in Lemma 2.1.*

Of course, the natural question is what would happen if $f_0 \notin (0, +\infty)$ or $f_\infty \notin (0, +\infty)$. Obviously, the previous results cannot deal with this case. The purpose of this work is to establish several results similar to those of [16]. The main methods used in this work are unilateral global bifurcation techniques and the approximation of connected components. Moreover, we consider the cases of $f_0, f_\infty \notin (0, \infty)$, while the authors of [14–16] only studied the cases of $f_0, f_\infty \in (0, \infty)$.

Remark 1.2 We also note that, in high-dimensional case, there are also a lot of fundamental papers on the global bifurcation for p-Laplacian [18–21].

Remark 1.3 For the abstract unilateral global bifurcation theory, we refer the reader to [3,10,14,16,22,23] and the references therein.

The rest of this paper is arranged as follows. In Section 2, we give some preliminaries. In Section 3, we investigate the existence of nodal solutions for a class of the half-quasilinear p-Laplacian problems under the conditions (H1)–(H8), respectively.

2. Preliminaries

Let $Y = C[0,1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Let $E = \{x \in C^1[0,1] | x(0) = x(1) = 0\}$ with the norm $\|x\| = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |x'(t)|$. Let $\mathbf{E} = \mathbb{R} \times E$ under the product topology. Let S_k^+ denote the set of functions in E which have exactly $k-1$ interior nodal (i.e., nondegenerate) zeros in $(0,1)$ and are positive near $t=0$. Set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E. Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$. Let $\mathscr{S}$ denote the closure of the set of nontrivial solutions of (1.2) in $\mathbb{R} \times E$, and $\mathscr{S}_k^\nu$ denote the subset of with $u \in S_k^\pm$ and $\mathscr{S}_k = \mathscr{S}_k^+ \cup \mathscr{S}_k^-$.

In [16], Dai established an important global bifurcation theorem from intervals for a class of second-order p-Laplacian problems involving non-differentiable nonlinearity. Furthermore, Dai established the spectrum for the following Half-quasilinear problem

$$\begin{aligned} &-(\varphi_p(x'))' = \lambda a(t)\varphi_p(x) + \alpha(t)\varphi_p(x^+) + \beta(t)\varphi_p(x^-),\ 0<t<1, \\ &x(0)=x(1)=0, \end{aligned} \tag{2.1}$$

and obtained the following result.

Lemma 2.1 ([16]) *Let (A1) hold. There exist two sequences of simple half-eigenvalues for (2.1)*

$$\lambda_1^+ < \lambda_2^+ < \cdots < \lambda_k^+ < \cdots,\ \lim_{k\to\infty} \lambda_k^+ = +\infty$$

and

$$\lambda_1^- < \lambda_2^- < \cdots < \lambda_k^- < \cdots,\ \lim_{k\to\infty} \lambda_k^- = +\infty.$$

The corresponding Half-quasilinear solutions are in $\{\lambda_k^+\} \times S_k^+$ and $\{\lambda_k^-\} \times S_k^-$. Further, aside from these solutions and the trivial ones, there are no other solutions of (2.1).

In order to prove our main results, by [16], we have

Lemma 2.2 ([16]) *If (λ, x) is a nontrivial solution of (1.2) under assumptions (A1), (A2), and (A3) and x has a double zero, then $x \equiv 0$.*

Lemma 2.3 ([16]) *Let $b_2(t) \geq \max\{b_1(t), b_1(t)+\alpha(t)+\beta(t), b_1(t)-\alpha(t)-\beta(t)\}$ for $t \in (0,1)$ and $b_i(t) \in C(0,1)$, $i=1,2$. Also let u_1, u_2 be solutions of the following differential equations*

$$\begin{aligned} &(\varphi_p(u'))' + b_i(t)\varphi_p(u) + \alpha(t)\varphi_p(u^+) + \beta(t)\varphi_p(u^-),\ 0<t<1, \\ &u(0)=u(1)=0, \end{aligned}$$

respectively. If $(c,d) \subseteq (0,1)$, and $u_1(c)=u_1(d)=0$, $u_1(t) \neq 0$ in (c,d), then either there exists $\tau \in (c,d)$ such that $u_2(\tau)=0$ or $b_2=b_1$ or $b_2=b_1+\alpha+\beta$ or $b_2=b_1-\alpha-\beta$ and $u_2(t)=\mu u_1(t)$ for some constant $\mu \neq 0$.

Lemma 2.4 ([16]) *Let $I=(a,b)$ be such that $I \subset (0,1)$ and*

$$\operatorname{meas}(I) > 0.$$

Let $g_n \in C((0,1),\mathbb{R})$ be such that

$$\lim_{n\to+\infty} g_n(t) = +\infty \text{ uniformly on } I.$$

Let $y_n \in E$ be a solution of the equation

$$\begin{cases} (\varphi_p(y_n'))' + b_i(t)\varphi_p(y_n) + \alpha(t)\varphi_p(y_n^+) + \beta(t)\varphi_p(y_n^-) = 0, & 0<t<1, \\ y_n(0)=y_n(1)=0. \end{cases}$$

Then y_n must change sign on I as $n \to +\infty$.

In this section, we need the following topological lemma.

Lemma 2.5 ([24]) *Let X be a Banach space and $\{C_n | n = 1, 2, \ldots\}$ be a family of closed connected subsets of X. Assume that*

(i) There exist $z_n \in C_n$, $n = 1, 2, \ldots$ and $z^* \in X$, such that $z_n \to z^*$;

(ii) $r_n = \sup\{\|x\| | x \in C_n\} = \infty$;

(iii) For all $R > 0$, $(\cup_{n=1}^{\infty} C_n) \cap B_R$ is a relative compact set of X, where

$$B_R = \{x \in X | \|x\| \leq R\}.$$

Then there exists an unbounded component C in $\mathbb{D}$ and $z^* \in C$, where

$$\mathbb{D} := \limsup_{n\to\infty} C_n = \{x \in X | \exists \{n_i\} \subset \mathbb{N} \text{ and } x_{n_i} \in C_{n_i}, \text{ such that } x_{n_i} \to x\} \text{ (see [25]).}$$

3. Nodal solutions for half–quasilinear p-Laplacian problems

In the section, we shall investigate the existence of nodal solutions for the problem (1.2), where a satisfies the condition (A1). Throughout this paper, we assume that f satisfies (A2) and the following assumptions:

(H1) $f_0 \in (0, \infty)$ and $f_\infty = 0$;

(H2) $f_0 = 0$ and $f_\infty \in (0, \infty)$;

(H3) $f_0 = \infty$ and $f_\infty \in (0, \infty)$;

(H4) $f_0 \in (0, \infty)$ and $f_\infty = \infty$;

(H5) $f_0 = \infty$ and $f_\infty = 0$;

(H6) $f_0 = 0$ and $f_\infty = \infty$;

(H7) $f_0 = 0$ and $f_\infty = 0$;

(H8) $f_0 = \infty$ and $f_\infty = \infty$.

We start this section by studying the following eigenvalue problem

$$\begin{aligned} &-(\varphi_p(x'))' = \alpha(t)\varphi_p(x^+) + \beta(t)\varphi_p(x^-) + \lambda r a(t) f(x), \ 0 < t < 1, \\ &x(0) = x(1) = 0, \end{aligned} \tag{3.1}$$

where $\lambda > 0$ is a parameter. Let $\zeta \in C(\mathbb{R}, \mathbb{R})$ be such that

$$f(x) = f_0 \varphi_p(x) + \zeta(x)$$

with $\lim_{|x|\to 0} \frac{\zeta(x)}{\varphi_p(x)} = 0$. Let us consider

$$\begin{aligned} &-(\varphi_p(x'))' = \lambda r a(t) f_0 \varphi_p(x) + \alpha(t)\varphi_p(x^+) + \beta(t)\varphi_p(x^-) + \lambda r a(t)\zeta(x), \ 0 < t < 1, \\ &x(0) = x(1) = 0, \end{aligned} \tag{3.2}$$

as a bifurcation problem from the trivial solution $x \equiv 0$. Dai [16] obtained the following Lemma.

Lemma 3.1 ([16, Lemma 4.1]) *Let (A1), (A2), and (A3) hold. For some $k \in \mathbb{N}$ and $\nu \in \{+, -\}$, $(\frac{\lambda_k^\nu}{r f_0}, 0)$ is a bifurcation point for problem ((3.2). Moreover, there exists an unbounded continuum*

$\mathscr{D}_k^\nu$ of solutions of problem (3.2), such that $\mathscr{D}_k^\nu \subset (\Phi_k^\nu \cup \{(\frac{\lambda_k^\nu}{rf_0}, 0)\})$.

Remark 3.2 Any solution of (3.1) of the form $(1, x)$ yields a solution x of (1.2). In order to prove our main results, one will only show that $\mathscr{D}_k^\nu$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$.

Clearly, $(A2)$ implies $f(0) = 0$. Hence, $x = 0$ is always the solution of (1.2). Applying Lemma 3.1 (or [16, Lemma 4.1]), we establish the existence of nodal solutions of (1.2) as follows.

Theorem 3.3 *Let (A1), (A2), and (H1) hold. For some $k \in \mathbb{N}$ and $\nu \in \{+, -\}$, assume that one of the following conditions holds*

(i) $r \in (\frac{\lambda_k^\nu}{f_0}, +\infty)$ for $\lambda_k^\nu > 0$;

(ii) $r \in (-\infty, \frac{\lambda_k^-}{f_0}) \cup (\frac{\lambda_k^+}{f_0}, +\infty)$ for $\nu\lambda_k^\nu > 0$;

(iii) $r \in (-\infty, \frac{\lambda_k^+}{f_0}) \cup (\frac{\lambda_k^-}{f_0}, +\infty)$ for $\nu\lambda_k^\nu < 0$;

(iv) $r \in (-\infty, \frac{\lambda_k^\nu}{f_0})$ for $\lambda_k^\nu < 0$.

Then the problem (1.2) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0 and x_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0.

Proof We only prove the case of (i) since the proofs of the cases for (ii), (iii) and (iv) can be given similarly.

In view of the proof to prove [16, Theorem 1.3], we only need to show that $\mathscr{D}_k^\nu$ joins $(\frac{\lambda_k^\nu}{rf_0}, 0)$ to (∞, ∞). To do this, it is enough to prove that $[\frac{\lambda_k^\nu}{rf_0}, +\infty) \subset \mathrm{Proj}_R \mathscr{D}_k^\nu$.

Assume on the contrary that $\sup\{\lambda | (\lambda, u) \in \mathscr{D}_k^\nu\} < +\infty$, then there exists a sequence $(\mu_n, x_n) \in \mathscr{D}_k^\nu$ such that

$$\lim_{n\to\infty} \|x_n\| = +\infty, \quad \mu_n \le c_0$$

for some positive constant c_0 independent of n.

By (H1), let $\overline{f}(x) = \max_{0\le|s|\le x} |f(s)|$. Then $\overline{f}$ is nondecreasing and

$$\lim_{x\to+\infty} \frac{\overline{f}(x)}{x^{p-1}} = 0. \tag{3.3}$$

We consider the equation

$$\begin{aligned} &-(\varphi_p(x_n'))' = \alpha(t)\varphi_p(x_n^+) + \beta(t)\varphi_p(x_n^-) + \lambda r a(t) f(x_n),\ 0 < t < 1, \\ &x_n(0) = x_n(1) = 0, \end{aligned}$$

Dividing the above problem by $\|x_n\|^{p-1}$, let $y_n = \frac{x_n}{\|x_n\|}$, y_n should be the solutions of problem

$$\begin{aligned} &-(\varphi_p(y_n'))' = \alpha(t)\varphi_p(y_n^+) + \beta(t)\varphi_p(y_n^-) + \mu_n r a(t)\frac{f(x_n)}{\|x_n\|^{p-1}}, \\ &y_n(0) = y_n(1) = 0. \end{aligned} \tag{3.4}$$

Since y_n is bounded in $C^2[0,1]$, choosing a subsequence and relabeling if necessary, we have that $y_n \to y$ for some $y \in E$ and $\|y\| = 1$.

Furthermore, from (3.3) and the fact that $\overline{f}$ is nondecreasing, we have that

$$\lim_{x\to+\infty}\frac{f(x)}{\|x\|^{p-1}}=0. \tag{3.5}$$

Since

$$\frac{f(x)}{\|x\|^{p-1}}\le\frac{\overline{f}(|x|)}{\|x\|^{p-1}}\le\frac{\overline{f}(\|x\|_\infty)}{\|x\|^{p-1}}\le\frac{\overline{f}(\|x\|)}{\|x\|^{p-1}}\to 0,\quad \|x\|\to+\infty,$$

by (3.4), (3.5) and the compactness of L^{-1}, we obtain that

$$\begin{aligned}&-(\varphi_p(y'))'=\alpha(t)\varphi_p(y^+)+\beta(t)\varphi_p(y^-),\ t\in(0,1),\\&y(0)=y(1)=0.\end{aligned}$$

By $y(0)=y(1)$, there exists $\xi\in(0,1)$ such that $y'(\xi)=0$. Furthermore, applying the similar method to prove [16, Lemma 2.2] with obvious changes, we may obtain $|y(t)|\equiv 0,\forall t\in[0,1]$. This contradicts $\|y(t)\|=1$. □

Theorem 3.4 *Let (A1), (A2) and (H2) hold. For some $k\in\mathbb{N}$ and $\nu\in\{+,-\}$, assume that one of the following conditions holds*

(i) $r\in(\frac{\lambda_k^\nu}{f_\infty},+\infty)$ *for* $\lambda_k^\nu>0$;

(ii) $r\in(\frac{\lambda_k^+}{f_\infty},+\infty)\cup(-\infty,\frac{\lambda_k^-}{f_\infty})$ *for* $\nu\lambda_k^\nu>0$;

(iii) $r\in(\frac{\lambda_k^-}{f_\infty},+\infty)\cup(-\infty,\frac{\lambda_k^+}{f_\infty})$ *for* $\nu\lambda_k^\nu<0$;

(iv) $r\in(-\infty,\frac{\lambda_k^\nu}{f_\infty})$ *for* $\lambda_k^\nu<0$.

Then the problem (1.2) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0 and x_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0.

Proof We only prove the case of (i) since the proof of (ii)–(iv) can be given similarly.

If (λ,x) is any nontrivial solution of problem (1.2), dividing problem (1.2) by $\|x\|^{2(p-1)}$ and setting $y=\frac{x}{\|x\|^2}$ yields

$$\begin{aligned}&-(\varphi_p(y'))'=\alpha(t)\varphi_p(y^+)+\beta(t)\varphi_p(y^-)+\lambda ra(t)\frac{f(x)}{\|x\|^{2(p-1)}},\ 0<t<1,\\&y(0)=y(1)=0.\end{aligned}\tag{3.6}$$

Define

$$\widetilde{f}(y):=\begin{cases}\|y\|^{2(p-1)}f(\frac{y}{\|y\|^2}), & \text{if } y\ne 0,\\ 0, & \text{if } y=0.\end{cases}$$

Evidently, problem (3.6) is equivalent to

$$\begin{aligned}&-(\varphi_p(y'))'=\alpha(t)\varphi_p(y^+)+\beta(t)\varphi_p(y^-)+\lambda ra(t)\widetilde{f}(y),\ 0<t<1,\\&y(0)=y(1)=0.\end{aligned}\tag{3.7}$$

It is obvious that $(\lambda,0)$ is always the solution of problem (3.7). By simple computation, we can show that $\widetilde{f}_0=f_\infty$ and $\widetilde{f}_\infty=f_0$. Now, applying Theorem 3.3, there exists an unbounded continuum $\mathscr{C}_k^\nu$ of solutions of the problem (3.7) emanating from $(0,0)$, such that $\mathscr{C}_k^\nu\subset(\Phi_k^\nu\cup\{(0,0)\})$, and $\mathscr{C}_k^\nu$ joins $(0,0)$ to $(\frac{\lambda^\nu}{r\widetilde{f}_\infty},\infty)$.

Under the inversion $y \to \frac{y}{\|y\|^2} = x$, we obtain $\mathscr{C}_k^\nu \to \mathscr{D}_k^\nu$ being solutions of the problem (1.2) such that $\mathscr{D}_k^\nu$ joins $(\frac{\lambda^\nu}{rf_0}, 0)$ to $(0, \infty)$. $\Box$

Theorem 3.5 *Let (A1), (A2) and (H3) hold. For some $k \in \mathbb{N}$ and $\nu \in \{+, -\}$, assume that one of the following conditions holds*

(i) $r \in (0, \frac{\lambda_k^\nu}{f_\infty})$ for $\lambda_k^\nu > 0$;

(ii) $r \in (0, \frac{\lambda_k^+}{f_\infty}) \cup (\frac{\lambda_k^-}{f_\infty}, 0)$ for $\nu\lambda_k^\nu > 0$;

(iii) $r \in (0, \frac{\lambda_k^-}{f_\infty}) \cup (\frac{\lambda_k^+}{f_\infty}, 0)$ for $\nu\lambda_k^\nu < 0$;

(iv) $r \in (\frac{\lambda_k^\nu}{f_\infty}, 0)$ for $\lambda_k^\nu < 0$.

Then the problem (1.2) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0 and x_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0.

Proof Inspired by the idea of [26] or see [11], we define the cut-off function of f as follows

$$f^{[n]}(s) := \begin{cases} n\varphi_p(s), & s \in [-\frac{1}{n}, \frac{1}{n}], \\ \big[f(\frac{2}{n}) - \frac{1}{n^p}\big](ns-2) + f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{2}{n}), \\ -\big[f(-\frac{2}{n}) + \frac{1}{n^p}\big](ns+2) + f(-\frac{2}{n}), & s \in (-\frac{2}{n}, \frac{1}{n}), \\ f(s), & s \in (-\infty, -\frac{2}{n}] \cup [\frac{2}{n}, +\infty). \end{cases}$$

We consider the following problem

$$\begin{aligned} &-(\varphi_p(x'))' = \alpha(t)\varphi_p(x^+) + \beta(t)\varphi_p(x^-) + \lambda r a(t) f^{[n]}(x),\ 0 < t < 1, \\ &x(0) = x(1) = 0. \end{aligned} \tag{3.8}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = n$ and $(f^{[n]})_\infty = f_\infty$.

Similarly to the proof of [16, Theorem 1.3], by Lemma 3.1 and Remark 3.2, there exists an unbounded continuum $\mathscr{D}_k^{\nu[n]}$ of solutions of the problem (3.8) emanating from $(\frac{\lambda_k^\nu}{rn}, 0)$, such that $\mathscr{D}_k^{\nu[n]} \subset (\Phi_k^\nu \cup \{(\frac{\lambda_k^\nu}{rn}, 0)\})$, and $\mathscr{D}_k^{\nu[n]}$ joins $(\frac{\lambda_k^\nu}{rn}, 0)$ to $(\frac{\lambda_k^\nu}{rf_\infty}, \infty)$.

Taking $z_n = (\frac{\lambda_k^\nu}{rn}, 0)$ and $z^* = (0,0)$, we have that $z_n \to z^*$.

So condition (i) in Lemma 2.5 is satisfied with $z^* = (0,0)$.

Obviously

$$r_n = \sup\{\lambda + \|x\| | (\lambda, x) \in \mathscr{D}_k^{\nu[n]}\} = \infty,$$

and accordingly, (ii) in Lemma 2.5 holds. (iii) in Lemma 2.5 can be deduced directly from the Arezela-Ascoli Theorem and the definition of $f^{[n]}$.

Therefore, by Lemma 2.5, $\limsup_{n\to\infty} \mathscr{D}_k^{\nu[n]}$ contains an unbounded connected component $\mathscr{D}_k^\nu$ with $(0,0) \in \mathscr{D}_k^\nu$.

From $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, (3.8) can be converted to the equivalent equation (3.1). Since $\mathscr{D}_k^{\nu[n]} \subset \Phi_k^\nu$, we conclude $\mathscr{D}_k^\nu \subset \Phi_k^\nu$. Moreover, $\mathscr{D}_k^\nu \subset \mathscr{S}_k^\nu$ by (1.2).

Similarly to the method of the proof of case 2 of [16, Theorem 1.3], we can obtain that $(\frac{\lambda_k^\nu}{rf_\infty}, \infty) \in \mathscr{D}_k^\nu$. $\Box$

Theorem 3.6 *Let (A1), (A2) and (H4) hold. For some $k \in \mathbb{N}$ and $\nu \in \{+, -\}$, assume that one* *of the following conditions holds*

(i) $r \in (0, \frac{\lambda_k^\nu}{f_0})$ *for* $\lambda_k^\nu > 0$;

(ii) $r \in (0, \frac{\lambda_k^+}{f_0}) \cup (\frac{\lambda_k^-}{f_0}, 0)$ *for* $\nu\lambda_k^\nu > 0$;

(iii) $r \in (0, \frac{\lambda_k^-}{f_0}) \cup (\frac{\lambda_k^+}{f_0}, 0)$ *for* $\nu\lambda_k^\nu < 0$;

(iv) $r \in (\frac{\lambda_k^\nu}{f_0}, 0)$ *for* $\lambda_k^\nu < 0$.

Then the problem (1.2) possesses two solutions x_k^+ *and* x_k^- *such that* x_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near* 0 *and* x_k^- *has exactly* $k-1$ *zeros in* $(0,1)$ *and is negative near* 0.

Proof By the method similar to the proofs of Theorems 3.4 and 3.5 with obvious changes, we can obtain the result. □

Theorem 3.7 *Let (A1), (A2) and (H5) hold. For some* $k \in \mathbb{N}$ *and* $\nu \in \{+,-\}$, *assume that one of the following conditions holds*

(i) $r \in (0, +\infty)$ *for* $\lambda_k^\nu > 0$ *or*

(ii) $r \in (0, +\infty) \cup (-\infty, 0)$ *for* $\nu\lambda_k^\nu > 0$, *or* $\nu\lambda_k^\nu < 0$;

(iii) $r \in (-\infty, 0)$ *for* $\lambda_k^\nu < 0$.

Then the problem (1.2) *possesses two solutions* x_k^+ *and* x_k^- *such that* x_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near* 0 *and* x_k^- *has exactly* $k-1$ *zeros in* $(0,1)$ *and is negative near* 0.

Proof Define

$$f^{[n]}(s) := \begin{cases} \frac{1}{n}\varphi_p(s), & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \frac{n\varphi_p(2n)+f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ \frac{n\varphi_p(2n)-f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ f(s), & s \in [-n, -\frac{2}{n}] \cup [\frac{2}{n}, n], \\ -\big[f(-\frac{2}{n}) + \frac{1}{n^p}\big](ns+2) + f(-\frac{2}{n}), & s \in (-\frac{2}{n}, -\frac{1}{n}), \\ \big[f(\frac{2}{n}) - \frac{1}{n^p}\big](ns-2) + f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{2}{n}), \\ n\varphi_p(s), & s \in [-\frac{1}{n}, \frac{1}{n}]. \end{cases}$$

We consider the following problem

$$\begin{aligned} &-(\varphi_p(x'))' = \alpha(t)\varphi_p(x^+) + \beta(t)\varphi_p(x^-) + \lambda r a(t) f^{[n]}(x),\ 0 < t < 1, \\ &x(0) = x(1) = 0. \end{aligned}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = n$ and $(f^{[n]})_\infty = \frac{1}{n}$.

Applying the similar method used in the proof of Theorem 3.5, we obtain an unbounded connected component $\mathscr{D}_k^\nu \subset \mathscr{S}_k^\nu$ with $(0,0) \in \mathscr{D}_k^\nu$.

Similarly to the proof of Theorem 3.3, we get $(\infty,\infty) \in \mathscr{D}_k^\nu$, and the result is obtained. □

Theorem 3.8 *Let (A1), (A2) and (H6) hold. For some* $k \in \mathbb{N}$ *and* $\nu \in \{+,-\}$, *assume that one of the following conditions holds*

(i) $r \in (0, +\infty)$ *for* $\lambda_k^\nu > 0$;

(ii) $r \in (0, +\infty) \cup (-\infty, 0)$ *for* $\nu\lambda_k^\nu > 0$*, or* $\nu\lambda_k^\nu < 0$*;*

(iii) $r \in (-\infty, 0)$ *for* $\lambda_k^\nu < 0$*.*

Then the problem (1.2) *possesses two solutions* x_k^+ *and* x_k^- *such that* x_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near* 0 *and* x_k^- *has exactly* $k-1$ *zeros in* $(0,1)$ *and is negative near* 0*.*

Proof By the method similar to the proofs of Theorems 3.4 and 3.7 with obvious changes, we can obtain the result. □

Theorem 3.9 *Let (A1), (A2) and (H7) hold. For some* $k \in \mathbb{N}$ *and* $\nu \in \{+, -\}$*, assume that one of the following conditions holds*

(i) There exists a $\lambda_{\nu k}^\nu > 0$ *for* $\lambda_k^\nu > 0$*, such that* $r \in (\lambda_{\nu k}^\nu, +\infty)$*;*

(ii) There exists a $\nu\lambda_{\nu k}^\nu > 0$ *for* $\nu\lambda_k^\nu > 0$*, such that* $r \in (-\infty, \lambda_{-k}^-) \cup (\lambda_{+k}^+, +\infty)$*;*

(iii) There exists a $\nu\lambda_{\nu k}^\nu < 0$ *for* $\nu\lambda_k^\nu < 0$*, such that* $r \in (-\infty, \lambda_{+k}^+) \cup (\lambda_{-k}^-, +\infty)$*;*

(iv) There exists a $\lambda_{\nu k}^\nu < 0$ *for* $\lambda_k^\nu < 0$*, such that* $r \in (-\infty, \lambda_{\nu k}^\nu)$*.*

Then the problem (1.2) *possesses two solutions* x_k^+ *and* x_k^- *such that* x_k^+ *has exactly* $k-1$ *zeros in* $(0,1)$ *and is positive near* 0 *and* x_k^- *has exactly* $k-1$ *zeros in* $(0,1)$ *and is negative near* 0*.*

Proof Define

$$f^{[n]}(s) := \begin{cases} \frac{1}{n}\varphi_p(s), & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \frac{n\varphi_p(2n)+f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ \frac{n\varphi_p(2n)-f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ f(s), & s \in [-n, -\frac{2}{n}] \cup [\frac{2}{n}, n], \\ -\big[f(-\frac{2}{n}) + \frac{1}{n^p}\big](ns+2) + f(-\frac{2}{n}), & s \in (-\frac{2}{n}, -\frac{1}{n}), \\ \big[f(\frac{2}{n}) - \frac{1}{n^p}\big](ns-2) + f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{2}{n}), \\ \frac{1}{n}\varphi_p(s), & s \in [-\frac{1}{n}, \frac{1}{n}]. \end{cases}$$

We consider the following problem

$$\begin{aligned} &-(\varphi_p(x'))' = \alpha(t)\varphi_p(x^+) + \beta(t)\varphi_p(x^-) + \lambda r a(t) f^{[n]}(x), \ 0 < t < 1, \\ &x(0) = x(1) = 0. \end{aligned}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = \frac{1}{n}$ and $(f^{[n]})_\infty = \frac{1}{n}$.

Applying the similar method used in the proof of Theorem 3.4, we obtain an unbounded connected component $\mathscr{D}_k^\nu \subset \mathscr{S}_k^\nu$ with $(\infty, 0) \in \mathscr{D}_k^\nu$.

Similarly to the proof of Theorem 3.3, we can show that $(\infty, \infty) \in \mathscr{D}_k^\nu$. □

Theorem 3.10 *Let (A1), (A2) and (H8) hold. For some* $k \in \mathbb{N}$ *and* $\nu \in \{+, -\}$*, assume that one of the following conditions holds*

(i) There exists a $\lambda_{\nu k}^\nu > 0$ *for* $\lambda_k^\nu > 0$*, such that* $r \in (\lambda_{\nu k}^\nu, +\infty)$*;*

(ii) There exists a $\nu\lambda_{\nu k}^\nu > 0$ *for* $\nu\lambda_k^\nu > 0$*, such that* $r \in (-\infty, \lambda_{-k}^-) \cup (\lambda_{+k}^+, +\infty)$*;*

(iii) There exists a $\nu\lambda_{\nu k}^\nu < 0$ *for* $\nu\lambda_k^\nu < 0$*, such that* $r \in (-\infty, \lambda_{+k}^+) \cup (\lambda_{-k}^-, +\infty)$*;*

(iv) There exists a $\lambda_{\nu k}^{\nu} < 0$ for $\lambda_k^{\nu} < 0$, such that $r \in (-\infty, \lambda_{\nu k}^{\nu})$.

Then the problem (1.2) possesses two solutions x_k^+ and x_k^- such that x_k^+ has exactly $k-1$ zeros in $(0,1)$ and is positive near 0 and x_k^- has exactly $k-1$ zeros in $(0,1)$ and is negative near 0.

Proof Define

$$f^{[n]}(s) := \begin{cases} n\varphi_p(s), & s \in (-\infty, -2n] \cup [2n, +\infty), \\ \frac{n\varphi_p(2n)+f(-n)}{n}(s+n) + f(-n), & s \in (-2n, -n), \\ \frac{n\varphi_p(2n)-f(n)}{n}(s-n) + f(n), & s \in (n, 2n), \\ f(s), & s \in [-n, -\frac{2}{n}] \cup [\frac{2}{n}, n], \\ -\big[f(-\frac{2}{n}) + \frac{1}{n^p}\big](ns+2) + f(-\frac{2}{n}), & s \in (-\frac{2}{n}, -\frac{1}{n}), \\ \big[f(\frac{2}{n}) - \frac{1}{n^p}\big](ns-2) + f(\frac{2}{n}), & s \in (\frac{1}{n}, \frac{2}{n}), \\ n\varphi_p(s), & s \in [-\frac{1}{n}, \frac{1}{n}]. \end{cases}$$

We consider the following problem

$$\begin{aligned} &-(\varphi_p(x'))' = \alpha(t)\varphi_p(x^+) + \beta(t)\varphi_p(x^-) + \lambda r a(t) f^{[n]}(x),\ 0 < t < 1, \\ &x(0) = x(1) = 0. \end{aligned}$$

Clearly, we can see that $\lim_{n\to+\infty} f^{[n]}(s) = f(s)$, $(f^{[n]})_0 = n$ and $(f^{[n]})_\infty = n$.

Applying the similar method used in the proof of Theorem 3.5, we obtain an unbounded connected component $\mathscr{D}_k^{\nu} \subset \Phi_k^{\nu}$ with $(0,0) \in \mathscr{D}_k^{\nu}$.

By Theorem 3.6, we can show that $(0,\infty) \in \mathscr{D}_k^{\nu}$. □

Remark 3.11 Clearly, if $p = 2$, $\alpha = \beta = 0$, the problem (1.2) was studied in [6,7] under the conditions (A2) and (H5) or (H6), and the results of Theorems 3.6 and 3.10 improve those of Theorems 1.1 and 1.2 in [6] or [7].

Remark 3.12 Note that if $\alpha = \beta \equiv 0$, the results of Theorems 3.3–3.10 are equivalent to those of Theorems 2.1–2.8 of [11], respectively. Hence, Theorems 3.1–3.8 extend the corresponding results of [11].

Remark 3.13 The nonlinear term of (1.2) is not necessarily homogeneous linearizable at the origin and infinity because of the influence of the term $\alpha x^+ + \beta x^-$. So the bifurcation results of [1–3,10,11] cannot be applied directly to obtain our results.

Remark 3.14 We consider the cases of $f_0, f_\infty \notin (0,\infty)$, while the authors of [16] only studied the cases of $f_0, f_\infty \in (0,\infty)$. Hence, Theorems 3.3–3.10 extend the Theorem 4.1 of [16].

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