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Commutative L^*-Rings

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Abstract We show that for an integral domain or a commutative local ring, it is an L^*-ring if and only if it is an O^*-ring. Some general conditions are also proved for a commutative ring that cannot be L^*.

Keywords division closed; lattice order; partial order; total order; F^*-ring; O^*-ring

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1. Introduction

In this paper, all rings are commutative and torsion-free as a group. A ring is called *unital* if it has an identity element. We review a few definitions and the reader is referred to [1,2] for more information on partially ordered rings and lattice-ordered rings (ℓ-rings). For a partially ordered ring R, the positive cone is defined as $R^+ = \{r \in R \mid r \geq 0\}$. Let $\leq$ and $\leq'$ be two partial orders on a ring R. We say that the partial order $\leq'$ extends the partial order $\leq$, if for any $a, b \in R$, $a \leq b$ implies that $a \leq' b$. Let P and P' be the positive cones of $\leq$ and $\leq'$. Then $\leq'$ extends $\leq$ just means $P \subseteq P'$.

A ring R is called an L^*-ring (O^*-ring), if each partial on R can be extended to a lattice order (total order) on R. A ring R is called a dir-L^*-ring (dir-O^*-ring) if each directed partial order on R can be extended to a lattice order (total order) [3]. Clearly an O^*-ring is L^* and dir-L^*. For a characterization of O^*-rings, the reader is referred to [4].

A partially ordered ring R is called division closed if for any $a, b \in R$, $ab > 0$ and $a > 0$ implies that $b > 0$. Each totally ordered ring is clearly division closed. In [3], a ring is defined to be consistently L^* if every partial order that is division closed can be extended to a lattice order that is division closed, and a ring is consistently O^* if each partial order that is division closed can be extended to a total order. A partially ordered ring R is called regular division closed if for any $a, b \in R$, $ab > 0$ and $a > 0$ is a regular element implies that $b > 0$. If a partially ordered ring is a domain, then clearly division closed and regular division closed are equivalent.

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Recently it is proved that a field is O^* if and only if it is L^*, if and only if it is consistently L^*, and a field is dir-O^* if and only if it is dir-L^* (see [5, Theorems 6,7]). In this paper, we continue the study on L^*-rings.

2. Main results

Let R be a commutative partially ordered ring with the positive cone that contains regular elements of R. Recall that an element a of R is called a regular element if $ab = 0$ implies that $b = 0$ for any $b \in R$. Define

$$P = \{a \in R \mid \exists \text{ a regular element } a_1 \in R^+ \text{ such that } a_1 a \in R^+\}.$$

Theorem 2.1 *Let R be a commutative partially ordered ring with R^+ containing regular element. The P defined above is a partial order on R to make R a partially ordered ring that is regular division closed, and P extends R^+.*

Proof It is clear that $P + P \subseteq P$, $PP \subseteq P$. Suppose that $a \in P \cap -P$. Then $a_1 a \in R^+$ and $a_2(-a) \in R^+$ for some regular elements $a_1, a_2 \in R^+$, so $a_1 a_2 a = 0$. Thus $a = 0$ since a_1 and a_2 are regular. Hence $P \cap -P = \{0\}$. Therefore P is a partial order on R.

Suppose that $ab > 0$ and $a > 0$ with respect to P, and a is regular. Then $b_1(ab) = c \in R^+$ and $a_1 a = d \in R^+$ for some regular elements $a_1, b_1 \in R^+$, and hence $(b_1 d)b = b_1(ab)a_1 = ca_1 \in R^+$. Since a_1 and a are regular, d is regular, so $b_1 d \in R^+$ is regular. Thus $b \in P$. Therefore P is regular division closed.

Let $z \in R^+$ be a regular element. For each $a \in R^+$, $za \in R^+$ implies that $a \in P$, so $R^+ \subseteq P$. □

Recall that R is called a domain if for any $a, b \in R$, $ab = 0$ implies that $a = 0$ or $b = 0$. The following result is an immediate consequence of Theorem 2.1.

Corollary 2.2 *Let R be a commutative domain and a partially ordered ring with $R^+ \neq \{0\}$. Then R^+ can be extended to a partial order on R that is division closed.*

Theorem 2.3 *Let R be an integral domain.*

(1) R is consistently L^* if and only if R is O^*, and hence R is L^* if and only if R is O^*.

(2) R is dir-L^* if and only if R is dir-O^*.

Proof We first notice that a unital ℓ-ring that is division closed must be an almost f-ring [2, Theorem 4.22].

(1) Suppose that R is consistently L^* and P is the positive cone of a partial order on R. Define $P_1 = \mathbb{Z}^+ 1 + P$, where $\mathbb{Z}^+$ is the set of all nonnegative integers. Then P_1 is a partial order on R, $P \subseteq P_1$ and $1 \in P_1$. We leave the verification of those facts to the reader. By Corollary 2.2, P_1 is extended to a partial order P_2 on R that is division closed. Thus P_2 is extended to a lattice order P' that is division closed by the assumption. Then (R, P') is a totally ordered ring. So R is O^*.

Now suppose that R is L^*. We show that R is consistently L^*. Then R is O^* by previous paragraph. Let P be the positive cone of a partial order on R that is division closed. By Zorn's Lemma, P is contained in a maximal partial order P_M. Certainly $1 \in P_M$. By Corollary 2.2, P_M is division closed, and P_M is a lattice order since R is L^*. Thus R is consistently L^*.

(2) The proof is similar to (1) and hence omitted. □

We look at an example.

Example 2.4 Let F be an integral domain and $R = F[x_1, ..., x_n]$ be the polynomial ring over F with n variables. Then R is not an L^*-ring. Otherwise R is O^* by Theorem 2.3, so by [6] each element in R is algebraic over $\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers, a contradiction.

Actually any integral domain that contains an element which is not algebraic over $\mathbb{Z}$ is not L^* since it is not O^*.

Let R be an ℓ-ring. Recall that an element $a \in R^+$ is called an f-element (d-element) if for any $x, y \in R$, $x \wedge y = 0$ implies that $ax \wedge y = 0$ ($ax \wedge ay = 0$). An ℓ-ring is called an f-ring (d-ring) if each positive element is an f-element (d-element). A ring R is called F^* if each partial order can be extended to a lattice order to make R into an f-ring.

A d-ring must be regular division closed. In fact, if $ab > 0$ and $a > 0$ is regular in a d-ring, then $ab = |ab| = a|b|$ implies that $b = |b| > 0$.

Theorem 2.5 *Let R be a unital commutative local ring and an ℓ-ring. Then R is regular division closed if and only if R is a d-ring.*

Proof Assume that R is regular division closed and M is the unique maximal ideal of R. We first notice that R^+ must contain a unit, otherwise $1 = 1^+ - 1^- \in M$. As a consequence, $1 > 0$. Take $a \in R^+$. If $a \notin M$, then a is a unit, so $aa^{-1} = 1 > 0$ implies that $a^{-1} > 0$ since R is regular division closed. Then a is a d-element by [2, Theorem 1.2]. If $a \in M$, $1 + a \notin M$, so $1 + a$ is a d-element. It follows that a is a d-element since $0 \leq a \leq 1 + a$. Therefore each positive element is a d-element and R is a d-ring. □

Corollary 2.6 *Let R be a unital commutative local ring.*

(1) R is L^* if and only if it is O^*.

(2) R is consistently L^* if and only if R is O^*.

(3) R is dir-L^* if and only if R is dir-O^*.

Proof (1) Suppose that R is a unital commutative local ring that is L^*. Let P be a maximal partial order on R. Then $1 \in P$ and P is a lattice order. By Theorem 2.1, P is regular division closed, so by Theorem 2.5, (R, P) is a d-ring. It is well-known that a unital d-ring must be an f-ring, and hence (R, P) is an f-ring. Thus each partial order on R can be extended to a lattice order on R to make it into an f-ring, that is, R is an F^*-ring. By [7], each F^*-ring is O^*. Therefore R is O^*.

The proof of (2) and (3) is similar to that of (1). We omit the detail and leave it to the reader. □

Example 2.7 (1) Let F be a field and $F[[x, y]]$ be the power series ring. Take $I = (xy)$, the ideal generated by xy, and then $R = F[[x, y]]/I$ is a reduced local ring with the unique maximal ideal $M = (\bar{x}, \bar{y})$, the ideal generated by $\bar{x} = x + I$ and $\bar{y} = y + I$. Then by Corollary 2.6, R is not L^* since R is not O^*.

(2) Let $R = \{a + bx \mid a, b \in \mathbb{Q}, x^2 = 0\}$. Then R is also a local ring with the unique maximal ideal $M = \mathbb{Q}x$. Then R is L^* since R is O^* (see [4]).

Theorem 2.8 *Let R be a finite direct product of unital rings $R_1, \ldots, R_k$ with $k \geq 1$. If R is L^*, then each R_i is L^*, $i = 1, \ldots, k$.*

Proof Let $R = R_1 \times R_2 \times \cdots \times R_k$ and R is L^*. Let us just show that R_1 is L^*. Take a maximal partial order P_1 on R_1. We need to show that P_1 is a lattice order on R_1. Take a maximal partial order P_i on R_i for $i = 2, \ldots, k$. Then clearly $P = P_1 \times P_2 \times \cdots \times P_k$ is a partial order on R, and hence $P \subseteq P^*$, where P^* is a maximal partial order on R, so P^* is a lattice order since R is L^*.

Define $Q_1 = \{a \in R_1 | (a, 0, \ldots, 0) \in P^*\}$. Clearly Q_1 is a partial order on R_1. If $x \in P_1$, then $(x, 0, \ldots, 0) \in P \subseteq P^*$, so $x \in Q_1$. Thus $P_1 \subseteq Q_1$, and hence $P_1 = Q_1$. We claim that Q_1 is a lattice order on R_1. Let $z \in R_1$ and

$$(z, 0, \ldots, 0) \vee_{P^*} 0 = (s_1, s_2, \ldots, s_k),$$

where $\vee_{P^*}$ is the sup with respect to the lattice order P^*. We show that s_1 is the least upper bound of z and 0 with respect to Q_1. From $(z, 0, \ldots, 0), 0 \leq_{P^*} (s_1, s_2, \ldots, s_k)$ and $(1, 0, \ldots, 0) \in P \subseteq P^*$, we have

$$(z, 0, \ldots, 0), 0 \leq_{P^*} (s_1, s_2, \ldots, s_k)(1, 0, \ldots, 0) = (s_1, 0, \ldots, 0),$$

so $(s_1, s_2, \ldots, s_k) \leq_{P^*} (s_1, 0, \ldots, 0)$. Since $(0, 1, \ldots, 1) \in P \subseteq P^*$, by multiplying it to the previous inequality, we obtain

$$(s_1, s_2, \ldots, s_k)(0, 1, \ldots, 1) = (0, s_2, \ldots, s_k) \leq_{P^*} (s_1, 0, \ldots, 0)(0, 1, \ldots, 1) = 0,$$

and hence $s_2 = \cdots = s_k = 0$. Then $(z, 0, \ldots, 0), 0 \leq_{P^*} (s_1, 0, \ldots, 0)$ implies that $s_1 \in Q_1$ and $(s_1 - z) \in Q_1$, that is, $z, 0 \leq_{Q_1} s_1$. Suppose that $s \in R_1$ such that $z, 0 \leq_{Q_1} s$. Then $(z, 0, \ldots, 0), 0 \leq_{P^*} (s, 0, \ldots, 0)$, so $(s_1, 0, \ldots, 0) \leq_{P^*} (s, 0, \ldots, 0)$. Therefore $s_1 \leq_{Q_1} s$. It follows that s_1 is the least upper bound of z and 0 with respect to Q_1, that is, $z \vee_{Q_1} 0 = s_1$. This completes the proof that Q_1 is a lattice order on R_1, so R_1 is L^*. Similarly each R_i is L^*. $\square$

A finite direct product of more than one unital ring cannot be O^*. In fact, it cannot even be a totally ordered ring since a unital totally ordered ring only has two idempotent elements 1 and 0. A finite direct product of unital commutative L^*-ring may not be L^*. This is a consequence of the following result that is motivated by [3, Proposition 2.2].

Theorem 2.9 *Let $S \neq \{0\}$ be a reduced commutative ring and $T \neq \{0\}$ be a commutative domain. Then the direct product $S \times T$ is not an L^*-ring.*

Proof We first show that T has a nontrivial partial order, namely a partial order with a nonzero positive cone. If T is unital, then $\mathbb{Z}^+1$ is a partial order on T with $1 > 0$. Suppose that T is not unital. Take $0 \neq t \in T$. If there are positive integers $n_1, \ldots, n_k$ such that $n_1 t^{a_1} + \cdots + n_k t^{a_k} = 0$ for $k > 1$ and $0 < a_1 < \cdots < a_k$, where a_i are positive integers, then T is a domain implies that for any $r \in R$, $n_1 r = zr$, where $z = -n_2 t^{(a_2 - a_1)} - \cdots - n_k t^{(a_k - a_1)}$. Hence $\mathbb{Z}^+ z$ is a nontrivial partial order. If t is not algebraic over $\mathbb{Z}^+$, then the set of all polynomials in t with no constant term in which each coefficient is a nonnegative integer is a nontrivial partial order on T. We leave the verification of this fact to the reader. Therefore T has a nontrivial maximal partial order, and we denote the positive cone of this partial order by T^+. By Corollary 2.2, T^+ is division closed.

Let $R = S \times T = \{(s,t) | s \in S, t \in T\}$ be the ring direct product of S and T. Define a positive cone P on R as follows

$$P = \{(s,t) \mid s \in S, 0 \neq t \in T^+\} \cup \{(0,0)\}.$$

It is clear that $P + P \subseteq P$, and $P \cap -P = \{0\}$. Since T is a domain, $PP \subseteq P$. Thus (R, P) is a partially ordered ring.

We claim that P is a maximal partial order on R. Suppose that $P \subseteq P^*$ and P^* is a maximal partial order on R. We show that $P = P^*$. Define

$$A = \{x \in T \mid (0,x) \in P^*\} \subseteq T.$$

Then $A + A \subseteq A$, $AA \subseteq A$, and $A \cap -A = \{0\}$, that is, A is a partial order on T. Since $P \subseteq P^*$, $T^+ \subseteq A$, and hence $T^+ = A$ since T^+ is a maximal partial order on T.

We show $P^* \subseteq P$. Take $0 \neq t \in T^+$, for $(w,z) \in P^*$. $(w,z)(0,t) = (0,zt) \in P^*$, so $zt \in A = T^+$. If $z \neq 0$, then $z \in T^+$ since T^+ is division closed, so $(w,z) \in P$ by the definition of P. If $z = 0$, then $(w,0) \in P^*$. Since $(0,0) \leq^* (w,0) \leq^* (0,t)$, where $\leq^*$ is the partial order on R with the positive cone P^*, we have

$$(0,0) \leq^* (w,0)^2 \leq^* (w,0)(0,t) = (0,0),$$

so $(w^2, 0) = (0,0)$. Thus $w^2 = 0$ and $w = 0$ since S is reduced. Hence $z = 0$ implies that $(w,z) = (0,0) \in P$. Therefore $P = P^*$ is a maximal partial order on R.

We finally show that P is not a lattice order. Suppose that P is a lattice order on R. We derive a contradiction. Assume that $(x,0) \vee (0,0) = (u,v)$, where $0 \neq x \in S$. If $(u,v) = (0,0)$, then $(x,0) \leq (0,0)$, a contradiction. Then $(u,v) \neq (0,0)$, so $0 \neq v \in T^+$. Then we have $(x,0), (0,0) \leq (u+x, v)$, so $(u,v) \leq (u+x,v)$ since (u,v) is the least supper bound of $(u,0), (0,0)$. It follows that $(x,0) \in P$, a contradiction. □

Here are some examples using Theorem 2.9.

Example 2.10 (1) Let $\mathbb{C}$ be the field of complex numbers. Then $\mathbb{C} \times \mathbb{C}$ is not L^*.

(2) Let F be a field and $G = \{1, e\}$ be a cyclic group of order 2. Then the group ring $R = F[G]$ is not L^*. In fact, $I = (1+e)F$ and $J = (1-e)F$ are two ideals of R with $R = I + J$ and $I \cap J = \{0\}$, now by Theorem 2.9, R is not L^*.

(3) A finite direct product of more than one unital integral domain is not L^*, for instance, a reduced commutative Artinian ring is not L^*.

Following result shows that under certain conditions, reduced unital commutative rings are not L^*. The proof is similiar to that of Theorem 2.9.

Theorem 2.11 *Let R be a reduced unital commutative algebra over a field F and $I_1, I_2, \ldots, I_k$ be nonzero ideals of R with $k \geq 2$. Suppose that I_1 admits a nontrivial partial order to make it into a partially ordered ring. If following conditions are satisfied, then R cannot be an L^*-ring.*

(1) $I_1 I_i = \{0\}$, $i = 2, \ldots, k$;

(2) $R = F1 + I_1 + \cdots + I_k$;

(3) $K = F1 + I_1$ *is a domain.*

Proof Take a maximal partial order P on K that contains the given nontrivial partial order on I_1. By Corollary 2.2, P is division closed and there is an nonzero element $x \in P \cap I_1$. Define

$$P^* = \{a + b \mid a \in P, a \neq 0 \ \&\ b \in I_2 + \cdots + I_k\} \cup \{0\}.$$

Clearly $P^* + P^* \subseteq P^*$, $P^* \cap -P^* = \{0\}$. Since K is a domain, $P^*P^* \subseteq P^*$. Thus P^* is a partial order on R.

We first show that P^* is a maximal partial order on R. Suppose that $P^* \subseteq Q$, where Q is a maximal partial order on R. Let $P_1 = Q \cap K$. Then P_1 is a partial order on K. If $w \in P$, then $w \in P^* \subseteq Q$, so $w \in P_1$. Hence $P \subseteq P_1$, so $P = P_1$ since P is a maximal partial order on K. Assume that $w + z \in Q$ with $w \in K$ and $z \in I_2 + \cdots + I_k$. Since $0 \neq x \in P \subseteq P^* \subseteq Q$, we have $wx = (w+z)x \in Q$ since $zx = 0$, so $wx \in Q \cap K = P_1 = P$. If $w \neq 0$, then that (K, P) is division closed implies $w > 0$ with respect to P in K, so $w + z \in P^*$. If $w = 0$, then $z \in Q$ and that $0 \leq_Q z \leq_Q x$, where $\leq_Q$ is the partial order with respect to Q, implies that $0 \leq_Q z^2 \leq_Q zx = 0$. Thus $z^2 = 0$ and hence $z = 0$, so $w + z = 0 \in P^*$. Therefore in any cases, we have $w + z \in P^*$, so $P^* = Q$, that is, P^* is a maximal partial order on R.

We claim that P^* is not a lattice order on R. Suppose that P^* is a lattice order on R and we get a contradiction. Take $0 \neq w \in I_2 + \cdots + I_k$ and suppose that $w \vee^* 0 = a + b$, where $a \in K, b \in I_2 + \cdots + I_k$, and the lattice order $\vee^*$ is with respect to P^*. If $a + b = 0$, then $w <^* 0$, where $\leq^*$ is the partial order with respect to P^*, a contradiction. Then $a + b \neq 0$, so $0 \neq a \in P$. It follows that $w, 0 \leq^* a + (b + w)$ and hence $a + b \leq^* a + (b + w)$, so $0 \leq^* w$, a contradiction. Therefore P^* is not a lattice order. $\square$

Example 2.12 Let F be a field and $A = F[x_1, \ldots, x_k]$ be the polynomial ring over F with $k \geq 2$. Suppose that I is the ideal generated by $\{x_i x_j | 1 \leq i, j \leq k, i \neq j\}$. Consider the ring $R = A/I$. Let $\overline{x_i} = x_i + I$, $i = 1, \ldots, k$ in R. Then $R = F1 + \overline{x_1}R + \cdots + \overline{x_k}R$, $(\overline{x_1}R)(\overline{x_i}R) = \{0\}$, for $i = 2, \ldots, k$, and $F1 + \overline{x_1}R$ is a domain. Thus R cannot be L^* by Theorem 2.11.

An important property of O^*-ring is that each nilpotent element has index at most two [6]. It is not known if this is true for an L^*-ring. In the following, we prove a similar result on L^*-ring. A unital partially ordered ring is called ℓ-unital if the identity element is positive.

Lemma 2.13 *Let R be an ℓ-unital commutative partially ordered ring that is regular division closed. If a is a positive nilpotent element, then $a < 1$.*

Proof We prove the result by the induction on the index of nilpotency. If $a^2 = 0$, then $(1-a)(1+a) = 1$ and regular division closed property implies that $1 - a > 0$, that is, $a < 1$. Now suppose it is true for index less than or equal to n and suppose $a^{n+1} = 0$. Then a^2 has the index less than or equal to n, so $a^2 < 1$. Thus $(1-a)(1+a) = 1 - a^2 > 0$ and regular division closed property gives us $1 - a > 0$, and hence $a < 1$. □

For an element $a \in R$, $\mathrm{Ann}(a)$ denotes the annihilator of a, that is, $\mathrm{Ann}(a) = \{w \in R | aw = 0\}$. Define $f(R) = \{a \in R | |a| \text{ is an } f\text{-element}\}$. Then $f(R)$ is an f-ring.

Theorem 2.14 *Let R be a unital commutative ring and $N = \{x \in R | x \text{ is nilpotent}\}$. Suppose there exists $a \in R$ such that $a^k = 0$ for some integer $k \geq 3$ and $Ann(a^{k-1}) \subseteq N$. Then R is not an L^*-ring.*

Proof Suppose that R is an L^*-ring. We derive a contradiction.

(1) First assume that k is odd. Let $b = -a^{k-1}$. Then $b^2 = 0$ and hence $\mathbb{Z}^+ b$ is a partial order on R. By Zorn's Lemma, $\mathbb{Z}^+ b \subseteq P$, where P is a maximal partial order on R, and hence P is a lattice order on R since R is L^*. Thus by Theorem 2.1, P is a regular division closed lattice order on R.

By Lemma 2.13, b is an f-element since $0 \leq b < 1$. Then $(a \vee 0)b = (ab \vee 0) = 0$, so $(a \vee 0) \in \mathrm{Ann}(a^{k-1}) \subseteq N$. Since $a \vee 0 \in N$, $|a| \in N$, so $|a|$ is nilpotent. By Lemma 2.13 again, $|a| < 1$, so $a \in f(R)$. It follows $a^{k-1} \geq 0$ since $f(R)$ is an f-ring and $k-1$ is even, a contradiction with $b = -a^{k-1} \geq 0$.

(2) Now assume that k is even, so $k \geq 4$. Let $b_1 = -a^{k-1}$ and $b_2 = -a^{k-2}$. Then $b_1^2 = b_1 b_2 = b_2^2 = 0$, so $\mathbb{Z}^+ b_1 + \mathbb{Z}^+ b_2$ is a partial order on R. Similar to the proof in (1), $\mathbb{Z}^+ b_1 + \mathbb{Z}^+ b_2 \subseteq P$, where P is a lattice order that is regular division closed. Also by the same proof in (1), $a \in f(R)$ and hence $a^{k-2} > 0$, which contradicts with $b_2 = -a^{k-2} > 0$.

Therefore R cannot be an L^*-ring. □

Example 2.15 Let F be a field and $F[x]$ be the polynomial ring over F and $n \geq 3$. Suppose (x^n) is the ideal generated by x^n and $R = F[x]/(x^n)$. Let $\bar{x} = x + (x^n)$ in R. Then $\bar{x}^{n-1} \neq 0$ and $\mathrm{Ann}(\bar{x}^{n-1}) = N$. Thus R is not L^*.

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