Journal of Mathematical Research with Applications
May, 2017, Vol. 37, No. 3, pp. 290–298
DOI:10.3770/j.issn:2095-2651.2017.03.006
Http://jmre.dlut.edu.cn

Coefficient Bounds for a New Subclass of Bi-Univalent Functions Defined by Salagean Operator

Fan CHEN[1], **Xiaofei LI**[2,3,*]

1. *College of Engineering and Technology, Yangtze University, Hubei* 434000*, P. R. China;*
2. *School of Information and Mathematics, Yangtze University, Hubei* 434000*, P. R. China;*
3. *Department of Mathematics, University of Macau, Macau* 999078*, P. R. China*

Abstract In this paper, a new subclass $\mathcal{N}_{\Sigma}^{h,p}(m,\lambda,\mu)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$ is defined by salagean operator. We obtain coefficients bounds $|a_2|$ and $|a_3|$ for functions of the class. Moreover, we verify Brannan and Clunie's conjecture $|a_2|\le\sqrt{2}$ for some of our classes. The results in this paper extend many results recently researched by many authors.

Keywords analytic functions; univalent functions; bi-univalent functions; coefficient bounds; Salagean operator

MR(2010) Subject Classification 30C45

1. Introduction

In this paper, we denote by $\mathcal{A}$ the class of functions of the form:

$$f(z)=z+\sum_{k=1}^{\infty}a_{k+1}z^{k+1} \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. Let $\mathcal{S}$ denote the class of functions in class $\mathcal{A}$ which are univalent in $\mathbb{U}$. For $f(z)\in\mathcal{A}$, Salagean operator is defined by

$$\begin{aligned} D^0f(z)&=f(z),\\ D^1f(z)&=Df(z)=zf'(z),\\ &\cdots\\ D^mf(z)&=D(D^{m-1}f(z)),\quad m\in\mathbb{N}=\{1,2,\ldots\}.\end{aligned}$$

If $f(z)\in\mathcal{A}$ is given by (1.1), then we see that

$$D^mf(z)=z+\sum_{n=1}^{\infty}(1+n)^m a_{n+1}z^{n+1},\quad m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}.$$

It is well known that every function $f\in\mathcal{S}$ has an inverse $g=f^{-1}$, which is defined by

$$f^{-1}(f(z))=z,\quad z\in\mathbb{U},$$

Received July 29, 2016; Accepted September 7, 2016
Supported by the National Natural Science Foundation of China (Grant No. 11401186) and the Research Fund from Engineering and Technology College Yangtze University (Grant No. 15J0802).
* Corresponding author
E-mail address: fanchen20160725@163.com (Fan CHEN); lxfei0828@163.com (Xiaofei LI)

$$f(f^{-1}(w)) = w, \quad |w| < r_0(f);\ r_0(f) \geq \frac{1}{4}.$$

In fact, the inverse function f^{-1} is given by

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{S}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote by Σ the class of all bi-univalent functions in $\mathbb{U}$ given by (1.1). In 1967, Lewin [1] first introduced the class Σ of bi-univalent functions and showed that $|a_2| \leq 1.51$ for every $f \in \Sigma$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [3] proved that $\max|a_2| = \frac{4}{3}$ for $f \in \Sigma$. In 2010, Srivastava et al. [4] introduced subclasses of bi-univalent function $\mathcal{H}_\Sigma(\alpha)$ and $\mathcal{H}_\Sigma(\beta)$, and obtained non-sharp estimates on the coefficients $|a_2|$ and $|a_3|$.

Definition 1.1 ([4]) *A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma(\alpha)$ if the following conditions are satisfied:*

$$f \in \Sigma \ \ \textit{and} \ \ |\arg f'(z)| < \frac{\alpha\pi}{2}, \quad 0 < \alpha \leq 1;\ z \in \mathbb{U},$$

$$|\arg g'(w)| < \frac{\alpha\pi}{2}, \quad 0 < \alpha \leq 1;\ w \in \mathbb{U},$$

where the function g is defined by (1.2).

Definition 1.2 ([4]) *A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{H}_\Sigma(\beta)$ if the following conditions are satisfied:*

$$f \in \Sigma, \ \ \textit{and} \ \ \Re(f'(z)) > \beta, \quad 0 \leq \beta < 1;\ z \in \mathbb{U},$$

$$\Re(g'(w)) > \beta, \quad 0 \leq \beta < 1;\ w \in \mathbb{U},$$

where the function g is defined by (1.2).

Frasin and Aouf [5] introduced the following subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the coefficients $|a_2|$ and $|a_3|$.

Definition 1.3 ([5]) *A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma \ \ \textit{and} \ \ \Big|\arg((1-\lambda)\frac{f(z)}{z} + \lambda f'(z))\Big| < \frac{\alpha\pi}{2}, \quad 0 < \alpha \leq 1;\ \lambda \geq 1;\ z \in \mathbb{U},$$

$$\Big|\arg((1-\lambda)\frac{g(w)}{w} + \lambda g'(w))\Big| < \frac{\alpha\pi}{2}, \quad 0 < \alpha \leq 1;\ \lambda \geq 1;\ w \in \mathbb{U},$$

where the function g is defined by (1.2).

Definition 1.4 ([5]) *A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{N}_\Sigma^\mu(\beta, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma \ \ \textit{and} \ \ \Re\big((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\big) > \beta, \quad 0 \leq \beta < 1;\ \lambda \geq 1;\ z \in \mathbb{U},$$

$$\Re\big((1-\lambda)\frac{g(w)}{w} + \lambda g'(w))\big) > \beta, \quad 0 \leq \beta < 1;\ \lambda \geq 1;\ \mu \geq 0;\ w \in \mathbb{U},$$

where the function g is defined by (1.2).

Xu et al. [6] introduced an interesting general subclass $\mathcal{B}_\Sigma^{h,p}(\lambda)$ of the analytic function class $\mathcal{A}$, and obtained the coefficients estimates on $|a_2|, |a_3|$ given by (1.1).

Definition 1.5 ([6]) *Let the functions h, $p : \mathbb{U} \to \mathbb{C}$ be so constrained that*

$$\min\{\Re(h(z)), \Re(p(z))\} > 0, \quad z \in \mathbb{U} \ \ \text{and} \ \ h(0) = p(0) = 1.$$

A function $f(z)$, defined by (1.1), is said to be in the class $\mathcal{B}_\Sigma^{h,p}(\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \ \ \text{and} \ \ (1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \in h(\mathbb{U}), \quad z \in \mathbb{U},$$

$$(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) \in p(\mathbb{U}), \quad w \in \mathbb{U},$$

where the function g is defined by (1.2).

Recently, Çağlar et al. [7] introduced the following two subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on coefficients $|a_2|$ and $|a_3|$.

Definition 1.6 ([7]) *A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{N}_\Sigma^{\mu}(\alpha, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma \ \ \text{and} \ \ \Big|\arg((1-\lambda)(\frac{f(z)}{z})^{\mu} + \lambda f'(z)(\frac{f(z)}{z})^{\mu-1})\Big| < \frac{\alpha\pi}{2}, \quad 0 < \alpha \le 1;\ \lambda \ge 1;\ \mu \ge 0;\ z \in \mathbb{U},$$

$$\Big|\arg((1-\lambda)(\frac{g(w)}{w})^{\mu} + \lambda g'(w)(\frac{g(w)}{w})^{\mu-1})\Big| < \frac{\alpha\pi}{2}, \quad 0 < \alpha \le 1;\ \lambda \ge 1;\ \mu \ge 0;\ w \in \mathbb{U},$$

where the function g is defined by (1.2).

Definition 1.7 ([7]) *A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{N}_\Sigma^{\mu}(\beta, \lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma \ \ \text{and} \ \ \Re\big((1-\lambda)(\frac{f(z)}{z})^{\mu} + \lambda f'(z)(\frac{f(z)}{z})^{\mu-1}\big) > \beta, \quad 0 \le \beta < 1;\ \lambda \ge 1;\ \mu \ge 0;\ z \in \mathbb{U},$$

$$\Re\big((1-\lambda)(\frac{g(w)}{w})^{\mu} + \lambda g'(w)(\frac{g(w)}{w})^{\mu-1}\big) > \beta, \quad 0 \le \beta < 1;\ \lambda \ge 1;\ \mu \ge 0;\ w \in \mathbb{U},$$

where the function g is defined by (1.2).

Following Çağlar et al.' work, Srivastava et al. [8] introduced the following subclasses of the bi-univalent function class Σ and also obtained non-sharp estimates on $|a_2|$ and $|a_3|$.

Definition 1.8 ([8]) *Let the functions h, $p : \mathbb{U} \to \mathbb{C}$ be so constrained that*

$$\min\{\Re(h(z)), \Re(p(z))\} > 0, \quad z \in \mathbb{U} \ \ \text{and} \ \ h(0) = p(0) = 1.$$

A function $f(z)$, defined by (1.1), is said to be in the class $\mathcal{N}_\Sigma^{h,p}(\lambda, \mu)(\lambda \ge 1; \mu \ge 0)$ if the following conditions are satisfied:

$$f \in \Sigma \ \ \text{and} \ \ (1-\lambda)(\frac{f(z)}{z})^{\mu} + \lambda f'(z)(\frac{f(z)}{z})^{\mu-1} \in h(\mathbb{U}), \quad z \in \mathbb{U},$$

$$(1-\lambda)(\frac{g(w)}{w})^{\mu} + \lambda g'(w)(\frac{g(w)}{w})^{\mu-1} \in p(\mathbb{U}), \quad w \in \mathbb{U},$$

where the function g is defined by (1.2).

Motivated by these papers, we introduce and investigate certain new subclass $\mathcal{N}_{\Sigma}^{h,p}(m,\lambda,\mu)$ of the analytic function class $\mathcal{A}$ defined by Salagean operator. Further we verify Brannan and Clunie's conjecture $|a_2| \leq \sqrt{2}$ for some of the subclass.

Definition 1.9 *Let the functions h, $p : \mathbb{U} \to \mathbb{C}$ be so constrained that*

$$\min\left\{\Re(h(z)), \Re(p(z))\right\} > 0, \quad z \in \mathbb{U} \ \text{ and } \ h(0) = p(0) = 1.$$

A function $f(z)$, defined by (1.1), is said to be in the class $\mathcal{N}_{\Sigma}^{h,p}(m,\lambda,\mu)$ if the following conditions are satisfied:

$$f \in \Sigma \ \text{ and } \ (1-\lambda)\Big(\frac{D^m f(z)}{z}\Big)^{\mu} + \lambda\Big(\frac{D^{m+1} f(z)}{D^m f(z)}\Big)\Big(\frac{D^m f(z)}{z}\Big)^{\mu} \in h(\mathbb{U}), \quad z \in \mathbb{U}, \tag{1.3}$$

$$(1-\lambda)\Big(\frac{D^m g(w)}{w}\Big)^{\mu} + \lambda\Big(\frac{D^{m+1} g(w)}{D^m g(w)}\Big)\Big(\frac{D^m g(w)}{w}\Big)^{\mu} \in p(\mathbb{U}), \quad w \in \mathbb{U}, \tag{1.4}$$

where $m \in \mathbb{N}_0$, $\lambda \geq 1$, $\mu \geq 0$ and the function g is defined by (1.2).

Remark 1.10 If we let $h(z) = p(z) = (\frac{1+z}{1-z})^{\alpha}$ $(0 < \alpha \leq 1)$, then the class $\mathcal{N}_{\Sigma}^{h,p}(m,\lambda,\mu)$ reduces to the new class denoted by $\mathcal{N}_{\Sigma}(m,\lambda,\mu,\alpha)$ which is the subclass of the functions $f(z) \in \Sigma$ satisfying

$$\Big|\arg((1-\lambda)\Big(\frac{D^m f(z)}{z}\Big)^{\mu} + \lambda\Big(\frac{D^{m+1} f(z)}{D^m f(z)}\Big)\Big(\frac{D^m f(z)}{z}\Big)^{\mu})\Big| < \frac{\alpha\pi}{2},$$

$$0 < \alpha \leq 1;\ \lambda \geq 1;\ \mu \geq 0;\ z \in \mathbb{U},$$

$$\Big|\arg(1-\lambda)\Big(\frac{D^m g(w)}{w}\Big)^{\mu} + \lambda\Big(\frac{D^{m+1} g(w)}{D^m g(w)}\Big)\Big(\frac{D^m g(w)}{w}\Big)^{\mu})\Big| < \frac{\alpha\pi}{2},$$

$$0 < \alpha \leq 1;\ \lambda \geq 1;\ \mu \geq 0;\ w \in \mathbb{U},$$

where the function g is defined by (1.2).

(i) For $\mu = 1, \lambda = 1, m = 0$, the class reduces to $\mathcal{H}_{\Sigma}(\alpha)$.

(ii) For $\mu = 1, m = 0$, the class reduces to $\mathcal{B}_{\Sigma}(\alpha,\lambda)$.

(iii) For $m = 0$, the class reduces to $\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda)$.

Remark 1.11 If we let $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$, then the class $\mathcal{N}_{\Sigma}^{h,p}(m,\lambda,\mu)$ reduces to the new class denoted by $\mathcal{N}_{\Sigma}(m,\lambda,\mu,\beta)$ which is the subclass of the functions $f(z) \in \Sigma$ satisfying

$$\Re\Big((1-\lambda)\Big(\frac{D^m f(z)}{z}\Big)^{\mu} + \lambda\Big(\frac{D^{m+1} f(z)}{D^m f(z)}\Big)\Big(\frac{D^m f(z)}{z}\Big)^{\mu}\Big) > \beta,$$

$$0 \leq \beta < 1;\ \lambda \geq 1;\ \mu \geq 0;\ z \in \mathbb{U},$$

$$\Re\Big((1-\lambda)\Big(\frac{D^m g(w)}{w}\Big)^{\mu} + \lambda\Big(\frac{D^{m+1} g(w)}{D^m g(w)}\Big)\Big(\frac{D^m g(w)}{w}\Big)^{\mu}\Big) > \beta,$$

$$0 \leq \beta < 1;\ \lambda \geq 1;\ \mu \geq 0;\ w \in \mathbb{U},$$

where the function g is defined by (1.2).

(i) For $\mu = 1, \lambda = 1, m = 0$, the class reduces to $\mathcal{H}_{\Sigma}(\beta)$.

(ii) For $\mu=1, m=0$, the class reduces to $\mathcal{B}_{\Sigma}(\beta,\lambda)$.

(iii) For $m=0$, the class reduces to $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$.

In order to derive our main results, we shall need the following lemma.

Lemma 1.12 ([9]) *If the function $h(z)\in\mathcal{P}$, then $|c_k|\le 2$ for each k, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which $\Re\{h(z)\}>0$, where*

$$h(z)=1+c_1z+c_2z^2+\cdots+c_kz^k+\cdots,\ c_k=\frac{h^{(k)}(0)}{k!},\quad z\in\mathbb{U}.$$

2. Coefficient estimates

In this section, we state and prove our general results involving the bi-univalent function class $\mathcal{N}_{\Sigma}^{h,p}(m,\lambda,\mu)$ given by Definition 1.4.

Theorem 2.1 *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathcal{N}_{\Sigma}^{h,p}(m,\lambda,\mu)$. Then*

$$|a_2|\le\min\Big\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2^{2m+1}(\mu+\lambda)^2}},\sqrt{\frac{|h''(0)|+|p''(0)|}{2(\mu+2\lambda)|2^{2m}(\mu-1)+2\cdot3^m|}}\Big\},\tag{2.1}$$

$$\begin{aligned}|a_3|\le\min\Big\{&\frac{|h'(0)|^2+|p'(0)|^2}{2^{2m+1}(\mu+\lambda)^2}+\frac{|h''(0)|+|p''(0)|}{4\cdot3^m(\mu+2\lambda)},\\&\frac{|2^{2m}(\mu-1)+4\cdot3^m||h''(0)|+2^{2m}|\mu-1||p''(0)|}{4\cdot3^m(\mu+2\lambda)|2^{2m}(\mu-1)+2\cdot3^m|}\Big\}.\end{aligned}\tag{2.2}$$

Proof First of all, we write the argument inequalities in (1.5) and (1.6) in their equivalent forms as follows:

$$(1-\lambda)(\frac{D^mf(z)}{z})^{\mu}+\lambda(\frac{D^{m+1}f(z)}{D^mf(z)})(\frac{D^mf(z)}{z})^{\mu}=h(z),\quad z\in\mathbb{U},\tag{2.3}$$

$$(1-\lambda)(\frac{D^mg(w)}{w})^{\mu}+\lambda(\frac{D^{m+1}g(w)}{D^mg(w)})(\frac{D^mg(w)}{w})^{\mu}=p(w),\quad w\in\mathbb{U},\tag{2.4}$$

respectively, where $h(z)$ and $p(w)$ satisfy the conditions of Definition 1.9. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z)=1+h_1z+h_2z^2+\cdots,\tag{2.5}$$

$$p(w)=1+p_1w+p_2w^2+\cdots,\tag{2.6}$$

respectively. Now, equating the coefficients in (2.3) and (2.4) with (1.4), (2.5), (2.6), we have

$$2^m(\mu+\lambda)a_2=h_1,\tag{2.7}$$

$$3^m(\mu+2\lambda)a_3+2^{2m-1}(\mu-1)(\mu+2\lambda)a_2^2=h_2,\tag{2.8}$$

$$-2^m(\mu+\lambda)a_2=p_1,\tag{2.9}$$

$$-3^m(\mu+2\lambda)a_3+(\mu+2\lambda)[2\cdot3^m+\frac{1}{2}\cdot3^{2m}(\mu-1)]a_2^2=p_2.\tag{2.10}$$

From (2.7) and (2.9), we obtain

$$h_1 = -p_1, \tag{2.11}$$

$$2^{2m+1}(\mu+\lambda)^2 a_2^2 = h_1^2 + p_1^2. \tag{2.12}$$

From (2.8) and (2.10), we find that

$$(\mu+2\lambda)[2^{2m}(\mu-1)+2\cdot 3^m]a_2^2 = h_2 + p_2. \tag{2.13}$$

Therefore, we find from the equations (2.12) and (2.13) that

$$|a_2| = \sqrt{\frac{h_1^2+p_1^2}{2^{2m+1}(\mu+\lambda)^2}} \leq \sqrt{\frac{|h'(0)^2|+|p'(0)^2}{2^{2m+1}(\mu+\lambda)^2}}, \tag{2.14}$$

$$|a_2| = \sqrt{\frac{|h_2+p_2|}{(\mu+2\lambda)|2^{2m}(\mu-1)+2\cdot 3^m|}} \leq \sqrt{\frac{|h''(0)|+|p''(0)|}{2(\mu+2\lambda)|2^{2m}(\mu-1)+2\cdot 3^m|}}, \tag{2.15}$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (2.10) from (2.8). We get

$$2\cdot 3^m(\mu+2\lambda)a_3 - 2\cdot 3^m(\mu+2\lambda)a_2^2 = h_2 - p_2. \tag{2.16}$$

Upon substituting the value of a_2^2 from (2.12) into (2.16), it follows that

$$a_3 = \frac{h_1^2+p_1^2}{2^{2m+1}(\mu+\lambda)^2} + \frac{h_2-p_2}{2\cdot 3^m(\mu+2\lambda)}.$$

We thus find that

$$|a_3| \leq \frac{|h'(0)|^2+|p'(0)|^2}{2^{2m+1}(\mu+\lambda)^2} + \frac{|h''(0)|+|p''(0)|}{4\cdot 3^m(\mu+2\lambda)}. \tag{2.17}$$

On the other hand, upon substituting the value of a_2^2 from (2.13) into (2.16), it follows that

$$\begin{aligned} a_3 &= \frac{h_2+p_2}{(\mu+2\lambda)[2^{2m}(\mu-1)+2\cdot 3^m]} + \frac{h_2-p_2}{2\cdot 3^m(\mu+2\lambda)} \\ &= \frac{[2^{2m}(\mu-1)+4\cdot 3^m]h_2 - 2^{2m}(\mu-1)p_2}{2\cdot 3^m(\mu+2\lambda)[2^{2m}(\mu-1)+2\cdot 3^m]}. \end{aligned}$$

Consequently, we have

$$|a_3| \leq \frac{|2^{2m}(\mu-1)+4\cdot 3^m||h''(0)| + 2^{2m}|\mu-1||p''(0)|}{4\cdot 3^m(\mu+2\lambda)|2^{2m}(\mu-1)+2\cdot 3^m|}. \tag{2.18}$$

This evidently completes the proof of Theorem 2.1. □

We note that for $h(z) \in \mathcal{P}$, it is easy to obtain that

$$|h'(0)| \leq 2, \quad |p'(0)| \leq 2$$

from Lemma 1.12. So if we let $\mu+\lambda \geq \sqrt{2}$, we can easily obtain the following Corollary 2.2 from Theorem 2.1.

Corollary 2.2 *If f given by (1.1) is in the class $\mathcal{N}_\Sigma^{h,p}(m,\lambda,\mu)$ for $\mu+\lambda \geq \sqrt{2}$, then $|a_2| \leq \sqrt{2}$.*

Proof For $\mu+\lambda\geq\sqrt{2}$, relation (2.1) indicates that

$$\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2^{2m+1}(\mu+\lambda)^2}}\leq\sqrt{\frac{8}{2^{2m+1}(\mu+\lambda)^2}}\leq\frac{1}{2^{m-1}(\mu+\lambda)}\leq\sqrt{2}.$$

Therefore, we complete the proof. □

From Corollary 2.2, we verify Brannan and Clunie's conjecture $|a_2|\leq\sqrt{2}$ for some of our class which satisfies the condition $\mu+\lambda\geq\sqrt{2}$.

If we let $h(z)=p(z)=(\frac{1+z}{1-z})^\alpha$ $(0<\alpha\leq 1)$ or $h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\leq\beta<1)$, then we can obtain Theorems 2.3 and 2.4.

Theorem 2.3 *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathcal{N}_\Sigma(m,\lambda,\mu,\alpha)$. Then*

$$|a_2|\leq\min\Big\{\frac{2\alpha}{2^m(\mu+\lambda)},\sqrt{\frac{4\alpha^2}{(\mu+2\lambda)|2^{2m}(\mu-1)+2\cdot 3^m|}}\Big\},\tag{2.19}$$

$$|a_3|\leq\min\Big\{\frac{4\alpha^2}{2^{2m}(\mu+\lambda)^2}+\frac{2\alpha^2}{3^m(\mu+2\lambda)},\frac{[|2^{2m}(\mu-1)+4\cdot 3^m|+2^{2m}|\mu-1|]\alpha^2}{3^m(\mu+2\lambda)|2^{2m}(\mu-1)+2\cdot 3^m|}\Big\}.\tag{2.20}$$

Theorem 2.4 *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the function class $\mathcal{N}_\Sigma(m,\lambda,\mu,\beta)$. Then*

$$|a_2|\leq\min\Big\{\frac{2(1-\beta)}{2^m(\mu+\lambda)},\sqrt{\frac{4(1-\beta)}{(\mu+2\lambda)|2^{2m}(\mu-1)+2\cdot 3^m|}}\Big\},\tag{2.21}$$

$$|a_3|\leq\min\Big\{\frac{4(1-\beta)^2}{2^{2m}(\mu+\lambda)^2}+\frac{2(1-\beta)}{3^m(\mu+2\lambda)},\frac{[|2^{2m}(\mu-1)+4\cdot 3^m|+2^{2m}|\mu-1|](1-\beta)}{3^m(\mu+2\lambda)|2^{2m}(\mu-1)+2\cdot 3^m|}\Big\}.\tag{2.22}$$

3. Corollaries and consequences

By setting $m=0$, in Theorem 2.1, we get Corollary 3.1 below.

Corollary 3.1 ([8]) *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{N}_\Sigma^{h,p}(\lambda,\mu)$. Then*

$$|a_2|\leq\min\Big\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(\mu+\lambda)^2}},\sqrt{\frac{|h''(0)|+|p''(0)|}{2(\mu+2\lambda)(\mu+1)}}\Big\},\tag{3.1}$$

$$|a_3|\leq\min\Big\{\frac{|h'(0)|^2+|p'(0)|^2}{2(\mu+\lambda)^2}+\frac{|h''(0)|+|p''(0)|}{4(\mu+2\lambda)},\frac{(\mu+3)|h''(0)|+|\mu-1||p''(0)|}{4(\mu+2\lambda)(\mu+1)}\Big\}.\tag{3.2}$$

There are many results generated by Corollary 3.1, for detail, see [4,6,7,10–12].

If $m=0$ in Theorems 2.3 and 2.4, we get Corollaries 3.2 and 3.3 below.

Corollary 3.2 *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{N}_\Sigma^\mu(\alpha,\lambda)$. Then*

$$|a_2|\leq\min\Big\{\frac{2\alpha}{\mu+\lambda},\sqrt{\frac{4\alpha^2}{(\mu+2\lambda)(\mu+1)}}\Big\},\tag{3.3}$$

$$|a_3| \le \min\{\frac{4\alpha^2}{(\mu+\lambda)^2}+\frac{2\alpha^2}{\mu+2\lambda}, \frac{(\mu+3+|\mu-1|)\alpha^2}{(\mu+2\lambda)(\mu+1)}\}. \tag{3.4}$$

Corollary 3.3 *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{N}_{\Sigma}^{\mu}(\beta, \lambda)$. Then*

$$|a_2| \le \min\Big\{\frac{2(1-\beta)}{\mu+\lambda}, \sqrt{\frac{4(1-\beta)}{(\mu+2\lambda)(\mu+1)}}\Big\}, \tag{3.5}$$

$$|a_3| \le \min\{\frac{4(1-\beta)^2}{(\mu+\lambda)^2}+\frac{2(1-\beta)}{(\mu+2\lambda)}, \frac{(\mu+3+|\mu-1|)(1-\beta)}{(\mu+2\lambda)(\mu+1)}\}. \tag{3.6}$$

If $m=0, \mu=1$ in Theorems 2.2 and 2.3, we get Corollaries 3.4 and 3.5 below.

Corollary 3.4 *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{B}_{\Sigma}(\alpha, \lambda)$. Then*

$$|a_2| \le \min\Big\{\frac{2\alpha}{1+\lambda}, \sqrt{\frac{2\alpha^2}{1+2\lambda}}\Big\}, \tag{3.7}$$

$$|a_3| \le \frac{2\alpha^2}{1+2\lambda}. \tag{3.8}$$

Corollary 3.5 *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{B}_{\Sigma}(\beta, \lambda)$. Then*

$$|a_2| \le \min\Big\{\frac{2(1-\beta)}{1+\lambda}, \sqrt{\frac{4(1-\beta)}{2(1+2\lambda)}}\Big\}, \tag{3.9}$$

$$|a_3| \le \frac{2(1-\beta)}{1+2\lambda}. \tag{3.10}$$

If $m=0, \mu=1, \lambda=1$ in Theorems 2.2 and 2.3, we get Corollaries 3.6 and 3.7 below.

Corollary 3.6 *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{H}_{\Sigma}(\alpha)$. Then*

$$|a_2| \le \min\Big\{\alpha, \sqrt{\frac{2}{3}}\alpha\Big\}, \tag{3.11}$$

$$|a_3| \le \frac{2\alpha^2}{3}. \tag{3.12}$$

Corollary 3.7 *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion (1.1) be in the bi-univalent function class $\mathcal{H}_{\Sigma}(\beta)$. Then*

$$|a_2| \le \min\Big\{1-\beta, \sqrt{\frac{2(1-\beta)}{3}}\Big\}, \tag{3.13}$$

$$|a_3| \le \frac{2(1-\beta)}{3}. \tag{3.14}$$

4. Conclusions

In this paper, a general subclass $\mathcal{N}_{\Sigma}^{h,p}(m, \lambda, \mu)$ of the analytic function class $\mathcal{A}$ involving Salagean operator D^m in the open unit disk $\mathbb{U}$ was introduced. The class extends many familiar subclasses of bi-univalent functions. We have derived estimates on the first two Taylor-Maclaurin coefficients $|a_2|, |a_3|$ for functions belonging to the class. Moreover, we verify Brannan and Clunie's conjecture $|a_2| \leq \sqrt{2}$ for some of our class. The results in our paper are more accurate than those in any other papers [4,7,10].

Acknowledgements We thank all the referees for their time and comments to improve this paper. We also thank the editors for handling this paper.

References

[1] M. LEWIN. *On a coefficient problem for bi-univalent functions*. Proc. Am. Math. Soc., 1967, **18**: 63–68.

[2] D. A. BRANNAN, J. G. CLUNIE. *Aspects of Contemporary Complex Analysis*. Academic Press, Inc., London-New York, 1980.

[3] E. NETANYAHU. *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in* $|z| < 1$. Arch. Ration. Mech. Anal., 1969, **32**: 100–112.

[4] H. M. SRIVASTAVA, A. K. MISHRA, P. GOCHHAYAT. *Certain subclasses of analytic and bi-univalent functions*. Appl. Math. Lett., 2010, **23**(10): 1188–1192.

[5] B. A. FRASIN, M. K. AOUF. *New subclasses of bi-univalent functions*. Appl. Math. Lett., 2011, **24**(9): 1569–1573.

[6] Qinghua XU, Haigen XIAO, H. M. SRIVASTAVA. *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems*. Appl. Math. Comput., 2012, **218**(23): 11461–11465.

[7] M. ÇAĞLAR, H. ORHAN, N. YAGMUR. *Coefficient bounds for new subclasses of bi-univalent functions*. Filomat, 2013, **27**(7): 1165–1171.

[8] H. M. SRIVASTAVA, S. BULUT, M. CAGLAR, et al. *Coefficient estimates for a general subclass of analytic and bi-univalent functions*. Filomat, 2013, **27**(5): 831–842.

[9] C. POMMERENKE. *Univalent Functions*. Vandenhoeck Ruprecht, Gottingen, 1975.

[10] S. BULUT. *Coefficient estimates for a class of analytic and bi-univalent functions*. Novi Sad J. Math., 2013, **43**(2): 59–65.

[11] Zhigang PENG, Qiuqiu HAN. *On the coefficients of several classes of bi-univalent functions*. Acta Math. Sci., 2014, **34**(1): 228–240.

[12] Qinghua XU, Yingchun GUI, H. M. SRIVASTAVA. *Coefficient estimates for a certain subclass of analytic and bi-univalent functions*. Appl. Math. Lett., 2012, **25**: 990–994.