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On the Addition of Two Cubes of Units and Nonunits mod p^α

Cuifang SUN*, **Zhi CHENG**
School of Mathematics and Computer Science, Anhui Normal University, Anhui 241003*, P. R. China*

Abstract Let $p \equiv 2 \pmod 3$ be an odd prime and α be a positive integer. In this paper, for any integer c, we obtain a formula for the number of solutions of the cubic congruence $x^3 + y^3 \equiv c \ (\mathrm{mod}\, p^\alpha)$ with x, y units, nonunits and mixed pairs, respectively. We resolve a problem posed by Yang and Tang.

Keywords residue classes; cubes of units; exponential sum

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1. Introduction

For any positive integer n, let $\mathbb{Z}_n = \{1, 2, \ldots, n\}$ be the ring of residue classes modulo n and $\mathbb{Z}_n^*$ be the group of its units, i.e., $\mathbb{Z}_n^* = \{s : s \in \mathbb{Z}_n \text{ and } \gcd(s, n) = 1\}$. Let $k \geq 1$ be an integer. For any integer c, we define

$$\begin{aligned}
U(n, k; , c) : &= \{(x, y) \in (\mathbb{Z}_n^*)^2 : x^k + y^k \equiv c \ (\mathrm{mod}\, n)\},\\
N(n, k; , c) : &= \{(x, y) \in (\mathbb{Z}_n \backslash \mathbb{Z}_n^*)^2 : x^k + y^k \equiv c \ (\mathrm{mod}\, n)\},\\
UN(n, k; , c) : &= \{(x, y) \in \mathbb{Z}_n^* \times (\mathbb{Z}_n \backslash \mathbb{Z}_n^*) : x^k + y^k \equiv c \ (\mathrm{mod}\, n)\},\\
NU(n, k; , c) : &= \{(x, y) \in (\mathbb{Z}_n \backslash \mathbb{Z}_n^*) \times \mathbb{Z}_n^* : x^k + y^k \equiv c \ (\mathrm{mod}\, n)\}.
\end{aligned}$$

In 2000, Deaconescu [1] obtained a formula for $|U(n, 1; c)|$. In 2009, Sander [2] gave a new proof of the formula for $|U(n, 1; c)|$ by using multiplicativity of $|U(n, 1; c)|$ with respect to n. Beyond this, the values of $|N(n, 1; c)|$, $|UN(n, 1; c)|$ and $|NU(n, 1; c)|$ were also obtained. Tóth [3] deduced formulas for the number of solutions of the quadratic congruence

$$a_1 x_1^2 + \ldots + a_t x_t^2 \equiv c \ (\mathrm{mod}\, n) \ \text{ with } \ x_1, \ldots, x_t \in \mathbb{Z}_n$$

in some special cases of t and c. Yang and Tang [4] gave a formula for $|U(n, 2; c)|$, $|N(n, 2; c)|$, $|UN(n, 2; c)|$ and $|NU(n, 2; c)|$, respectively. They also posed several problems for further research. Recently, Sun and Cheng [5] obtained a formula for the number of representations of c as the sum of two weighted squares of units modulo n. For the number of solutions of diagonal equations over finite fields, one can refer to [6, Chapter 10] and [7, Chapter 8].

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* Corresponding author
E-mail address: cuifangsun@163.com (Cuifang SUN); chengzhimath@126.com (Zhi CHENG)

Throughout this paper, we use the following notation: $e(x) = e^{2\pi ix}$; p always denotes an odd prime; $p^\alpha \| n$ denotes $p^\alpha | n$ while $p^{\alpha+1} \nmid n$; $\phi(n)$ is the Euler's totient function; $\sum\limits_{x=1}^{q}{}'$ denotes the summation over all integers x with $1 \le x \le q$ such that $\gcd(x, q) = 1$.

In this paper, we study the cubic congruence $x^3 + y^3 \equiv c \pmod{p^\alpha}$ and give a formula for $|U(p^\alpha, 3; c)|$, $|N(p^\alpha, 3; c)|$, $|UN(p^\alpha, 3; c)|$ and $|NU(p^\alpha, 3; c)|$, respectively. This solves a problem posed by Yang and Tang [4].

Theorem 1.1 *Let $p \equiv 2 \pmod 3$ be an odd prime and α be a positive integer. For any integer c, we have*

$$|U(p^\alpha, 3; c)| = \begin{cases} p^{\alpha-1}(p-1), & \text{if } p \mid c, \\ p^{\alpha-1}(p-2), & \text{if } p \nmid c. \end{cases}$$

Theorem 1.2 *Let $p \equiv 2 \pmod 3$ be an odd prime and α be a positive integer. Let $c = p^\beta c_1$ with $\beta \ge 0$ and $p \nmid c_1$. We have*

$$|N(p^\alpha, 3; c)| = \begin{cases} 0, & \text{if } \beta < 3; \\ p^{2\alpha-2}, & \text{if } \beta \ge 3 \text{ and } \alpha \le 3; \\ \sum\limits_{j=2}^{m} p^{\alpha-2j} T(j) + p^{\alpha+1}, & \text{if } \beta \ge 3 \text{ and } \alpha = 3m \text{ with } m \ge 2; \\ \sum\limits_{j=2}^{m} p^{\alpha-2j} T(j) + p^{\alpha+1}, & \text{if } \beta \ge 3 \text{ and } \alpha = 3m+1 \text{ with } m \ge 1; \\ p^m \sum\limits_{t_1=1}^{p^\alpha}{}' e(\frac{-ct_1}{p^\alpha}) + \sum\limits_{j=2}^{m} p^{\alpha-2j} T(j) + p^{\alpha+1}, & \text{if } \beta \ge 3 \text{ and } \alpha = 3m+2 \text{ with } m \ge 1, \end{cases}$$

where

$$T(j) = \sum_{t_1=1}^{p^{3j}}{}' e(\frac{-ct_1}{p^{3j}}) + \sum_{t_1=1}^{p^{3j-1}}{}' e(\frac{-ct_1}{p^{3j-1}}).$$

Theorem 1.3 *Let $p \equiv 2 \pmod 3$ be an odd prime and α be a positive integer. For any integer c, we have*

$$|UN(p^\alpha, 3; c)| = \begin{cases} 0, & \text{if } p \mid c, \\ p^{\alpha-1}, & \text{if } p \nmid c. \end{cases}$$

Remark 1.4 Since $(x, y) \mapsto (y, x)$ is a one-to-one correspondence between $UN(p^\alpha, 3; c)$ and $NU(p^\alpha, 3; c)$, we have $|UN(p^\alpha, 3; c)| = |NU(p^\alpha, 3; c)|$.

2. Preliminary lemmas

Lemma 2.1 ([7, Proposotion 4.2.1]) *If n possesses primitive root and $\gcd(a, n) = 1$, then a is m-th power residue mod n if and only if $a^{\phi(n)/d} \equiv 1 \pmod n$, where $d = \gcd(m, \phi(n))$. Moreover, if $x^m \equiv a \pmod n$ is solvable, there are exactly $\gcd(m, \phi(n))$ solutions.*

Lemma 2.2 *Let p be an odd prime and α be a positive integer. Let $t = p^\gamma t_1$ with $0 \le \gamma \le \alpha - 1$*

and $p \nmid t_1$. Then for $0 \le \gamma \le \alpha - 1$, we have

$$\sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{xt}{p^\alpha}) = \begin{cases} -p^{\alpha-1}, & \text{if } \gamma = \alpha - 1, \\ 0, & \text{if } 0 \le \gamma < \alpha - 1. \end{cases}$$

Proof If $\gamma = \alpha - 1$, then

$$\sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{xt}{p^\alpha}) = \sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{xt_1}{p}) = p^{\alpha-1}\sideset{}{'}\sum_{x=1}^{p} e(\frac{xt_1}{p}) = p^{\alpha-1}\Big(\sum_{x=1}^{p} e(\frac{xt_1}{p}) - 1\Big) = -p^{\alpha-1}.$$

If $\gamma < \alpha - 1$, then

$$\begin{aligned} \sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{xt}{p^\alpha}) &= \sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{xt_1}{p^{\alpha-\gamma}}) = p^\gamma \sideset{}{'}\sum_{x=1}^{p^{\alpha-\gamma}} e(\frac{xt_1}{p^{\alpha-\gamma}}) \\ &= p^\gamma\Big(\sum_{x=1}^{p^{\alpha-\gamma}} e(\frac{xt_1}{p^{\alpha-\gamma}}) - \sum_{\substack{x=1 \\ p\|x}}^{p^{\alpha-\gamma}} e(\frac{xt_1}{p^{\alpha-\gamma}}) - \sum_{\substack{x=1 \\ p^2\|x}}^{p^{\alpha-\gamma}} e(\frac{xt_1}{p^{\alpha-\gamma}}) - \cdots - \sum_{\substack{x=1 \\ p^{\alpha-\gamma-1}\|x}}^{p^{\alpha-\gamma}} e(\frac{xt_1}{p^{\alpha-\gamma}}) - 1\Big) \\ &= -p^\gamma\Big(\sideset{}{'}\sum_{x=1}^{p^{\alpha-\gamma-1}} e(\frac{xt_1}{p^{\alpha-\gamma-1}}) + \sideset{}{'}\sum_{x=1}^{p^{\alpha-\gamma-2}} e(\frac{xt_1}{p^{\alpha-\gamma-2}}) + \cdots + \sideset{}{'}\sum_{x=1}^{p} e(\frac{xt_1}{p}) + 1\Big) \\ &= 0. \end{aligned}$$

This completes the proof of Lemma 2.2. □

3. Proofs

Proof of Theorem 1.1 Suppose a is an integer with $\gcd(a,p) = 1$. By Lemma 2.1, we know the congruence $x^3 \equiv a \pmod{p^\alpha}$ has a unique solution since $\gcd(3,\phi(p^\alpha)) = 1$. Further, if $x_1,\dots,x_{\phi(p^\alpha)}$ forms a reduced residue system modulo p^α, then $x_1^3,\dots,x_{\phi(p^\alpha)}^3$ again forms a reduced residue system modulo p^α. By Lemma 2.2, we have

$$\begin{aligned} |U(p^\alpha,3;c)| =& \frac{1}{p^\alpha}\sideset{}{'}\sum_{x=1}^{p^\alpha}\sideset{}{'}\sum_{y=1}^{p^\alpha}\sum_{t=1}^{p^\alpha} e(\frac{(x^3+y^3-c)t}{p^\alpha}) = \frac{1}{p^\alpha}\sum_{t=1}^{p^\alpha}\Big(\sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{x^3t}{p^\alpha})\Big)^2 e(\frac{-ct}{p^\alpha}) \\ =& \frac{1}{p^\alpha}\sideset{}{'}\sum_{t=1}^{p^\alpha}\Big(\sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{x^3t}{p^\alpha})\Big)^2 e(\frac{-ct}{p^\alpha}) + \frac{1}{p^\alpha}\sum_{\substack{t=1 \\ p\|t}}^{p^\alpha}\Big(\sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{x^3t}{p^\alpha})\Big)^2 e(\frac{-ct}{p^\alpha}) + \cdots + \\ & \frac{1}{p^\alpha}\sum_{\substack{t=1 \\ p^{\alpha-1}\|t}}^{p^\alpha}\Big(\sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{x^3t}{p^\alpha})\Big)^2 e(\frac{-ct}{p^\alpha}) + \frac{1}{p^\alpha}(\phi(p^\alpha))^2 \\ =& \frac{1}{p^\alpha}\sideset{}{'}\sum_{t=1}^{p^\alpha}\Big(\sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{xt}{p^\alpha})\Big)^2 e(\frac{-ct}{p^\alpha}) + \frac{1}{p^\alpha}\sideset{}{'}\sum_{t=1}^{p^{\alpha-1}}\Big(\sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{xt}{p^{\alpha-1}})\Big)^2 e(\frac{-ct}{p^{\alpha-1}}) + \cdots + \\ & \frac{1}{p^\alpha}\sideset{}{'}\sum_{t=1}^{p}\Big(\sideset{}{'}\sum_{x=1}^{p^\alpha} e(\frac{xt}{p})\Big)^2 e(\frac{-ct}{p}) + p^{\alpha-2}(p-1)^2 \end{aligned}$$

$$=p^{\alpha-2}\sum_{t=1}^{p}{}' e(\frac{-ct}{p})+p^{\alpha-2}(p-1)^2.$$

If $p \mid c$, then $|U(p^\alpha,3;c)| = p^{\alpha-2}(p-1)+p^{\alpha-2}(p-1)^2 = p^{\alpha-1}(p-1)$. If $p\nmid c$, then $|U(p^\alpha,3;c)| = -p^{\alpha-2}+p^{\alpha-2}(p-1)^2 = p^{\alpha-1}(p-2)$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 If $x,y\in \mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}$, then $x^3+y^3\equiv 0 \pmod{p^3}$. Thus we only need to calculate $N(p^\alpha,3;c)$ on the condition that $p^3\mid c$. Let $c=p^\beta c_1$ with $\beta\geq 3$ and $p\nmid c_1$. Then

$$|N(p^\alpha,3;c)| =\frac{1}{p^\alpha}\sum_{\substack{x=1\\ x\in\mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha}\sum_{\substack{y=1\\ y\in\mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha}\sum_{t=1}^{p^\alpha} e(\frac{(x^3+y^3-c)t}{p^\alpha})$$
$$=\frac{1}{p^\alpha}\sum_{t=1}^{p^\alpha-1}\Big(\sum_{\substack{x=1\\ x\in\mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha} e(\frac{x^3t}{p^\alpha})\Big)^2 e(\frac{-ct}{p^\alpha})+p^{\alpha-2}.$$

Now we divide into two cases according to the value of α.

Case 1 $\alpha\leq 3$. Since $\beta\geq 3$, we have

$$|N(p^\alpha,3;c)| = \frac{1}{p^\alpha}\sum_{t=1}^{p^\alpha-1}\Big(\sum_{\substack{x=1\\ x\in\mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha} 1\Big)^2+p^{\alpha-2}=p^{2\alpha-2}.$$

Case 2 $\alpha>3$. Now we first calculate the inner summation $\sum\limits_{\substack{x=1\\ x\in\mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha} e(\frac{x^3t}{p^\alpha})$. Let $t=p^\gamma t_1$ with $0\leq\gamma<\alpha$ and $p\nmid t_1$. If $\alpha-3\leq\gamma\leq\alpha-1$, then

$$\sum_{\substack{x=1\\ x\in\mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha} e(\frac{x^3t}{p^\alpha}) = \sum_{\substack{x=1\\ x\in\mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha} e(\frac{x^3t_1}{p^{\alpha-\gamma}}) = \sum_{\substack{x=1\\ x\in\mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha} 1 = p^{\alpha-1}.$$

If $0\leq\gamma\leq\alpha-4$, then

$$\sum_{\substack{x=1\\ x\in\mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha} e(\frac{x^3t}{p^\alpha}) = \sum_{\substack{x=1\\ x\in\mathbb{Z}_{p^\alpha}\backslash\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha} e(\frac{x^3t_1}{p^{\alpha-\gamma}}) = p^\gamma \sum_{\substack{x=1\\ x\in\mathbb{Z}_{p^{\alpha-\gamma}}\backslash\mathbb{Z}^*_{p^{\alpha-\gamma}}}}^{p^{\alpha-\gamma}} e(\frac{x^3t_1}{p^{\alpha-\gamma}})$$
$$=p^\gamma\Big(\sum_{\substack{x=1\\ p\| x}}^{p^{\alpha-\gamma}} e(\frac{x^3t_1}{p^{\alpha-\gamma}})+\sum_{\substack{x=1\\ p^2\| x}}^{p^{\alpha-\gamma}} e(\frac{x^3t_1}{p^{\alpha-\gamma}})+\cdots+\sum_{\substack{x=1\\ p^{\alpha-\gamma-1}\| x}}^{p^{\alpha-\gamma}} e(\frac{x^3t_1}{p^{\alpha-\gamma}})+1\Big)$$
$$=p^\gamma\Big(\sum_{x=1}^{p^{\alpha-\gamma-1}}{}' e(\frac{x^3t_1}{p^{\alpha-\gamma-3}})+\sum_{x=1}^{p^{\alpha-\gamma-2}}{}' e(\frac{x^3t_1}{p^{\alpha-\gamma-6}})+\cdots+\sum_{x=1}^{p}{}' e(\frac{x^3t_1}{p^{\alpha-\gamma-3(\alpha-\gamma-1)}})+1\Big).$$

Let

$$S(\gamma)=\sum_{x=1}^{p^{\alpha-\gamma-1}}{}' e(\frac{x^3t_1}{p^{\alpha-\gamma-3}})+\sum_{x=1}^{p^{\alpha-\gamma-2}}{}' e(\frac{x^3t_1}{p^{\alpha-\gamma-6}})+\cdots+\sum_{x=1}^{p}{}' e(\frac{x^3t_1}{p^{\alpha-\gamma-3(\alpha-\gamma-1)}})+1.$$

Since $\beta \geq 3$, we have

$$\begin{aligned}
&|N(p^\alpha,3;c)|\\
&=\frac{1}{p^\alpha}\sideset{}{'}\sum_{t_1=1}^{p^\alpha}S^2(0)e(\frac{-ct_1}{p^\alpha})+\frac{1}{p^\alpha}\sideset{}{'}\sum_{t_1=1}^{p^{\alpha-1}}p^2S^2(1)e(\frac{-ct_1}{p^{\alpha-1}})+\cdots+\frac{1}{p^\alpha}\sideset{}{'}\sum_{t_1=1}^{p^4}p^{2\alpha-8}S^2(\alpha-4)e(\frac{-ct_1}{p^4})+\\
&\quad\frac{1}{p^\alpha}\sideset{}{'}\sum_{t_1=1}^{p^3}p^{2\alpha-2}e(\frac{-ct_1}{p^3})+\frac{1}{p^\alpha}\sideset{}{'}\sum_{t_1=1}^{p^2}p^{2\alpha-2}e(\frac{-ct_1}{p^2})+\frac{1}{p^\alpha}\sideset{}{'}\sum_{t_1=1}^{p}p^{2\alpha-2}e(\frac{-ct_1}{p})+p^{\alpha-2}\\
&=p^{-\alpha}\sideset{}{'}\sum_{t_1=1}^{p^\alpha}S^2(0)e(\frac{-ct_1}{p^\alpha})+p^{-\alpha+2}\sideset{}{'}\sum_{t_1=1}^{p^{\alpha-1}}S^2(1)e(\frac{-ct_1}{p^{\alpha-1}})+\cdots+p^{\alpha-8}\sideset{}{'}\sum_{t_1=1}^{p^4}S^2(\alpha-4)e(\frac{-ct_1}{p^4})+\\
&\quad p^{\alpha+1}.
\end{aligned}$$

Subcase 2.1 Let $\alpha=3m$ with $m\geq 2$. For $j=0,1,\ldots,m-2$ and $i=0,1$, we have

$$\begin{aligned}
S(3j+i)&=\sideset{}{'}\sum_{x=1}^{p^{3(m-j)-i-(m-j)}}1+\cdots+\sideset{}{'}\sum_{x=1}^{p}1+1=p^{2(m-j)-i},\\
S(3j+2)&=\sideset{}{'}\sum_{x=1}^{p^{3(m-j)-2-(m-j-1)}}e(\frac{x^3t_1}{p})+\sideset{}{'}\sum_{x=1}^{p^{3(m-j)-2-(m-j)}}1+\cdots+\sideset{}{'}\sum_{x=1}^{p}1+1\\
&=-p^{2(m-j-1)}+p^{2(m-j-1)}=0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|N(p^\alpha,3;c)|=&p^m\Big(\sideset{}{'}\sum_{t_1=1}^{p^{3m}}e(\frac{-ct_1}{p^{3m}})+\sideset{}{'}\sum_{t_1=1}^{p^{3m-1}}e(\frac{-ct_1}{p^{3m-1}})\Big)+\\
&p^{m+2}\Big(\sideset{}{'}\sum_{t_1=1}^{p^{3(m-1)}}e(\frac{-ct_1}{p^{3(m-1)}})+\sideset{}{'}\sum_{t_1=1}^{p^{3(m-1)-1}}e(\frac{-ct_1}{p^{3(m-1)-1}})\Big)+\cdots+\\
&p^{3m-4}\Big(\sideset{}{'}\sum_{t_1=1}^{p^6}e(\frac{-ct_1}{p^6})+\sideset{}{'}\sum_{t_1=1}^{p^5}e(\frac{-ct_1}{p^5})\Big)+p^{\alpha+1}\\
=&\sum_{j=2}^{m}p^{\alpha-2j}T(j)+p^{\alpha+1}.
\end{aligned}$$

Subcase 2.2 Let $\alpha=3m+1$ with $m\geq 1$. For $j=0,1,\ldots,m-1$, by Lemma 2.2, we have

$$S(3j)=\sideset{}{'}\sum_{x=1}^{p^{3(m-j)+1-(m-j)}}e(\frac{x^3t_1}{p})+\sideset{}{'}\sum_{x=1}^{p^{3(m-j)+1-(m-j+1)}}1+\cdots+\sideset{}{'}\sum_{x=1}^{p}1+1=0.$$

For $j=0,1,\ldots,m-2$ and $i=1,2$, we have

$$S(3j+i)=\sideset{}{'}\sum_{x=1}^{p^{3(m-j)+(1-i)-(m-j)}}1+\cdots+\sideset{}{'}\sum_{x=1}^{p}1+1=p^{2(m-j)+(1-i)}.$$

Therefore, we have

$$\begin{aligned}|N(p^\alpha,3;c)| =& p^{m+1}\Big(\sum_{t_1=1}^{p^{3m}}{}' e(\frac{-ct_1}{p^{3m}}) + \sum_{t_1=1}^{p^{3m-1}}{}' e(\frac{-ct_1}{p^{3m-1}})\Big) + \\ & p^{m+3}\Big(\sum_{t_1=1}^{p^{3(m-1)}}{}' e(\frac{-ct_1}{p^{3(m-1)}}) + \sum_{t_1=1}^{p^{3(m-1)-1}}{}' e(\frac{-ct_1}{p^{3(m-1)-1}})\Big) + \cdots + \\ & p^{3m-3}\Big(\sum_{t_1=1}^{p^{6}}{}' e(\frac{-ct_1}{p^{6}}) + \sum_{t_1=1}^{p^{5}}{}' e(\frac{-ct_1}{p^{5}})\Big) + p^{\alpha+1} \\ =& \sum_{j=2}^{m} p^{\alpha-2j}T(j) + p^{\alpha+1}.\end{aligned}$$

Subcase 2.3 Let $\alpha = 3m+2$ with $m \geq 1$. For $j = 0, 1, \ldots, m-1$, by Lemma 2.2, we have

$$S(3j) = \sum_{x=1}^{p^{3(m-j)+2-(m-j+1)}}{}' 1 + \cdots + \sum_{x=1}^{p}{}' 1 + 1 = p^{2(m-j)+1},$$

$$S(3j+1) = \sum_{x=1}^{p^{3(m-j)+1-(m-j)}}{}' e(\frac{x^3t_1}{p}) + \sum_{x=1}^{p^{3(m-j)+1-(m-j+1)}}{}' 1 + \cdots + \sum_{x=1}^{p}{}' 1 + 1 = 0.$$

For $j = 0, 1, \ldots, m-2$, we have

$$S(3j+2) = \sum_{x=1}^{p^{3(m-j)-(m-j)}}{}' 1 + \cdots + \sum_{x=1}^{p}{}' 1 + 1 = p^{2(m-j)}.$$

Therefore, we have

$$\begin{aligned}|N(p^\alpha,3;c)| =& p^{m}\sum_{t_1=1}^{p^{3m+2}}{}' e(\frac{-ct_1}{p^{3m+2}}) + p^{m+2}\Big(\sum_{t_1=1}^{p^{3m}}{}' e(\frac{-ct_1}{p^{3m}}) + \sum_{t_1=1}^{p^{3m-1}}{}' e(\frac{-ct_1}{p^{3m-1}})\Big) + \\ & p^{m+4}\Big(\sum_{t_1=1}^{p^{3(m-1)}}{}' e(\frac{-ct_1}{p^{3(m-1)}}) + \sum_{t_1=1}^{p^{3(m-1)-1}}{}' e(\frac{-ct_1}{p^{3(m-1)-1}})\Big) + \cdots + \\ & p^{3m-2}\Big(\sum_{t_1=1}^{p^{6}}{}' e(\frac{-ct_1}{p^{6}}) + \sum_{t_1=1}^{p^{5}}{}' e(\frac{-ct_1}{p^{5}})\Big) + p^{\alpha+1} \\ =& p^{m}\sum_{t_1=1}^{p^{3m+2}}{}' e(\frac{-ct_1}{p^{3m+2}}) + \sum_{j=2}^{m} p^{\alpha-2j}T(j) + p^{\alpha+1}.\end{aligned}$$

This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. By Theorems 1.1, 1.2 and Lemma 2.2, we have

$$|UN(p^\alpha,3;c)| = \frac{1}{p^\alpha}\sum_{x=1}^{p^\alpha}{}' \sum_{\substack{y=1 \\ y\in\mathbb{Z}_{p^\alpha}\setminus\mathbb{Z}^*_{p^\alpha}}}^{p^\alpha} \sum_{t=1}^{p^\alpha} e(\frac{(x^3+y^3-c)t}{p^\alpha})$$

$$=\frac{1}{p^\alpha}\sum_{t=1}^{p^\alpha}{}'\Big(\sum_{x=1}^{p^\alpha}{}'e(\frac{x^3t}{p^\alpha})\sum_{\substack{y=1\\ y\in\mathbb{Z}_{p^\alpha}\setminus\mathbb{Z}_{p^\alpha}^*}}^{p^\alpha}e(\frac{y^3t}{p^\alpha})\Big)e(\frac{-ct}{p^\alpha})+$$

$$\frac{1}{p^\alpha}\sum_{\substack{t=1\\ p\|t}}^{p^\alpha}\Big(\sum_{x=1}^{p^\alpha}{}'e(\frac{x^3t}{p^\alpha})\sum_{\substack{y=1\\ y\in\mathbb{Z}_{p^\alpha}\setminus\mathbb{Z}_{p^\alpha}^*}}^{p^\alpha}e(\frac{y^3t}{p^\alpha})\Big)e(\frac{-ct}{p^\alpha})+\cdots+$$

$$\frac{1}{p^\alpha}\sum_{\substack{t=1\\ p^{\alpha-1}\|t}}^{p^\alpha}\Big(\sum_{x=1}^{p^\alpha}{}'e(\frac{x^3t}{p^\alpha})\sum_{\substack{y=1\\ y\in\mathbb{Z}_{p^\alpha}\setminus\mathbb{Z}_{p^\alpha}^*}}^{p^\alpha}e(\frac{y^3t}{p^\alpha})\Big)e(\frac{-ct}{p^\alpha})+p^{\alpha-2}(p-1)$$

$$=-p^{\alpha-2}\sum_{t_1=1}^{p}{}'e(\frac{-ct_1}{p})+p^{\alpha-2}(p-1).$$

If $p\mid c$, then $|UN(p^\alpha,3;c)|=-p^{\alpha-2}(p-1)+p^{\alpha-2}(p-1)=0$. If $p\nmid c$, then $|UN(p^\alpha,3;c)|=p^{\alpha-2}+p^{\alpha-2}(p-1)=p^{\alpha-1}$. This completes the proof of Theorem 1.3. □

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