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The Signless Dirichlet Spectral Radius of Unicyclic Graphs

Guangjun ZHANG[1,*], **Weixia LI**[2]
1. *School of Mathematics and Physics, Qingdao University of Science and Technology, Shandong* 266061*, P. R. China;*
2. *School of Mathematics and Statistics, Qingdao University, Shandong* 266071*, P. R. China*

Abstract Let G be a simple connected graph with pendant vertex set ∂V and nonpendant vertex set V_0. The signless Laplacian matrix of G is denoted by $Q(G)$. The signless Dirichlet eigenvalue is a real number λ such that there exists a function $f \neq 0$ on $V(G)$ such that $Q(G)f(u) = \lambda f(u)$ for $u \in V_0$ and $f(u) = 0$ for $u \in \partial V$. The signless Dirichlet spectral radius $\lambda(G)$ is the largest signless Dirichlet eigenvalue. In this paper, the unicyclic graphs with the largest signless Dirichlet spectral radius among all unicyclic graphs with a given degree sequence are characterized.

Keywords signless Dirichlet spectral radius; unicyclic graph; degree sequence

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1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Let $d(x)$ denote the degree of a vertex x. A non-increasing positive integers sequence $\pi = (d_0, d_1, \ldots, d_{n-1})$ is called a unicyclic graphic degree sequence if there exists at least a unicyclic graph with degree sequence π. Denote by $A(G)$ the adjacency matrix of G. The Laplacian matrix and signless Laplacian matrix of G is defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively, where $D(G)$ is the diagonal matrix of vertex degrees of G. For a long time, most scholars have been interested in the spectra of adjacency matrix and Laplacian matrix of the graphs with a prescribed graphic degree sequence. For example, Biyikoğlu et al. [1] determined the graphs with the maximal spectral radius among all trees with a given degree sequence. Zhang [2,3] determined the graphs with the largest signless Laplacian spectral radius among all trees and unicyclic graphs with a given degree sequence, respectively. Belardo et al. [4] determined the graphs with the largest spectral radius in the set of unicyclic graphs with a given degree sequence. Huang et al. [5] determined the graphs with the largest signless Laplacian spectral radius in the set of bicyclic graphs with a given degree sequence.

Recently there is an increasing interest in the Dirichlet eigenvalue of graphs. Friedman in [6] introduced the idea of a graph with boundary and formulated the Dirichlet eigenvalue problem

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* Corresponding author
E-mail address: guangjunzhang@126.com (Guangjun ZHANG); liweixia99@163.com (Weixia LI)

for graphs involving Laplacian. Biyikoğlu and Leydold [7] determined the trees with the smallest first Dirichlet eigenvalue among all the trees with the same degree sequence. Zhang et al. [8] determined the graphs with the smallest first Dirichlet eigenvalue among the unicyclic graphs with a given degree sequence under minor conditions. Let ∂V be the set of pendant vertices of G and $V_0 = V(G)\backslash\partial V$. In this paper, we always assume that ∂V is a nonempty set. A real number λ is called a signless Dirichlet eigenvalue of G if there exists a function $f \neq 0$ on $V(G)$ such that for $u \in V(G)$,

$$\begin{cases} Q(G)f(u) = \lambda f(u), & u \in V_0, \\ f(u) = 0, & u \in \partial V. \end{cases}$$

The largest signless Dirichlet eigenvalue of $Q(G)$, denoted by $\lambda(G)$, is called the signless Dirichlet spectral radius [9]. Zhang et al. [9] determined the graphs with the largest signless Dirichlet spectral radius among the trees with a given degree sequence. Let Γ_π be the set of unicyclic graphs with a given degree sequence π. In this paper, we will characterize the graphs with the largest signless Dirichlet spectral radius in Γ_π. The main result of this paper is as follows:

Theorem 1.1 *For a given unicyclic degree sequence π, Γ_π^* (see in Section 3) is the unique graph with the largest signless Dirichlet spectral radius in Γ_π.*

2. The signless Dirichlet spectral radius

The Rayleigh quotient of signless Laplacian matrix $Q(G)$ is denoted by

$$\Delta_G(f) = \frac{\langle Qf, f\rangle}{\langle f, f\rangle} = \frac{\sum_{uv\in E}(f(u)+f(v))^2}{\sum_{v\in V} f^2(v)}.$$

Then we have

Proposition 2.1 ([9]) *Let G be a graph such that ∂V is not empty. Then*

$$\lambda(G) = \max_{f\in\mathcal{S}} \Delta_G(f) = \max_{f\in\mathcal{S}} \frac{\langle Qf, f\rangle}{\langle f, f\rangle}.$$

Moreover, if $\Delta_G(f) = \lambda(G)$ for a function $f \in \mathcal{S}$, then f is an eigenfunction of $\lambda(G)$, where $\mathcal{S}$ denote the set of all real-valued functions f on $V(G)$ with $f(u) = 0$ for any $u \in \partial V$.

Lemma 2.2 ([9]) *Let G be a graph such that ∂V is not empty. Then the signless Dirichlet spectral radius $\lambda(G)$ of G is positive. Moreover, if f is an eigenfunction of $\lambda(G)$, then $f(v) > 0$ for all $v \in V_0(G)$ or $f(v) < 0$ for all $v \in V_0(G)$.*

In [9], the unit eigenvector f of $\lambda(G)$ is called a Dirichlet Perron vector of G if $f(v) > 0$ for all $v \in V_0(G)$.

3. Main result

Let $G - uv$ denote the graph obtained from G by deleting an edge uv in G and $G + uv$ denote the graph obtained from G by adding an edge uv.

Lemma 3.1 ([9]) *Let G be a graph such that ∂V is not empty. Assume $u, v, x \in V_0$ and $y \in V(G)$ such that $uv, xy \in E(G)$ and $ux, yv \notin E(G)$. Let f be the Dirichlet Perron vector of G and $G' = G - uv - xy + ux + yv$. Then $\lambda(G') \geq \lambda(G)$ if $f(u) \geq f(y)$ and $f(x) \geq f(v)$. Moreover, $\lambda(G') > \lambda(G)$ if one of the two inequalities is strict.*

Lemma 3.2 ([9]) *Let G be a graph such that ∂V is not empty, and P be a path from a non-pendant vertex v_1 to another non-pendant vertex v_2. Suppose that $v_1u_i \in E(G)$, $v_2u_i \notin E(G)$ and u_i is not on the path P for $i = 1, 2, \ldots, t$ with $t \leq d(v_1) - 2$. By deleting the t edges $v_1u_1, v_1u_2, \ldots, v_1u_t$ and adding the t edges $v_2u_1, v_2u_2, \ldots, v_2u_t$ we get a new graph G'. Let f be the Dirichlet Perron vector of G. Then if $f(v_1) \leq f(v_2)$, we have $\lambda(G') > \lambda(G)$.*

Corollary 3.3 *Let G be a graph with the largest signless Dirichlet spectral radius in Γ_π and f be the Dirichlet Perron vector of G. If $f(x) \geq f(y)$ for any $x, y \in V(G)$, then $d(x) \geq d(y)$.*

Proof Clearly, the assertion holds for $d(y) = 1$. If $d(y) = 2$, then $f(x) \geq f(y) > 0$. So x is not a pendant vertex and $d(x) \geq d(y) = 2$. In the following we prove that the assertion holds for $d(y) \geq 3$. Assume $d(x) < d(y)$. Let $t = d(y) - d(x)$ and $u_1, u_2, \ldots, u_t$ be the vertices which are adjacent to y and not in any path from x to y. Let $G_1 = G - \bigcup_{s=1}^{t} yu_s + \bigcup_{s=1}^{t} xu_s$. By Lemma 3.2, we have $\lambda(G_1) > \lambda(G)$. It is a contradiction to our assumption. So $d(x) \geq d(y)$. The proof is completed. □

Lemma 3.4 *Let G be a graph with the largest signless Dirichlet spectral radius in Γ_π and f be the Dirichlet Perron vector of G. Then we have $f(u) > f(v)$ for any $u \in V(C)$ and $v \notin V(C)$.*

Proof Let C be the cycle of G. Assume the assertion does not hold. Then there exist $x \in V(C)$ and $y \notin V(C)$ such that $f(x) \leq f(y)$. Clearly, y is not a pendant vertex. There exists a vertex $w \in V(C)$ such that $xw \in E(C)$ and $yw \notin E(G)$. There also exists a hanging path $yy_1y_2\cdots y_p$ such that $y_1, y_2, \ldots, y_{p-1} \notin V(C)$ and y_p is a pendant vertex. Since G is a unicyclic graph, we have $wy_i \notin E(G)$ and $xy_i \notin E(G)$ for all $1 \leq i \leq p$. Let $G_1 = G - wx - yy_1 + wy + xy_1$. Then $G_1 \in \Gamma_\pi$. Furthermore, we have $f(w) \leq f(y_1)$. Otherwise, we have $\lambda(G_1) > \lambda(G)$ by Lemma 3.1. It is a contradiction to our assumption that G is the graph with the largest signless Dirichlet spectral radius in Γ_π. Let $G_2 = G - wx - y_1y_2 + wy_2 + xy_1$. Then $G_2 \in \Gamma_\pi$. If $f(x) > f(y_2)$, we have $\lambda(G_2) > \lambda(G)$ by Lemma 3.1, a contradiction. So we have $f(x) \leq f(y_2)$. By repeating the similar discussion as above, we have $f(w) \leq f(y_p)$ or $f(x) \leq f(y_p)$. Then $f(y_p) > 0$. It is a contradiction to our assumption that y_p is a pendant vertex. So $f(x) > f(y)$. The proof is completed. □

Lemma 3.5 *Let G be a graph with the largest signless Dirichlet spectral radius in Γ_π and f be the Dirichlet Perron vector of G. Let v_0, v_1 and v_2 be the three vertices such that $f(v_0) \geq f(v_1) \geq f(v_2) \geq f(x)$ for any $x \in V(G)$. Then we have $v_0v_1, v_1v_2, v_0v_2 \in E(G)$.*

Proof Let C be the cycle of G. By Lemma 3.4, $v_0, v_1, v_2 \in V(C)$. Since $d(v_0) \geq 3$, there exists $x \notin V(C)$ such that $xv_0 \in E(G)$. If $v_0v_1 \notin E(G)$, there exists $y \in V(C)$ such that $v_1y \in E(G)$.

Let $G_1 = G - v_0x - v_1y + v_0v_1 + xy$. Note $f(v_0) \geq f(y)$ and $f(v_1) > f(x)$ by Lemma 3.4. Then we have $G_1 \in \Gamma_\pi$ and $\lambda(G_1) > \lambda(G)$ by Lemma 3.1. It is a contradiction to our assumption that G is the graph with the largest signless Dirichlet spectral radius in Γ_π. So $v_0v_1 \in E(G)$. By similar proof, we have $v_0v_2 \in E(G)$. Now assume $v_1v_2 \notin E(G)$. There exists $z \in V(C)$ such that $v_1z \in E(C)$ and $z \neq v_2$. Let $G_2 = G - v_0x - v_1z + v_0z + xv_1$. Note $f(v_0) \geq f(v_1)$ and $f(z) > f(x)$ by Lemma 3.4. Then we have $G_2 \in \Gamma_\pi$ and $\lambda(G_2) > \lambda(G)$ by Lemma 3.1. It is also a contradiction to our assumption that G is the graph with the largest signless Dirichlet spectral radius in Γ_π. So $v_1v_2 \in E(G)$. The proof is completed. $\square$

Let $\pi = (d_0, d_1, \ldots, d_{n-1})$ be a unicyclic graphic degree sequence with $d_0 \geq d_1 \geq \cdots \geq d_{n-1}$. In the following we will construct a unicyclic graph Γ^*_π with degree sequence π by the recursion. Select n vertices $v_0, v_1, \ldots, v_{n-1}$ such that v_0 is adjacent to $v_1, v_2, \ldots, v_{d_0}$, and v_1 is adjacent to v_2. Let v_1 be adjacent to v_k for $k = d_0 + 1, d_0 + 2, \ldots, d_0 + d_1 - 2$, v_2 be adjacent to v_l for $l = d_0 + d_1 - 1$, $d_0 + d_1, \ldots, d_0 + d_1 + d_2 - 4$, and v_3 be adjacent to v_b for $b = d_0 + d_1 + d_2 - 3, d_0 + d_1 + d_2 - 2, \ldots, d_0 + d_1 + d_2 + d_3 - 5$. Now assume that v_k is adjacent to v_h for $h = c_k + 1, c_k + 2, \ldots, c_k + d_k - 1$, where $c_k = \sum_{m=0}^{k-1} d_m - k - 1$ and $4 \leq k \leq i$. We let v_{i+1} be adjacent to v_g for $g = c_i + d_i, c_i + d_i + 1, \ldots, c_i + d_i + d_{i+1} - 2$. In this way we obtain a unicyclic graph Γ^*_π such that $V(\Gamma^*_\pi) = \{v_0, v_1, \ldots, v_{n-1}\}$ with $d(v_i) = d_i$ for $i = 0, 1, \ldots, n-1$.

Proof of Theorem 1.1 Let G be the graph with the largest signless Dirichlet spectral radius in Γ_π. We label the vertices of G as $V(G) = \{v_0, v_1, \ldots, v_{n-1}\}$ such that $f(v_0) \geq f(v_1) \geq \cdots \geq f(v_{n-1})$. Then we have $d(v_0) \geq d(v_1) \geq \cdots \geq d(v_{n-1})$ by Corollary 3.3 and $v_0v_1v_2$ is the cycle of G by Lemma 3.5. If $v_0v_3 \notin E(G)$, there exists $v_0v_p \in E(G)$ with $p > 3$. Let P_1 be the path from v_0 to v_3. If $f(v_3) = f(v_p)$, we may exchange the labeling of v_3 and v_p. In the following we assume that $f(v_3) > f(v_p)$. Then $d(v_3) \geq 2$. If $v_p \in V(P_1)$, there exists v_q such that $v_3v_q \in E(G)$ and $v_q \notin V(P_1)$. Let $G_1 = G - v_0v_p - v_3v_q + v_0v_3 + v_pv_q$. Then we have $G_1 \in \Gamma_\pi$ and $\lambda(G_1) > \lambda(G)$ by Lemma 3.1, since $f(v_3) > f(v_p)$ and $f(v_0) \geq f(v_q)$, a contradiction. If $v_p \notin V(P_1)$, there exists $v_{q'}$ such that $v_3v_{q'} \in E(P_1)$. Let $G'_1 = G - v_0v_p - v_3v_{q'} + v_0v_3 + v_pv_{q'}$. Note that $f(v_3) > f(v_p)$ and $f(v_0) \geq f(v_{q'})$. Then we have $G'_1 \in \Gamma_\pi$ and $\lambda(G'_1) > \lambda(G)$ by Lemma 3.1, a contradiction. So $v_0v_3 \in E(G)$. By the similar discussion as above, we have $v_0v_m \in E(G)$ for $m = 4, 5, \ldots, d_0$.

If $v_1v_{d_0+1} \notin E(G)$, there is a vertex v_s such that $v_1v_s \in E(G)$ and $s > d_0 + 1$. Let P_2 be the path from v_1 to v_{d_0+1}. Without loss of generality, assume that $f(v_{d_0+1}) > f(v_s)$. Then $d(v_{d_0+1}) \geq 2$. If $v_s \in V(P_2)$, there exists $v_t \notin V(P_2)$ such that $v_{d_0+1}v_t \in E(G)$. Let $G_2 = G - v_1v_s - v_{d_0+1}v_t + v_1v_{d_0+1} + v_sv_t$. Note that $f(v_{d_0+1}) > f(v_s)$ and $f(v_1) \geq f(v_t)$. Then we have $G_2 \in \Gamma_\pi$ and $\lambda(G_2) > \lambda(G)$ by Lemma 3.1, a contradiction. If $v_s \notin V(P_2)$, there exists $v_{t'} \in V(P_2)$ such that $v_{d_0+1}v_{t'} \in E(G)$. Let $G'_2 = G - v_1v_s - v_{d_0+1}v_{t'} + v_1v_{d_0+1} + v_sv_{t'}$. Since $f(v_{d_0+1}) > f(v_s)$ and $f(v_1) \geq f(v_{t'})$, we have $G'_2 \in \Gamma_\pi$ and $\lambda(G'_2) > \lambda(G)$ by Lemma 3.1, a contradiction. So $v_1v_{d_0+1} \in E(G)$. By similar proof, we have $v_1v_k \in E(G)$ for $k = d_0 + 2, d_0 + 3, \ldots, d_0 + d_1 - 2$, $v_2v_l \in E(G)$ for $l = d_0 + d_1 - 1, d_0 + d_1, \ldots, d_0 + d_1 + d_2 - 4$, and $v_3v_b \in E(G)$ for $b = d_0 + d_1 + d_2 - 3, d_0 + d_1 + d_2 - 2, \ldots, d_0 + d_1 + d_2 + d_3 - 5$.

Let $c_k = \sum_{m=0}^{k-1} d_m - k - 1$. Now assume $v_kv_h \in E(G)$ for $h = c_k + 1, c_k + 2, \ldots, c_k + d_k - 1$

for $4 \leq k \leq i$. If $v_{i+1}v_{c_i+d_i} \notin E(G)$, there is a vertex $v_{i+1}v_r \in E(G)$ such that $r > c_i + d_i$. Let P_{i+2} be the path from v_{i+1} to $v_{c_i+d_i}$. Without loss of generality, assume that $f(v_{c_i+d_i}) > f(v_r)$. Then $d(v_{c_i+d_i}) \geq 2$. If $v_r \in V(P_{i+2})$, there exists $v_j \notin V(P_{i+2})$ such that $v_{c_i+d_i}v_j \in E(G)$. Let $G_{i+2} = G - v_{i+1}v_r - v_{c_i+d_i}v_j + v_{i+1}v_{c_i+d_i} + v_rv_j$. Since $v_{c_i+d_i}v_j \in E(G)$ and $v_0, v_1, \ldots, v_i$ have already been adjacent to the proper vertex, we have $j > i+1$. So $f(v_{i+1}) \geq f(v_j)$. Note that $f(v_{c_i+d_i}) > f(v_r)$. Then $G_{i+2} \in \Gamma_\pi$ and $\lambda(G_{i+2}) > \lambda(G)$ by Lemma 3.1, a contradiction. If $v_r \notin V(P_{i+2})$, there exists $v_{j'} \in V(P_{i+2})$ such that $v_{c_i+d_i}v_{j'} \in E(P_{i+2})$. Then we have $j' > i+1$ by the same reason as above. Let $G'_{i+2} = G - v_{i+1}v_r - v_{c_i+d_i}v_{j'} + v_{i+1}v_{c_i+d_i} + v_rv_{j'}$. Since $f(v_{c_i+d_i}) > f(v_r)$ and $f(v_{i+1}) \geq f(v_{j'})$, we have $G'_{i+2} \in \Gamma_\pi$ and $\lambda(G'_{i+2}) > \lambda(G)$ by Lemma 3.1, a contradiction. So $v_{i+1}v_{c_i+d_i} \in E(G)$. By similar proof, we have $v_{i+1}v_g \in E(G)$ for $g = c_i+d_i+1, c_i+d_i+2, \ldots, c_i+d_i+d_{i+1}-2$. So G is isomorphic to Γ^*_π. The proof is completed. $\square$

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