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Estimates for the Lower Order Eigenvalues of Elliptic Operators in Weighted Divergence Form

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Abstract In this paper, we firstly give a general inequality for the lower order eigenvalues of elliptic operators in weighted divergence form on compact smooth metric measure spaces with boundary (possibly empty). Then using this general inequality, we get some universal inequalities for the lower order eigenvalues of elliptic operators in weighted divergence form on a connected bounded domain in the smooth metric measure spaces.

Keywords universal inequalities; drifting Laplacian; elliptic operators in weighted divergence form; smooth metric measure space

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1. Introduction

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M, and let Δ be the Laplace operator on M. We consider the following eigenvalue problem for the Laplace operator

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

It is well known that (1) has a discrete spectrum

$$0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_k \le \cdots,$$

where each eigenvalue is repeated with its multiplicity.

In 1955, Payne, Pólya and Weinberger showed that for any open bounded domain in an 2-dimensional Euclidean space $\mathbb{R}^2$ the bound $\frac{\lambda_2}{\lambda_1} \le 3$ holds [1,2]. Based on exact calculations for simple domains they also conjectured that

$$\frac{\lambda_2}{\lambda_1} \le \frac{\lambda_2(\mathbb{S}^1)}{\lambda_1(\mathbb{S}^1)} = \frac{j_{1,1}^2}{j_{0,1}^2} \approx 2.539, \tag{2}$$

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where, $\mathbb{S}^1 \subset \mathbb{R}^2$ is a circular disk, and $j_{n,m}$ denotes the m^{th} positive zero of the Bessel function $j_n(x)$. This conjecture and the corresponding inequalities in n-dimensions were proven in 1991 by Ashbaugh and Benguria [3–5]. Furthermore, when $M = \mathbb{R}^n, \Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, Ashbaugh and Benguria [6] in 1993 proved

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \le n(1 + \frac{4}{n}). \tag{3}$$

In 2008, when M are complex projective spaces, unit spheres, and compact complex submanifolds of a complex projective space, by making use of the orthogonalization of Gram-Schmidt (QR-factorization theorem), Sun, Cheng and Yang [7] gave some universal inequalities such as (3). More results, we refer to [8–10]. Let $(M, \langle , \rangle)$ be an n-dimensional complete Riemannian manifold with boundary ∂M and Ω be a bounded connected domain in M, and let $A : \Omega \to \mathrm{End}(T\Omega)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of $T\Omega$. Denote by ∇ the gradient operator. Define

$$L = -\mathrm{div}(A\nabla), \tag{4}$$

where $\mathrm{div}X$ denotes the divergence of a vector field X on M. The operator L defined in (4) is an elliptic operator in divergence form. It is easy to see that the Laplace operator is the special case when A is identity map.

In 2010, do Carmo, Wang and Xia [11] considered the eigenvalue problem of the elliptic operator in divergence form with weight such that

$$Lu + Vu = \lambda \rho u \ \text{ in } \ M, \qquad \text{and } \ u = 0 \ \text{ on } \ \partial M,$$

where M is a compact Riemannian manifold with boundary ∂M (possibly empty), V is a non-negative continuous function on M and ρ is a weight function which is positive and continuous on M. They got a Yang type inequality

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \le \frac{4\xi_2^2 \rho_2^2}{n\rho_1^2} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)\Big(\frac{1}{\xi_1}(\lambda_i - \frac{V_0}{\rho_2}) + \frac{n^2 H_0^2}{4\rho_1}\Big), \tag{5}$$

where $\xi_1 I \le A$ and $tr(A) \le n\xi_2$ throughout M, $\rho_1 \le \rho(x) \le \rho_2$, $\forall x \in M$, I is the identity map, $\xi_1, \xi_2, \rho_1, \rho_2$ are positive constants, $H_0 = \max_{x \in M} |\mathbf{H}|(x)$, $V_0 = \min_{x \in M} V(x)$, and $\mathbf{H}$ is the mean curvature vector field of M immersed into an Euclidean space $\mathbb{R}^N$. Recently, Sun and Chen gave some universal inequalities for the lower order eigenvalues of the elliptic operator in divergence form. For more recent developments about universal inequalities of the eigenvalue of elliptic operator in divergence form on Riemannian manifolds, we refer to [12–15] and the references therein.

A smooth metric measure space (also known as the weighted measure space) is actually a Riemannian manifold equipped with some measure which is absolutely continuous with respect to the usual Riemannian measure. More precisely, for a given complete n-dimensional Riemannian manifold $(M, \langle , \rangle)$ with the metric $\langle , \rangle$, the triple $(M, \langle , \rangle, e^{-f} \mathrm{d}\nu)$ is called a smooth metric measure space, where f is a smooth real-valued function on M and $\mathrm{d}\nu$ is the Riemannian volume element related to $\langle , \rangle$ (sometimes, we also call $\mathrm{d}\nu$ the volume density). On a smooth metric measure space $(M, \langle,\rangle, e^{-f}\mathrm{d}\nu)$, we can define the elliptic operator in weighted divergence form as

$$\mathfrak{L}_f = -\mathrm{div}_f A\nabla, \tag{6}$$

where $\mathrm{div}_f X = e^f \mathrm{div}\left(e^{-f}X\right)$ is the weighted divergence of vector field X, and A and ∇ are defined as before. When A is an identity map, $-\mathfrak{L}_f$ becomes the drifting Laplacian Δ_f, for the drifting Laplacian, some universal inequalities have been given in [16–20]. As briefly mentioned above, it is a natural problem how to get the universal inequalities of the eigenvalues of elliptic operator in weighted divergence form. In this paper, we will give some universal inequalities for the lower order eigenvalues of the elliptic operator in weighted divergence form on smooth metric measure space.

2. A key lemma

In this section, we will prove some general inequalities which play the key role in the proof of the main results.

Lemma 2.1 *Let $(M, \langle,\rangle, e^{-f}\mathrm{d}\nu)$ be an n-dimensional compact smooth metric measure space with boundary ∂M (possibly empty). Let λ_i be the i^{th} eigenvalue of the eigenvalue problem of the fourth-order elliptic operator in weighted divergence form with weight ρ such that*

$$\begin{cases} (a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)u = \lambda\rho u, & \text{in } M, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial M, \end{cases}$$

and u_i be the orthonormal eigenfunction corresponding to λ_i, that is,

$$\begin{cases} (a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)u_i = \lambda_i\rho u_i, & \text{in } M, \\ u_i = \frac{\partial u_i}{\partial \nu} = 0, & \text{on } \partial M, \\ \int_M \rho u_i u_j = \delta_{ij}, & \forall\, i,j = 1,2,\ldots. \end{cases}$$

If $g_i \in C^4(\overline{M})$ satisfies $\int_M \rho g_i u_1 u_{j+1} = 0$ for $1 \le j < i$, then we have

$$(\lambda_{i+1} - \lambda_1)^{\frac{1}{2}} \|u_1 \nabla g_i\|^2 \le \mathrm{d} \int_M g_i u_1 p_i \mathrm{d}\mu + \frac{1}{\mathrm{d}} \Big\| \frac{1}{\sqrt{\rho}} (\langle \nabla u_1, \nabla g_i\rangle + \frac{1}{2} u_1 \Delta_f g_i) \Big\|^2, \tag{7}$$

where

$$\begin{aligned} p_i = & -2a\langle \nabla g_i, A\nabla(\mathfrak{L}_f u_1)\rangle + a\mathfrak{L}_f g_i \mathfrak{L}_f u_1 - 2a\mathfrak{L}_f\left(\langle \nabla g_i, A\nabla u_1\rangle\right) + \\ & a\mathfrak{L}_f\left(u_1 \mathfrak{L}_f g_i\right) - 2b\langle \nabla g_i, A\nabla u_1\rangle + b u_1 \mathfrak{L}_f g_i, \end{aligned}$$

d *is any positive constant and $\|f\|^2 = \int_M f^2 \mathrm{d}\mu$.*

Proof Let $\varphi_i = (g_i - a_i)u_1$, where $a_i = \int_M \rho g_i u_1^2 \mathrm{d}\mu$. We have $\int_M \rho\varphi_i u_1 \mathrm{d}\mu = 0$. Noticing

$$\int_M \rho g_i u_1 u_{j+1} \mathrm{d}\mu = 0, \quad \text{for} \ \ 1 \le j < i,$$

we infer

$$\varphi_i|_{\partial M} = \frac{\partial \varphi_i}{\partial \nu}\Big|_{\partial M} = 0, \quad \text{and} \quad \int_M \rho\varphi_i u_{j+1} \mathrm{d}\mu = 0, \ \text{for} \ \ 0 \le j < i.$$

Then according to Rayleigh-Ritz inequality, we have

$$\lambda_{i+1} \leq \frac{\int_M \varphi_i(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)(\varphi_i)\mathrm{d}\mu}{\int_M \rho\varphi_i^2\mathrm{d}\mu}. \tag{8}$$

From the definition of φ_i, we have

$$\int_M \rho\varphi_i^2\mathrm{d}\mu = \int_M \rho\varphi_i(g_i - a_i)u_1\mathrm{d}\mu = \int_M \rho\varphi_i g_i u_1\mathrm{d}\mu, \tag{9}$$

and

$$\begin{aligned}\int_M \varphi_i(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)\varphi_i\mathrm{d}\mu &= \int_M \varphi_i(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)((g_i - a_i)u_1)\mathrm{d}\mu \\ &= \int_M \varphi_i(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)(g_i u_1)\mathrm{d}\mu. \end{aligned}\tag{10}$$

By direct computation, we have

$$\begin{aligned}\mathfrak{L}_f(g_i u_1) &= -\mathrm{div}_f(A\nabla(g_i u_1)) = -\mathrm{div}_f(A(u_1\nabla g_i + g_i\nabla u_1)) \\ &= -\langle\nabla u_1, A\nabla g_i\rangle - g_i\mathrm{div}_f(A\nabla u_1) - \langle\nabla g_i, A\nabla u_1\rangle - u_1\mathrm{div}_f(A\nabla g_i) \\ &= g_i\mathfrak{L}_f u_1 - 2\langle\nabla g_i, A\nabla u_1\rangle + u_1\mathfrak{L}_f g_i, \\ \mathfrak{L}_f^2(g_i u_1) &= \mathfrak{L}_f(g_i L u_1 - 2\langle\nabla g_i, A\nabla u_1\rangle + u_1\mathfrak{L}_f g_i) \\ &= g_i\mathfrak{L}_f^2 u_1 - 2\langle\nabla g_i, \nabla(\mathfrak{L}_f u_1)\rangle + \mathfrak{L}_f g_i\mathfrak{L}_f u_1 + \mathfrak{L}_f(-2\langle\nabla g_i, A\nabla u_1\rangle + u_1\mathfrak{L}_f g_i).\end{aligned}$$

So, we infer from above equalities that

$$(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)(g_i u_1) = \lambda_1\rho g_i u_1 + p_i, \tag{11}$$

where

$$\begin{aligned}p_i =& -2a\langle\nabla g_i, A\nabla(\mathfrak{L}_f u_1)\rangle + a\mathfrak{L}_f g_i\mathfrak{L}_f u_1 - 2a\mathfrak{L}_f\left(\langle\nabla g_i, A\nabla u_1\rangle\right) + \\ & a\mathfrak{L}_f\left(u_1\mathfrak{L}_f g_i\right) - 2b\langle\nabla g_i, A\nabla u_1\rangle + bu_1\mathfrak{L}_f g_i.\end{aligned}$$

It follows from (10) and (11) that

$$\begin{aligned}\int_M \varphi_i(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)\varphi_i\mathrm{d}\mu &= \int_M \varphi_i(a\mathfrak{L}_f^2 + b\mathfrak{L}_f + V)(g_i u_1)\mathrm{d}\mu \\ &= \lambda_1\int_M \varphi_i\rho g_i u_1\mathrm{d}\mu + \int_M \varphi_i p_i\mathrm{d}\mu \\ &= \lambda_1\int_M \rho\varphi_i^2\mathrm{d}\mu + \int_M g_i u_1 p_i\mathrm{d}\mu - a_i b_i, \end{aligned}\tag{12}$$

where

$$\begin{aligned}b_i =& \int_M p_i u_1\mathrm{d}\mu \\ =& \int_M -2au_1\langle\nabla g_i, A\nabla(L(u_1))\rangle\mathrm{d}\mu + \int_M au_1\mathfrak{L}_f g_i\mathfrak{L}_f u_1\mathrm{d}\mu - \int_M 2au_1\mathfrak{L}_f\left(\langle\nabla g_i, A\nabla u_1\rangle\right)\mathrm{d}\mu + \\ & \int_M au_1\mathfrak{L}_f\left(u_1\mathfrak{L}_f g_i\right) - \int_M 2bu_1\langle\nabla g_i, A\nabla u_1\rangle\mathrm{d}\mu + \int_M bu_1^2\mathfrak{L}_f g_i\mathrm{d}\mu \\ =& \int_M 2a\mathfrak{L}_f u_1\langle\nabla g_i, A\nabla u_1\rangle\mathrm{d}\mu + \int_M au_1\mathfrak{L}_f g_i\mathfrak{L}_f u_1\mathrm{d}\mu - \int_M 2a\mathfrak{L}_f u_1\left(\langle\nabla g_i, A\nabla u_1\rangle\right)\mathrm{d}\mu +\end{aligned}$$

$$\int_M a\mathfrak{L}_f u_1\,(u_1\mathfrak{L}_f g_i)\,\mathrm{d}\mu - \int_M 2bu_1\langle\nabla g_i, A\nabla u_1\rangle\mathrm{d}\mu + \int_M 2bu_1\langle\nabla g_i, A\nabla u_1\rangle\mathrm{d}\mu = 0. \tag{13}$$

Combining (8), (12) and (13), we have

$$(\lambda_{i+1}-\lambda_1)\int_M \rho\varphi_i^2\mathrm{d}\mu \le \int_M g_i u_1 p_i\mathrm{d}\mu. \tag{14}$$

Observing that $\int_M u_1(\langle\nabla u_1,\nabla g_i\rangle + \frac{1}{2}u_1\Delta g_i)\mathrm{d}\mu = 0$, we have

$$\int_M (-2)\varphi_i(\langle\nabla u_1,\nabla g_i\rangle + \frac{1}{2}u_1 g_i)\mathrm{d}\mu = -2\int_M g_i u_1(\langle\nabla u_1,\nabla g_i\rangle + \frac{1}{2}u_1\Delta_f g_i)\mathrm{d}\mu = \|u_1\nabla g_i\|^2.$$

On the other hand, we have

$$\begin{aligned}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\|u_1\nabla g_i\|^2 &= (\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\int_M (-2)\sqrt{\rho}\varphi_i(\frac{1}{\sqrt{\rho}}(\langle\nabla u_1,\nabla g_i\rangle + \frac{1}{2}u_1\Delta_f g_i))\mathrm{d}\mu\\ &\le \mathrm{d}(\lambda_{i+1}-\lambda_1)\|\sqrt{\rho}\varphi_i\|^2 + \frac{1}{\mathrm{d}}\Big\|\frac{1}{\sqrt{\rho}}(\langle\nabla u_1,\nabla g_i\rangle + \frac{1}{2}u_1\Delta_f g_i)\Big\|^2\\ &\le \mathrm{d}\int_M g_i u_1 p_i + \frac{1}{\mathrm{d}}\Big\|\frac{1}{\sqrt{\rho}}(\langle\nabla u_1,\nabla g_i\rangle + \frac{1}{2}u_1\Delta_f g_i)\Big\|^2,\end{aligned}$$

where d is any positive constant. This completes the proof of Lemma 2.1. □

3. Universal inequalities for lower order eigenvalues

In this section, using Lemma 2.1, we will give some universal inequalities for lower order eigenvalues of the elliptic operators in weighted divergence form on a connected bounded domain in complete smooth metric measure spaces. Firstly, we have

Theorem 3.1 *Let Ω be a connected bounded domain in an n-dimensional complete smooth metric measure space $\big(M,\langle,\rangle,e^{-f}\mathrm{d}\nu\big)$. Assume that $\xi_1 I \le A, tr(A) \le n\xi_2$ throughout Ω, and $\rho_1 \le \rho(x) \le \rho_2, |\nabla f|(x) \le C_0,\ \forall x\in\Omega$, here I is the identity map, $\xi_1,\xi_2,\rho_1,\rho_2,C_0$ are positive constants and $tr(A)$ denotes the trace of A. Let λ_i be the i^{th} eigenvalue of the following problem:*

$$\begin{cases}(\mathfrak{L}_f + V)u = \lambda\rho u, & in\ \ \Omega,\\ u = 0, & on\ \ \partial\Omega.\end{cases}$$

Then we have

$$\sum_{i=1}^n(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}} \le \frac{\rho_2}{\rho_1}\Big\{n\xi_2\big(\frac{\lambda_1-\rho_2^{-1}V_0}{\xi_1} + C_0\big(\frac{\lambda_1-\rho_2^{-1}V_0}{\xi_1}\big)^{\frac{1}{2}} + \frac{n^2H_0^2+C_0^2}{4\rho_1}\big)\Big\}^{\frac{1}{2}}, \tag{15}$$

where $H_0 = \max_{x\in\Omega}|\mathbf{H}|(x)$, $V_0 = \min_{x\in\Omega}V(x)$, and $\mathbf{H}$ is the mean curvature vector field of M immersed into an Euclidean space $\mathbb{R}^N$.

Proof Since M is a complete Riemannian manifold, from Nash embedding theorem, we know that there exists an isometric immersion from M into an Euclidean space $\mathbb{R}^N$. Thus, M can be considered as an n-dimensional complete isometrically immersed submanifold in $\mathbb{R}^N$. Let $y_1,y_2,\ldots,y_N$ be the standard coordinate functions of $\mathbb{R}^N$. Then from (2.2)–(2.5) in [21], we have

$$\sum_{i=1}^{N}|\nabla y_i|^2=n,\quad \Delta(y_1,y_2,\ldots,y_N)=n\mathbf{H}, \tag{16}$$

$$\sum_{i=1}^{N}\langle\nabla y_i,\nabla u_j\rangle^2=|\nabla u_j|^2,\quad \sum_{i=1}^{N}\langle\nabla y_i,\nabla f\rangle^2=|\nabla f|^2, \tag{17}$$

$$\sum_{i=1}^{N}\langle\nabla y_i,\nabla u_j\rangle\langle\nabla y_i,\nabla f\rangle=\sum_{i=1}^{N}\nabla u_j(y_i)\nabla f(y_i)=\langle\nabla u_j,\nabla f\rangle, \tag{18}$$

$$\sum_{i=1}^{N}\Delta y_i\langle\nabla y_i,\nabla u_j\rangle=\sum_{i=1}^{N}\Delta y_i\nabla u_j(y_i)=\langle n\mathbf{H},\nabla u_j\rangle=0, \tag{19}$$

and

$$\sum_{i=1}^{N}\Delta y_\alpha\langle\nabla y_\alpha,\nabla f\rangle=\sum_{i=1}^{N}\Delta y_\alpha\nabla f(y_\alpha)=\langle n\mathbf{H},\nabla f\rangle=0. \tag{20}$$

We also have

$$\sum_{i=1}^{N}\Delta_f y_i\langle\nabla y_i,\nabla u_j\rangle=\sum_{i=1}^{N}(\Delta y_i-\langle\nabla y_i,\nabla f\rangle)\langle\nabla y_i,\nabla u_j\rangle=\langle\nabla u_j,\nabla f\rangle, \tag{21}$$

and

$$\begin{aligned}\sum_{i=1}^{N}(\Delta_f y_i)^2&=\sum_{i=1}^{N}(\Delta y_i-\langle\nabla y_i,\nabla f\rangle)^2=\sum_{i=1}^{N}\big((\Delta y_i)^2-2\Delta y_i\langle\nabla y_i,\nabla f\rangle+\langle\nabla y_i,\nabla f\rangle^2\big)\\&=n^2|\mathbf{H}|^2+|\nabla f|^2.\end{aligned} \tag{22}$$

By using the QR-factorization theorem, we know that there exists an orthogonal $N\times N$ matrix $T=(T_{ij})$ such that

$$\sum_{k=1}^{N}T_{ik}\int_M y_k u_1 u_{j+1}=\sum_{k=1}^{N}\int_M T_{ik}y_k u_1 u_{j+1}=0,\quad \text{for } 1\le j<i\le N.$$

Set $g_i=\sum_{k=1}^{N}T_{ik}y_k$, we get

$$\int_M g_i u_1 u_{j+1}=\int_M\sum_{k=1}^{N}T_{ik}y_k u_1 u_{j+1}=0,\quad 1\le j<i\le N. \tag{23}$$

Since T is an orthogonal matrix, g_i also satisfies (16)–(22).

Let $a=0,b=1$ in (7). Taking $h=g_i$ and summing for i from 1 to N, we have

$$\sum_{i=1}^{N}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\|u_1\nabla g_i\|^2\le \mathrm{d}\int_M\sum_{i=1}^{N}g_i u_1 p_i\mathrm{d}\mu+\frac{1}{\mathrm{d}}\sum_{i=1}^{N}\Big\|\frac{1}{\sqrt{\rho}}(\langle\nabla u_1,\nabla g_i\rangle+\frac{1}{2}u_1\Delta g_i)\Big\|^2, \tag{24}$$

where $p_i=-2\langle\nabla g_i,A\nabla u_1\rangle+u_1\mathfrak{L}_f g_i$. Since

$$-2\int_\Omega g_i u_1\langle\nabla g_i,A\nabla u_1\rangle\mathrm{d}\mu=\int_\Omega u_1^2\langle\nabla g_i,A\nabla g_i\rangle\mathrm{d}\mu-\int_\Omega g_i u_1^2\mathfrak{L}_f g_i\mathrm{d}\mu,$$

we infer from above equality and $\sum_{i=1}^{N}\langle\nabla g_i, A\nabla g_i\rangle = \mathrm{tr}(A) \le n\xi_2$ that

$$\begin{aligned}\int_\Omega \sum_{i=1}^{N} g_i u_1 p_i \mathrm{d}\mu &= \int_\Omega \sum_{i=1}^{N} g_i u_1 \left(-2\langle\nabla g_i, A\nabla u_1\rangle + u_1\mathfrak{L}_f g_i\right)\mathrm{d}\mu \\ &= \int_\Omega \sum_{i=1}^{N} u_1^2\langle\nabla g_i, A\nabla g_i\rangle \mathrm{d}\mu \le n\xi_2\|u_1\|^2 \le n\xi_2\rho_1^{-1}. \end{aligned} \tag{25}$$

From $\rho_2^{-1} \le \|u_1\|^2 \le \rho_1^{-1}$ and $A \ge \xi_1 I$, we have

$$\begin{aligned}\lambda_1 &= \int_\Omega u_1\left(\mathfrak{L}_f + V\right)u_1\mathrm{d}\mu = \int_\Omega -u_1\mathrm{div}_f(A\nabla u_1)\mathrm{d}\mu + \int_\Omega V u_1^2\mathrm{d}\mu \\ &= \int_\Omega \langle\nabla u_1, A\nabla u_1\rangle\mathrm{d}\mu + \int_\Omega V u_1^2 \mathrm{d}\mu \ge \xi_1\|\nabla u_1\|^2 + \rho_2^{-1}V_0,\end{aligned}$$

which implies

$$\|\nabla u_1\|^2 \le \frac{\lambda_1 - \rho_2^{-1}V_0}{\xi_1}. \tag{26}$$

From Schwarz inequality and above inequality, we have

$$\int_\Omega \langle\nabla f, \nabla u_1\rangle\mathrm{d}\mu \le \int_\Omega |\nabla f||\nabla u_1|\mathrm{d}\mu \le C_0\left\{\|\nabla u_1\|^2\right\}^{\frac{1}{2}} \le C_0\left(\frac{\lambda_1 - \rho_2^{-1}V_0}{\xi_1}\right)^{\frac{1}{2}}. \tag{27}$$

Combining (23), (26) and (27), we have

$$\begin{aligned}&\int_\Omega \frac{1}{\rho}\sum_{i=1}^{N}\left(\langle\nabla g_i, \nabla u_1\rangle + \frac{u_1\Delta_f g_i}{2}\right)^2\mathrm{d}\mu \\ &= \int_\Omega \frac{1}{\rho}\sum_{i=1}^{N}\left(\langle\nabla g_i, \nabla u_1\rangle^2 + u_1\Delta_f g_i\langle\nabla g_i, \nabla u_1\rangle + \frac{u_1^2\left(\Delta_f g_i\right)^2}{4}\right)\mathrm{d}\mu \\ &= \int_\Omega \frac{1}{\rho}\left(|\nabla u_1|^2 + \langle\nabla f, \nabla u_1\rangle + \frac{u_1^2}{4}\left(n^2|\mathbf{H}|^2 + |\nabla f|^2\right)\right)\mathrm{d}\mu \\ &\le \frac{1}{\rho_1}\left\{\frac{\lambda_1 - \rho_2^{-1}V_0}{\xi_1} + C_0\left(\frac{\lambda_1 - \rho_2^{-1}V_0}{\xi_1}\right)^{\frac{1}{2}} + \frac{1}{4\rho_1}\left(n^2H_0^2 + C_0^2\right)\right\}.\end{aligned} \tag{28}$$

For any point $p \in M$, by a transformation of coordinates if necessary, we have $|\nabla g_i|^2 \le 1$ for any i. Then we have

$$\begin{aligned}\sum_{i=1}^{N}(\lambda_{i+1} - \lambda_1)^{\frac{1}{2}}|\nabla g_i|^2 &= \sum_{j=1}^{n}(\lambda_{j+1} - \lambda_1)^{\frac{1}{2}}|\nabla g_i|^2 + \sum_{k=n+1}^{N}(\lambda_{k+1} - \lambda_1)^{\frac{1}{2}}|\nabla g_k|^2 \\ &\ge \sum_{j=1}^{n}(\lambda_{j+1} - \lambda_1)^{\frac{1}{2}}|\nabla g_i|^2 + (\lambda_{n+1} - \lambda_1)^{\frac{1}{2}}(n - \sum_{l=1}^{n}|\nabla g_l|^2) \\ &\ge \sum_{j=1}^{n}(\lambda_{j+1} - \lambda_1)^{\frac{1}{2}}|\nabla g_i|^2 + \sum_{l=1}^{n}(\lambda_{l+1} - \lambda_1)^{\frac{1}{2}}(1 - |\nabla g_l|^2) \\ &\ge \sum_{i=1}^{n}(\lambda_{i+1} - \lambda_1)^{\frac{1}{2}}.\end{aligned} \tag{29}$$

Taking (25), (28) and (29) into (24), we have

$$\sum_{i=1}^{n}\frac{1}{\rho_2}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\leq\frac{\mathrm{d}n\xi_2}{\rho_1}+\frac{1}{\mathrm{d}}\frac{1}{\rho_1}\{\frac{\lambda_1-\rho_2^{-1}V_0}{\xi_1}+C_0(\frac{\lambda_1-\rho_2^{-1}V_0}{\xi_1})^{\frac{1}{2}}+\frac{1}{4\rho_1}\left(n^2H_0^2+C_0^2\right)\}.$$

Taking $\delta=\{\frac{1}{n\xi_1}(\frac{\lambda_1-\rho_2^{-1}V_0}{\xi_1}+C_0(\frac{\lambda_1-\rho_2^{-1}V_0}{\xi_1})^{\frac{1}{2}}+\frac{1}{4\rho_1}(n^2H_0^2+C_0^2))\}^{\frac{1}{2}}$, we have

$$\sum_{i=1}^{n}(\lambda_{i+1}-\lambda_1)^{\frac{1}{2}}\leq\frac{\rho_2}{\rho_1}\{n\xi_2(\frac{\lambda_1-\rho_2^{-1}V_0}{\xi_1}+C_0(\frac{\lambda_1-\rho_2^{-1}V_0}{\xi_1})^{\frac{1}{2}}+\frac{1}{4\rho_1}(n^2H_0^2+C_0^2))\}^{\frac{1}{2}}.$$

This completes the proof of Theorem 3.1. $\Box$

In the following, we will give a universal inequality for eigenvalues of the fourth order elliptic operator in weighted divergence form.

Theorem 3.2 *Let Ω be a connected bounded domain in an n-dimensional complete smooth metric measure space $\left(M,\langle,\rangle,e^{-f}\mathrm{d}\nu\right)$. Assume that $\xi_1 I\leq A\leq\xi_2 I$ throughout Ω, and $|\nabla f|(x)\leq C_0$, $\forall x\in\Omega$, here I is the identity map, ξ_1,ξ_2,C_0 are positive constants. Let Λ_i be the i^{th} eigenvalue of the following problem:*

$$\mathfrak{L}_f^2u=\Lambda u\ \ \text{in}\ \ \Omega,\quad\text{and}\ \ u=\frac{\partial u}{\partial\nu}=0\ \ \text{on}\ \ \partial\Omega.$$

Then we have

$$\begin{aligned}\sum_{i=1}^{n}(\Lambda_{i+1}-\Lambda_1)^{\frac{1}{2}}\leq&\{\frac{\xi_2}{\xi_1}((2n+4)\Lambda_1^{\frac{1}{2}}+4C_0\xi_2^{\frac{1}{2}}\Lambda_1^{\frac{1}{4}}+\xi_2(n^2H_0^2+C_0^2))\times\\&(4\Lambda_i^{\frac{1}{2}}+4C_0\xi_1^{\frac{1}{2}}\Lambda_i^{\frac{1}{4}}+\xi_1(n^2H_0^2+C_0^2))\}^{\frac{1}{2}},\end{aligned}\tag{30}$$

where $H_0=\max_{x\in\Omega}|\mathbf{H}|(x)$, $V_0=\min_{x\in\Omega}V(x)$, and $\mathbf{H}$ is the mean curvature vector field of M immersed into an Euclidean space $\mathbb{R}^N$.

Proof Let $a=1$, $b=0$, $V\equiv0$ and $\rho\equiv1$ in (7). Taking $h=g_i$ and summing for i from 1 to N, where g_i is defined as above, we have

$$\sum_{i=1}^{N}(\Lambda_{i+1}-\Lambda_1)^{\frac{1}{2}}\|u_1\nabla g_i\|^2\leq\mathrm{d}\int_M\sum_{i=1}^{N}g_iu_1p_i+\frac{1}{\mathrm{d}}\sum_{i=1}^{N}\left\|\frac{1}{\sqrt{\rho}}(\langle\nabla u_1,\nabla g_i\rangle+\frac{1}{2}u_1\Delta_fg_i)\right\|^2,\tag{31}$$

where

$$p_i=-2\langle\nabla g_i,A\nabla(\mathfrak{L}_fu_1)\rangle+\mathfrak{L}_fg_i\mathfrak{L}_fu_1-2\mathfrak{L}_f\left(\langle\nabla g_i,A\nabla u_1\rangle\right)+\mathfrak{L}_f\left(u_1\mathfrak{L}_fg_i\right).$$

By direct computation, we have

$$\begin{aligned}&\int_\Omega g_iu_1p_i\mathrm{d}\mu=\int_\Omega g_iu_1\{-2\langle\nabla g_i,A\nabla(\mathfrak{L}_fu_1)\rangle+\mathfrak{L}_fg_i\mathfrak{L}_fu_1-2\mathfrak{L}_f\left(\langle\nabla g_i,A\nabla u_1\rangle\right)+\mathfrak{L}_f\left(u_1\mathfrak{L}_fg_i\right)\}\mathrm{d}\mu\\&=\int_\Omega2\{u_1\mathfrak{L}_fu_1\langle\nabla g_i,A\nabla g_i\rangle+g_i\mathfrak{L}_fu_1\langle\nabla u_1,A\nabla g_i\rangle-g_iu_1\mathfrak{L}_fg_i\mathfrak{L}_fu_1\}\mathrm{d}\mu+\int_\Omega g_iu_1\mathfrak{L}_fg_i\mathfrak{L}_fu_1\mathrm{d}\mu+\\&\quad\int_\Omega\{\mathfrak{L}_fg_iu_1+g_i\mathfrak{L}_fu_i-2\langle\nabla g_i,A\nabla u_1\rangle\}\{-2\langle\nabla g_i,A\nabla u_1\rangle+u_1\mathfrak{L}_fg_i\}\mathrm{d}\mu\\&=\int_\Omega2u_1\mathfrak{L}_fu_1\langle\nabla g_1,A\nabla g_i\rangle\mathrm{d}\mu+\int_\Omega4\langle\nabla g_i,A\nabla u_1\rangle^2\mathrm{d}\mu-\int_\Omega4u_1\mathfrak{L}_fg_i\langle\nabla g_i,A\nabla u_1\rangle\mathrm{d}\mu+\\&\quad\int_\Omega(u_1\mathfrak{L}_fg_i)^2\mathrm{d}\mu.\end{aligned}\tag{32}$$

Since $\xi_1 I \leq A \leq \xi_2 I$, we can infer from (16)–(22) that

$$\sum_{i=1}^{N}\int_\Omega 2u_1\mathfrak{L}_f u_1\langle\nabla g_i, A\nabla g_i\rangle \mathrm{d}\mu \leq 2n\xi_2\int_\Omega u_1\mathfrak{L}_f u_1\mathrm{d}\mu \leq 2n\xi_2\{\|u_1\|^2\|\mathfrak{L}_f u_1\|^2\}^{\frac{1}{2}} = 2n\xi_2\Lambda_1^{\frac{1}{2}}, \quad (33)$$

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega 4\langle\nabla g_i, A\nabla u_1\rangle^2\mathrm{d}\mu &= 4\|A\nabla u_1\|^2 \leq 4\xi_2\int_\Omega\langle\nabla u_1, A\nabla u_1\rangle\\ &= 4\xi_2\int_\Omega u_1\mathfrak{L}_f u_1\mathrm{d}\mu \leq 4\xi_2\Lambda_1^{\frac{1}{2}},\end{aligned} \quad (34)$$

$$\begin{aligned}&\sum_{i=1}^{N} -\int_\Omega 4u_1\mathfrak{L}_f g_i\langle\nabla g_i, A\nabla u_1\rangle\mathrm{d}\mu \leq \Big|4\xi_2\int_\Omega u_1\Delta_f g_i\langle\nabla g_i, A\nabla u_1\rangle\mathrm{d}\mu\Big|\\ &= \Big|4\xi_2\int_\Omega u_1\langle\nabla f, A\nabla u_1\rangle\mathrm{d}\mu\Big| \leq 4\xi_2\int_\Omega u_1|\nabla f||\nabla u_1|\mathrm{d}\mu\\ &\leq 4C_0\xi_2\{\|u_1\|^2\|A\nabla u_1\|^2\}^{\frac{1}{2}} = 4C_0\xi_2^{\frac{3}{2}}\Big\{\int_\Omega\langle\nabla u_1, A\nabla u_1\rangle\mathrm{d}\mu\Big\}^{\frac{1}{2}} \leq 4C_0\xi_2^{\frac{3}{2}}\Lambda_1^{\frac{1}{4}},\end{aligned} \quad (35)$$

and

$$\begin{aligned}\sum_{i=1}^{N}\int_\Omega(u_1\mathfrak{L}_f g_i)^2\mathrm{d}\mu\xi_2^2 \leq &\sum_{i=1}^{N}\int_\Omega u_1^2(\Delta_f g_i)^2\mathrm{d}\mu = \xi_2^2\int_\Omega u_1^2\left(n^2|\mathbf{H}|^2+|\nabla f|^2\right)\mathrm{d}\mu\\ \leq &\xi_2^2\left(n^2H_0^2+C_0^2\right).\end{aligned} \quad (36)$$

Combining (32)–(36), we have

$$\sum_{i=1}^{N}\int_\Omega g_i u_1 p_i \leq \xi_2((2n+4)\Lambda_1^{\frac{1}{2}} + 4C_0\xi_2^{\frac{1}{2}}\Lambda_1^{\frac{1}{4}} + \xi_2(n^2H_0^2+C_0^2)). \quad (37)$$

Since $\|\nabla u_1\|^2 \leq \frac{1}{\xi_1}\int_\Omega\langle\nabla u_1, A\nabla u_1\rangle\mathrm{d}\mu = \frac{1}{\xi_1}\int_\Omega u_1\mathfrak{L}_f u_1\mathrm{d}\mu \leq \frac{\Lambda_1^{\frac{1}{2}}}{\xi_1}$, we have

$$\begin{aligned}&\int_\Omega\sum_{i=1}^{N}\big(\langle\nabla g_i,\nabla u_1\rangle + \frac{u_1\Delta_f g_i}{2}\big)^2\mathrm{d}\mu\\ &= \int_\Omega\big(|\nabla u_1|^2 + \langle\nabla f,\nabla u_1\rangle + \frac{u_1^2}{4}\left(n^2|\mathbf{H}|^2+|\nabla f|^2\right)\big)\mathrm{d}\mu\\ &\leq \frac{\Lambda_1^{\frac{1}{2}}}{\xi_1} + \frac{C_0\Lambda_1^{\frac{1}{4}}}{\xi_1^{\frac{1}{2}}} + \frac{n^2H_0^2+C_0^2}{4}.\end{aligned} \quad (38)$$

Taking (37) and (38) into (31), we have

$$\begin{aligned}\sum_{i=1}^{n}(\Lambda_{i+1}-\Lambda_1)^{\frac{1}{2}} \leq &\delta\xi_2((2n+4)\Lambda_1^{\frac{1}{2}} + 4C_0\xi_2^{\frac{1}{2}}\Lambda_1^{\frac{1}{4}} + \xi_2(n^2H_0^2+C_0^2)) +\\ &\frac{1}{\delta}\Big(\frac{\Lambda_i^{\frac{1}{2}}}{\xi_1} + \frac{C_0\Lambda_i^{\frac{1}{4}}}{\xi_1^{\frac{1}{2}}} + \frac{n^2H_0^2+C_0^2}{4}\Big).\end{aligned} \quad (39)$$

Let

$$\delta = \left\{\frac{\{\frac{\Lambda_i^{\frac{1}{2}}}{\xi_1} + \frac{C_0\Lambda_i^{\frac{1}{4}}}{\xi_1^{\frac{1}{2}}} + \frac{n^2H_0^2+C_0^2}{4}\}}{\xi_2((2n+4)\Lambda_1^{\frac{1}{2}} + 4C_0\xi_2^{\frac{1}{2}}\Lambda_1^{\frac{1}{4}} + \xi_2(n^2H_0^2+C_0^2))}\right\}^{\frac{1}{2}}$$

in (39). We have

$$\sum_{i=1}^{n}(\Lambda_{i+1}-\Lambda_1)^{\frac{1}{2}} \leq \frac{1}{n}\Big\{\frac{\xi_2}{\xi_1}((2n+4)\Lambda_1^{\frac{1}{2}}+4C_0\xi_2^{\frac{1}{2}}\Lambda_1^{\frac{1}{4}}+\xi_2(n^2H_0^2+C_0^2))\times (4\Lambda_i^{\frac{1}{2}}+4C_0\xi_1^{\frac{1}{2}}\Lambda_i^{\frac{1}{4}}+\xi_1(n^2H_0^2+C_0^2))\Big\}^{\frac{1}{2}}.$$

This completes the proof of Theorem 3.2. □

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