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Maxima of the Q-Index for Halin Graphs

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Abstract The Q-index of a graph G is the largest eigenvalue $q(G)$ of its signless Laplacian matrix $Q(G)$. In this paper, we prove that the wheel graph $W_n = K_1 \vee C_{n-1}$ is the unique graph with maximal Q-index among all Halin graphs of order n. Also we obtain the unique graph with second maximal Q-index among all Halin graphs of order n.

Keywords Halin graph; signless Laplacian spectral radius; wheel graph

MR(2010) Subject Classification 05C50; 15A18

1. Introduction

As usual, let $G = (V(G), E(G))$ be a finite, undirected and simple graph with order n. Let $A(G)$ be the adjacency matrix of G and $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees of G, or simply A, D and Q, respectively. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G. The largest eigenvalue of $A(G)$ or $Q(G)$ is called the spectral radius or signless Laplacian spectral radius of G, and denoted by $\rho(G)$ or $q(G)$, respectively. By Perron-Frobenius Theorem, we know that if G is a simple connected graph, then $q(G)$ is nonnegative and simple, and there is a unique positive unit eigenvector $x = \mathbf{x}(G)$ (called Perron eigenvector) of $Q(G)$ corresponding to $q(G)$. In recent years, the study of the spectral properties of the signless Laplacian spectral radius has attracted much attention, and the reader may consult [1–4].

Set $\Gamma_G(v) = \{u|uv \in E(G)\}$, and $d_G(v) = |\Gamma_G(v)|$; or simply $\Gamma(v)$ and $d(v)$, respectively. Let $\delta = \delta(G)$ and $\Delta = \Delta(G)$ denote the minimum degree and maximum degree of the graph G. And $T(v) = \sum_{uv\in E(G)} d(u)$ denotes the sum of degrees of neighbours of v. Further, for a matrix M, denote the v-th row sum of M by $s_v(M)$. Denote by K_n, C_n, P_n a complete graph, a cycle and a path of order n, respectively. The join of $G \vee H$ of disjoint graphs G and H is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. For terminology and notations of graphs undefined here, we refer the reader to [5].

Let T be a tree with $n \geq 4$ vertices and without vertices of degree two. If T is embeded in the plane with the leaves $v_1, v_2, \ldots, v_t$ arranged in clockwise direction, then T, together with the

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new edges v_iv_{i+1} (where $v_{t+1} = v_1$) that induce a cycle on the set of leaves, forms a 3-connected planar graph G called a Halin graph. The leaves v_i of T are called the outer vertices, while the remaining vertices are called inner vertices.

Obviously, the wheel graph $W_n = K_1 \vee C_{n-1}$ (as shown in Figure 1) is the unique Halin graph with one inner vertex. The graph $G_3 = G(s,t)$ (as shown in Figure 2) is the Halin graph with order $n = s + t + 2 \geq 6$ and two inner vertices. For $G_3 = G(s,t)$, when $t = 2$ or $s = 2$, $G_3 \cong G_2$ (as shown in Figure 1). Graphs G_4 and G_5 (as shown in Figure 3) are the Halin graphs with order $n \geq 8$ and three inner vertices.

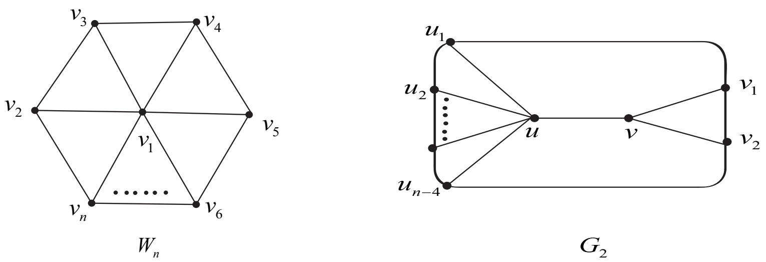

Figure 1 The wheel W_n with order $n \geq 4$ and the graph G_2 with order $n \geq 6$

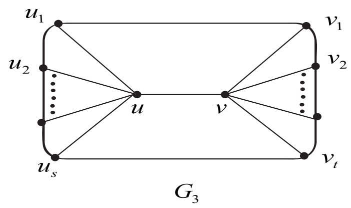

Figure 2 The graph $G_3 = G(s,t)$ with order $n = s + t + 2 \geq 6$

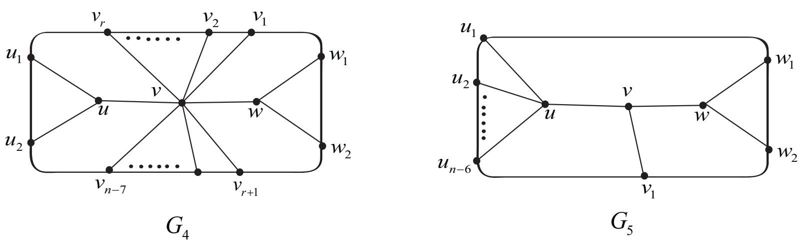

Figure 3 The graphs G_4 and G_5 with order $n \geq 8$, where $0 \leq r \leq n - 7$

In 1969, Halin [6] introduced Halin graphs when they researched the minimum 3-connected planar graphs. Later, Li, Zhang and Wang et al. [7,8] have done a lot of work for the chromatic number, the edge chromatic number and the total chromatic number of Halin graphs. Next Shu and Hong [9] studied the spectral radius of outerplanar graphs and Halin graphs, and gave the upper bound for the spectral radius of Halin graphs and the corresponding extremal graphs. Furthermore, Yuan and Shu [10] gave a new upper bound for the spectral radius of Halin graphs and the corresponding extremal graphs. Recently, Lin [11] obtained the upper bound on the signless Laplacian spectral radius of Halin graphs in terms of its genus and the maximum degree.

Then in 2013, Feng et al. [12] obtained the upper bound on the signless Laplacian spectral radius of Halin graphs in terms of its genus and the order of the graph, with further improvements in the case that the graph is Halin graphs. In 2015, Yu et al. [13] presented a better upper bound on the signless Laplacian spectral radius of Halin graphs in terms of order and maximum degree.

In this paper, we prove that the wheel graph W_n is the unique graph with maximal Q-index among all Halin graphs of order n. Also we prove that G_2 (as shown in Figure 1) is the unique graph with second maximal Q-index among all Halin graphs of order n. We also obtain a lower bound of the Q-index $q(G)$ of the graph $G = K_1 \vee P_{n-1}$.

2. Main lemmas

Before our proofs we give some lemmas which are used in the proofs.

Lemma 2.1 ([12]) *Let G be a Halin graph of order $n \geq 7$. If G has $t \geq 1$ inner vertices. Then*

$$q(G) \leq \frac{1}{2}(n - 2t + 6 + \sqrt{(n - 2t + 2)^2 + 24}).$$

Lemma 2.2 ([14]) *Let Q be the signless Laplacian matrix of a graph G and let $P(x)$ be an arbitrary real-valued polynomial in x. Then*

$$\min_{v \in V(G)} s_v(P(Q)) \leq q(P(Q)) \leq \max_{v \in V(G)} s_v(P(Q)).$$

Moreover, if the row sums of $P(Q)$ are not all equal then both inequalities are strict.

Lemma 2.3 ([3]) *Let G be a connected graph containing at least one edge. Then*

$$q(G) \geq \Delta + 1$$

with equality if and only if $G \cong K_{1,n-1}$.

Lemma 2.4 ([15]) *Let G be a connected graph of order n and $q(G)$ be the signless Laplacian spectral radius of G. Let u, v be two vertices of G and $d(v)$ be the degree of vertex v. Assume $v_1, v_2, \ldots, v_s (1 \leq s \leq d(v))$ are some vertices of $\Gamma(v) \setminus (\Gamma(u) \cup \{u\})$ and $X = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $Q(G)$, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let $G^* = G - \{vv_1, vv_2, \ldots, vv_s\} + \{uv_1, uv_2, \ldots, uv_s\}$ be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \leq i \leq s)$, If $x_u \geq x_v$, then $q(G) < q(G^*)$.*

Lemma 2.5 ([1]) *Let G be a graph on n vertices with vertex degrees $d_1, d_2, \ldots, d_n$ and largest Q-eigenvalue q_1. Then*

$$\min(d_i + d_j) \leq q_1 \leq \max(d_i + d_j),$$

where (i, j) runs over all pairs of adjacent vertices of G. For a connected graph G, equality holds in either of these inequalities if and only if G is regular or semi-regular bipartite.

3. Main results

First we give a lower bound of the Q-index $q(G)$ of the graph $G = K_1 \vee P_{n-1}$. Then we will prove some basic results.

Lemma 3.1 *Let $G = K_1 \vee P_{n-1}$ and $n \geq 3$. Then*

$$q(G) \geq \frac{n+1+\sqrt{(n-7)^2+32}}{2}.$$

Proof $G = K_1 \vee P_{n-1}$ is shown in Figure 4. Let $q = q(G)$ be the signless Laplacian spectral radius of G. For $Q = D + A$, then we have $s_v(Q) = 2d(v)$ and

$$s_v(D^2) = s_v(DA) = d^2(v), \quad s_v(AD) = s_v(A^2) = T(v).$$

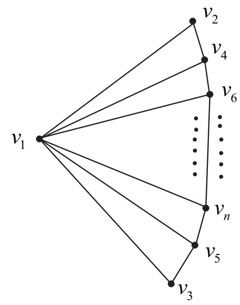

Figure 4 $G = K_1 \vee P_{n-1}$

Therefore,

$$\begin{aligned} s_v(Q^2) - (n+1)s_v(Q) &= s_v(D^2 + DA + AD + A^2) - (n+1)s_v(Q) \\ &= 2d^2(v) + 2T(v) - 2(n+1)d(v). \end{aligned}$$

From the graph G, we obtain the following equations

$$\begin{cases} 2d^2(v_1) + 2T(v_1) - 2(n+1)d(v_1) = 2n - 6, \\ 2d^2(v_i) + 2T(v_i) - 2(n+1)d(v_i) = -2n + 8, \quad i = 2, 3, \\ 2d^2(v_j) + 2T(v_j) - 2(n+1)d(v_j) = -4n + 20, \ j = 4, 5, \\ 2d^2(v_k) + 2T(v_k) - 2(n+1)d(v_k) = -4n + 22, \quad k = 6, 7, \ldots, n. \end{cases}$$

So for $i = 1, 2, \ldots, n$, we have $s_{v_i}(Q^2) - (n+1)s_{v_i}(Q) = 2d^2(v_i) + 2T(v_i) - 2(n+1)d(v_i) \geq -4n+20$.

From Lemma 2.2, it follows that

$$\begin{aligned} q^2 - (n+1)q = q(Q^2 - (n+1)Q) &\geq \min_{v \in V(G)} s_v(Q^2 - (n+1)Q) \\ &= \min_{v \in V(G)} [s_v(Q^2) - (n+1)s_v(Q)]. \end{aligned}$$

So we have $q^2 - (n+1)q + 4n - 20 \geq 0$, which implies $q(G) \geq \frac{n+1+\sqrt{(n-7)^2+32}}{2}$. $\square$

Lemma 3.2 *Let G be a Halin graph with order $n \geq 6$ and two inner vertices. Then*

$$q(G) \leq q(G_2),$$

where G_2 is shown in Figure 1.

Proof Because the Halin graph G has two inner vertices, then $G \cong G_3$ (as shown in Figure 2).

If $s = 2$ or $t = 2$, then $G_3 \cong G_2$.

If $s \geq 3$ and $t \geq 3$, let $x = \mathbf{x}(G_3)$ be the Perron eigenvector corresponding to $q(G_3)$. Without loss of generality, suppose that $x_u \geq x_v$, let $G' = G_3 - \{vv_3, vv_4, \ldots, vv_t\} + \{uv_3, uv_4, \ldots, uv_t\}$. By Lemma 2.4, we have $q(G) < q(G') = q(G_2)$.

This completes the proof. □

Lemma 3.3 *Let G be a Halin graph with order $n \geq 8$ and three inner vertices. Then*

$$q(G) \leq \max\{q(G_i) | i = 4, 5\},$$

where G_i $(i = 4, 5)$ are shown in Figure 3.

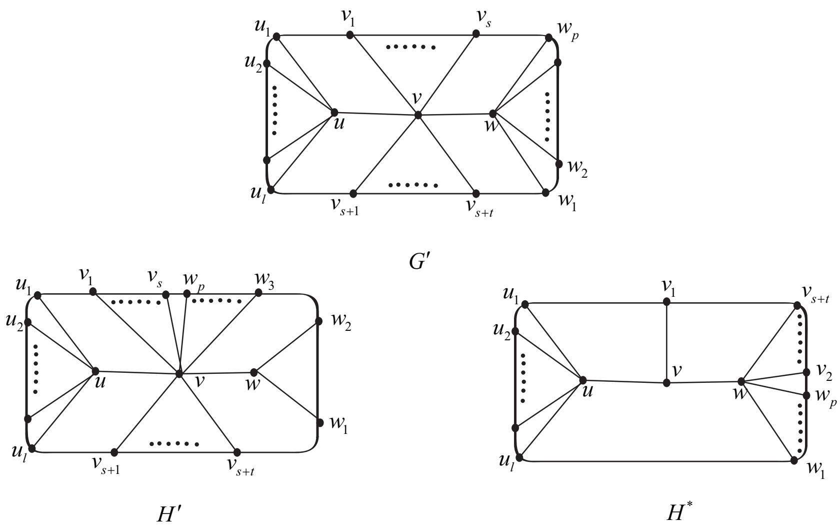

Figure 5 Halin graphs G', H' and H^*

Proof Let G be a Halin graph with three inner vertices. Then $G \cong G'$ (G' is shown in Figure 5), where $n = l + s + t + p + 3$. Let $x = \mathbf{x}(G')$ be the Perron eigenvector corresponding to $q(G')$.

If $x_v > x_w$, let $H' = G' - \{ww_3, \ldots, ww_p\} + \{vw_3, \ldots, vw_p\}$, by Lemma 2.4, then we have $q(G') \leq q(H')$. If $x_v \leq x_w$, let $H^* = G' - \{vv_2, \ldots, vv_{s+t}\} + \{wv_2, \ldots, wv_{s+t}\}$, by Lemma 2.4, then we have $q(G') \leq q(H^*)$.

Then let $x' = \mathbf{x}(H')$ be the Perron eigenvector corresponding to $q(H')$. When $x'_u \leq x'_v$, let $H_1 = H' - \{uu_3, \ldots, uu_l\} + \{vu_3, \ldots, vu_l\}$. By Lemma 2.4, we have $q(H') \leq q(H_1)$. Also we know that $H_1 \cong G_4$, where $r = s + p - 2$ for G_4. So $q(H') \leq q(G_4)$. When $x'_u > x'_v$, let $H_2 = H' - \{vv_1, \ldots, vv_{s+t}, vw_4, \ldots, vw_p\} + \{uv_1, \ldots, uv_{s+t}, uw_4, \ldots, uw_p\}$. By Lemma 2.4, we have $q(H') \leq q(H_2)$. Also we know that $H_2 \cong G_5$. So $q(H') \leq q(G_5)$.

Then let $x^* = \mathbf{x}(H^*)$ be the Perron eigenvector corresponding to $q(H^*)$. Without loss of generality, suppose that $x^*_w \geq x^*_u$, let $H_3 = H^* - \{uu_3, \ldots, uu_l\} + \{wu_3, \ldots, wu_l\}$. By Lemma 2.4, we have $q(H^*) \leq q(H_3)$. Also we know that $H_3 \cong G_5$. So $q(H^*) \leq q(G_5)$.

From the above all, we obtain $q(G) \leq \max\{q(G_i) | i = 4, 5\}$. □

Lemma 3.4 *Let G_4 and G_5 (as shown in Figure 3) be Halin graphs with order $n \geq 8$ and three* *inner vertices. Then* $q(G_i) < q(G_2)$ *for* $i = 4, 5$.

Proof Method 1 of the proof of Lemma 3.4:

We will prove the results in the following two cases.

1) First we will prove that $q(G_4) < q(G_2)$.

Because $\Delta(G_4) = n-5$, by Lemma 2.3, we know that $q(G_4) > n-4 \geq 4$. Let $x = \mathbf{x}(G_4)$ be the Perron eigenvector corresponding to $q(G_4)$.

Note that $G_2 = G_4 - \{w_1w_2\} + \{vw_1, vw_2\}$. Then

$$\begin{aligned} q(G_2) - q(G_4) &\geq x^TQ(G_2)x - x^TQ(G_4)x \\ &= (x_v + x_{w_1})^2 + (x_v + x_{w_2})^2 - (x_{w_2} + x_{w_1})^2 \\ &= 2x_v^2 + 2x_vx_{w_1} + 2x_vx_{w_2} - 2x_{w_1}x_{w_2}. \end{aligned}$$

Next we will prove that $2x_vx_{w_1} > 2x_{w_1}x_{w_2}$, i.e., $x_v > x_{w_2}$. By the symmetry of G_4, we have

$$x_{w_1} = x_{u_1}, x_{w_2} = x_{u_2}, x_u = x_w, x_{v_1} = x_{v_r}, x_{v_{r+1}} = x_{v_{n-7}}.$$

Let $q = q(G_4)$. From the eigenequations for $Q(G_4)$ we see that

$$\begin{cases} qx_v = (n-5)x_v + 2x_w + \sum_{i=1}^{n-7} x_{v_i}, \\ qx_{w_2} = 3x_{w_2} + x_{v_{n-7}} + x_{w_1} + x_w, \\ qx_w = 3x_w + x_{w_1} + x_{w_2} + x_v, \\ qx_{w_1} = 3x_{w_1} + x_w + x_{w_2} + x_{v_1}. \end{cases}$$

Then

$$qx_v - qx_{w_2} = (n-5)x_v + x_w + \sum_{i=1}^{n-8} x_{v_i} - 3x_{w_2} - x_{w_1},$$

$$(q-3)(x_v - x_{w_2}) = (n-8)x_v + x_w + \sum_{i=1}^{n-8} x_{v_i} - x_{w_1}.$$

When $n \geq 9$, we have $(q-3)(x_v - x_{w_2}) \geq x_v + x_w - x_{w_1}$.

Next we will prove that $x_v + x_w - x_{w_1} > 0$,

$$qx_v + qx_w - qx_{w_1} = (n-4)x_v + 4x_w + \sum_{i=2}^{n-7} x_{v_i} - 2x_{w_1},$$

$$(q-2)(x_v + x_w - x_{w_1}) = (n-6)x_v + 2x_w + \sum_{i=2}^{n-7} x_{v_i} > 0.$$

It follows immediately that $x_v > x_{w_2}$. So when $n \geq 9$, we prove $q(G_4) < q(G_2)$. When $n = 8$, by careful calculation, we obtain $6.000 = q(G_4) < q(G_2) = 7.1078$.

Hence we have $q(G_4) < q(G_2)$.

2) Next we will prove that $q(G_5) < q(G_2)$.

Because $\Delta(G_5) = n-5$, by Lemma 2.3, we know that $q(G_5) > n-4 \geq 4$. Let $x = \mathbf{x}(G_5)$ be the Perron eigenvector corresponding to $q(G_5)$. Let $G' = G_5 - \{w_1w_2\} + \{uw_1, vw_2\}$. Then

$$q(G') - q(G_5) \geq x^TQ(G')x - x^TQ(G_5)x$$

$$= (x_u + x_{w_1})^2 + (x_v + x_{w_2})^2 - (x_{w_2} + x_{w_1})^2$$
$$= x_u^2 + x_v^2 + 2x_u x_{w_1} + 2x_v x_{w_2} - 2x_{w_1} x_{w_2}.$$

Let $B = 2x_u x_{w_1} + 2x_v x_{w_2} - 2x_{w_1} x_{w_2}$. Next we will prove that $B > 0$.

Let $q = q(G_5)$. From the eigenequations for $Q(G_5)$, we see that

$$\begin{cases} qx_u = (n-5)x_u + x_v + \sum_{i=1}^{n-6} x_{u_i}, \\ qx_v = 3x_v + x_u + x_{v_1} + x_w, \\ qx_{w_1} = 3x_{w_1} + x_{u_1} + x_{w_2} + x_w, \\ qx_{w_2} = 3x_{w_2} + x_{v_1} + x_{w_1} + x_w. \end{cases}$$

(1) Let $x_{w_1} \geq x_{w_2}$. Then

$$B \geq 2x_u x_{w_2} + 2x_v x_{w_2} - 2x_{w_1} x_{w_2} = 2x_{w_2}(x_u + x_v - x_{w_1}).$$

Next we will prove $x_u + x_v - x_{w_1} > 0$. Note that

$$q(x_u + x_v - x_{w_1}) = (n-4)x_u + 4x_v + \sum_{i=2}^{n-6} x_{u_i} + x_{v_1} - 3x_{w_1} - x_{w_2}$$
$$\geq (n-4)x_u + 4x_v + \sum_{i=2}^{n-6} x_{u_i} + x_{v_1} - 4x_{w_1},$$
$$(q-4)(x_u + x_v - x_{w_1}) \geq (n-8)x_u + \sum_{i=2}^{n-6} x_{u_i} + x_{v_1} > 0.$$

It follows immediately that $x_u + x_v - x_{w_1} > 0$, then $q(G') - q(G_5) > 0$.

(2) Let $x_{w_1} < x_{w_2}$. Then

$$B \geq 2x_u x_{w_1} + 2x_v x_{w_1} - 2x_{w_1} x_{w_2} = 2x_{w_1}(x_u + x_v - x_{w_2}).$$

Next we will prove $x_u + x_v - x_{w_2} > 0$. Note that

$$q(x_u + x_v - x_{w_2}) = (n-4)x_u + 4x_v + \sum_{i=1}^{n-6} x_{u_i} - 3x_{w_2} - x_{w_1}$$
$$\geq (n-4)x_u + 4x_v + \sum_{i=1}^{n-6} x_{u_i} - 4x_{w_2},$$
$$(q-4)(x_u + x_v - x_{w_2}) \geq (n-8)x_u + \sum_{i=1}^{n-6} x_{u_i} > 0.$$

It follows immediately that $x_u + x_v - x_{w_2} > 0$, then $q(G') - q(G_5) > 0$.

Hence we have $q(G_5) < q(G')$. From Lemma 3.2, we have $q(G_5) < q(G_2)$.

From the above two cases, we prove that $q(G_i) < q(G_2)$ $(i = 4, 5)$.

Method 2 of the proof of Lemma 3.4:

By Lemma 2.3, we have $q(G_2) \geq \Delta + 1 = n-2$. By Lemma 2.5, we have $q(G_i) \leq n-5+3 = n-2$ $(i = 4, 5)$. Therefore, we have that $q(G_i) < q(G_2)$ $(i = 4, 5)$. $\square$

Theorem 3.5 *Let G be a Halin graph with order $n \geq 6$ and $t \geq 2$ inner vertices. Then*

$q(G) \le q(G_2)$ with equality if and only if $G \cong G_2$, where G_2 is shown in Figure 1.

Proof Because the Halin graph G has $t \ge 2$ inner vertices, we have

1) When $t = 2$ or $t = 3$, by Lemmas 3.2–3.4, we have $q(G) \le q(G_2)$.

2) When $t \ge 4$, by Lemma 2.1, $q(G) \le \frac{1}{2}(n-2+\sqrt{(n-6)^2+24})$. The Halin graph G has $t \ge 4$ inner vertices, then $n \ge 10$. So we obtain

$$\begin{aligned} q(G) &\le \frac{1}{2}(n-2+\sqrt{(n-6)^2+24}) = \frac{1}{2}(n-2+\sqrt{n^2-12n+60}) \\ &\le \frac{1}{2}(n-2+\sqrt{n^2-6n}) \le \frac{1}{2}(n-2+n-3)) \\ &\le n-\frac{5}{2}. \end{aligned}$$

Also from Lemma 2.3, we have $q(G_2) \ge \Delta + 1 = n-2 > n-\frac{5}{2} \ge q(G)$. When $t \ge 4$, then $q(G) < q(G_2)$.

From all above, we obtain when a Halin graph G has $t \ge 2$ inner vertices, $q(G) \le q(G_2)$ with equality if and only if $G \cong G_2$. $\square$

Theorem 3.6 *Let G be a Halin graph with order n (≥ 4). Then $q(G) \le q(W_n)$ with equality if and only if $G \cong W_n$.*

Proof Method 1 of the proof of Theorem 3.6:

It is easy to check that when $n = 4, 5$, then $G \cong W_n$. Next we assume $n \ge 6$. The graph G_2 is shown in Figure 1. Because $\Delta(G_2) = n-3$, by Lemma 2.3, we know that $q(G_2) > n-2 \ge 4$. Let $x = \mathbf{x}(G_2)$ be the Perron eigenvector corresponding to $q(G_2)$. Also we have $W_n = G_2 - \{v_1v_2\} + \{uv_1, uv_2\}$. Then

$$\begin{aligned} q(W_n) - q(G_2) &\ge x^TQ(W_n)x - x^TQ(G_2)x \\ &= (x_u + x_{v_1})^2 + (x_u + x_{v_2})^2 - (x_{v_2} + x_{v_1})^2 \\ &= 2x_u^2 + 2x_ux_{v_1} + 2x_ux_{v_2} - 2x_{v_1}x_{v_2}. \end{aligned}$$

Next we will prove that $2x_ux_{v_2} > 2x_{v_1}x_{v_2}$, i.e., $x_u > x_{v_1}$. By the symmetry of G_2, we have

$$x_{u_1} = x_{u_{n-4}}, x_{v_1} = x_{v_2}.$$

Let $q = q(G_2)$. From the eigenequations for $Q(G_2)$, we see that

$$\begin{cases} qx_u = (n-3)x_u + x_v + \sum\limits_{i=1}^{n-4} x_{u_i}, \\ qx_{v_1} = 3x_{v_1} + x_{v_1} + x_v + x_{u_1}. \end{cases}$$

Then

$$qx_u - qx_{v_1} = (n-3)x_u + \sum_{i=2}^{n-4} x_{u_i} - 4x_{v_1},$$

$$(q-4)(x_u - x_{v_1}) = (n-7)x_u + \sum_{i=2}^{n-4} x_{u_i}.$$

When $n \geq 7$, it follows immediately that $x_u > x_{v_1}$. Then when $n = 6$, by careful calculation, we obtain $6.000 = q(G_2) < q(W_n) = 7.2361$. So $q(G_2) < q(W_n)$.

When G is a Halin graph with order n, from Theorem 3.5, we obtain that $q(G) \leq q(W_n)$ with equality if and only if $G \cong W_n$.

Method 2 of the proof of Theorem 3.6:

It is easy to check that when $n = 4, 5$, then $G \cong W_n$. Next we assume $n \geq 6$. By Lemma 2.3, we have $q(G_1) \geq \Delta + 1 = n$. By Lemma 2.5, we have $q(G_2) \leq n - 3 + 3 = n$. Therefore, we have that $q(G_2) < q(G_1)$.

When G is a Halin graph with order n, from Theorem 3.5, we obtain that $q(G) \leq q(W_n)$ with equality if and only if $G \cong W_n$. This completes the proof. $\square$

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