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Zero Products and Finite Rank of Toeplitz Operators on the Harmonic Bergman Space

Qian DING[1,*], **Yinyin HU**[2], **Liu LIU**[1], **Yufeng LU**[1]
1. *School of Mathematical Sciences, Dalian University of Technology, Liaoning* 116024*, P. R. China;*
2. *Department of Mathematics, Dalian Maritime University, Liaoning* 116023*, P. R. China*

Abstract We study zero product problem and finite rank of the Brown-Halmos type results involving products of Toeplitz operators acting on the harmonic Bergman space. We use the Berezin transform and invariant Laplacian in this paper.

Keywords Toeplitz operator; finite rank; harmonic Bergman space

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and dA the normalized Lebesgue area measure on $\mathbb{D}$. As usual, L^2 denotes the Hilbert space of Lebesgue square integral functions on $\mathbb{D}$ with the inner product

$$\langle u, v\rangle = \int_{\mathbb{D}} u\bar{v}\mathrm{d}A,$$

where $u, v \in L^2$. The Bergman space L_a^2 is the closed subspace of L^2 consisting of the analytic functions on $\mathbb{D}$. Let P be the orthogonal projection from L^2 onto L_a^2 which is given explicitly by

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\bar{w}z)^2}\mathrm{d}A(w),$$

where $z, w \in \mathbb{D}$, $f \in L^2$. For $z \in \mathbb{D}$, the reproducing kernel function in Bergman space will be denoted by K_z which is given explicitly by

$$K_z(w) = \frac{1}{(1-\bar{z}w)^2}.$$

Since $\|K_z\|^2 = K_z(z) = \frac{1}{(1-|z|^2)^2}$, it follows that the normalized reproducing kernel is equal to

$$k_z(w) = \frac{1-|z|^2}{(1-\bar{z}w)^2}.$$

For $f \in L^\infty$, the Toeplitz operator $\tilde{T}_f$ on the Bergman space is defined by

$$\tilde{T}_f u = P(fu),$$

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* Corresponding author
E-mail address: dingqian@mail.dlut.edu.cn (Qian DING); 534480549@qq.com (Yinyin HU); beth.liu@dlut.edu.cn (Liu LIU); lyfdlut@dlut.edu.cn (Yufeng LU)

where $u \in L_a^2$. The small Hankel operator h_f on the Bergman space is defined by

$$h_f u = P(fUu),$$

where U is the unitary operator defined by $(Uf)(z) = f(\bar{z})$.

As is well-known, the harmonic Bergman space b^2 is the collection of harmonic functions on $\mathbb{D}$ which are in $L^2 = L^2(\mathbb{D}, \mathrm{d}A)$. Since each point evaluation functional is bounded on b^2, for each $z \in \mathbb{D}$, there exists a unique function $R_z \in b^2$ which has the reproducing property

$$f(z) = \langle f, R_z \rangle = \int_{\mathbb{D}} f\overline{R_z}\mathrm{d}A,$$

where $f \in b^2$. R_z is also related to the Bergman kernel

$$R_z = K_z + \overline{K_z} - 1,$$

where $z \in \mathbb{D}$.

Let Q be the orthogonal projection from L^2 onto b^2. By the reproducing property, we have

$$Qf(z) = \int_{\mathbb{D}} f(w)\overline{R_z(w)}\mathrm{d}A(w),$$

where $f \in L^2$. Also,

$$Qf = P(f) + \overline{P(\bar{f})} - P(f)(0),$$

where $f \in L^2$. Hence, $\overline{Q(f)} = Q(\bar{f})$.

For $f \in L^\infty$, the Toeplitz operator T_f on b^2 with symbol f is defined by

$$T_f u = Q(fu),$$

where u in b^2. It is clear that T_f is a bounded linear operator.

In classical function theory of the unit disk, Toeplitz operators were defined on the Hardy space H^2 by $T_\varphi f = \widetilde{P}(\varphi f)$, where φ is a bounded measurable function on the unit circle $\mathbb{T} = \partial\mathbb{D}$ and $\widetilde{P}$ is the Szegö projection from L^2 (of the unit circle) to H^2. On the Hardy space, bounded Toeplitz operators arise from bounded symbols only and there are no compact Toeplitz operators other than the zero operator. Likewise, the product of two such operators is zero if and only if the symbol of one of them is zero [1]. The following natural and basic conjecture about finite products of Toeplitz operators has been well known for a long time: If a product of n Toeplitz operators is the zero operator, then at least one of these operators must be zero. This was shown to hold for three operators by Axler [2] in the 1970s, but the method used in [1] becomes quite complicated for handling more operators. Although the question has received some attention, only recently has the conjecture been verified for $n = 5$ and $n = 6$ by Guo [3] and Gu [4], respectively. In 2009, they [5] used some new vector-valued techiniques and proved that the product of finitely many Toeplitz operators on the Hardy space is zero if and only if at least one of the operators is zero. Ding [6] solved the problem for two factors on the Hardy space of the polydisk. The ball case seems to have not been studied yet.

Returning to the Bergman space case, Ahern and Čučković [7] solved the zero-product problem for two Toeplitz operators with harmonic symbols and the problem for arbitrary symbols still remains open. The higher dimensional cases have been also studied on the ball and polydisk. Recently, the polydisk case was solved by Choe et al. [8] for two factors with pluriharmonic symbols by extending the method in [7]. In [9], Choe et al. solved the zero-product problem for two Toeplitz operators with n-harmonic symbols that have local continuous extension property up to the distinguished boundary on the Bergman space of the unit polydisk. On the setting of the unit ball, they [10] used an entirely different method to solve the zero-product problem for two factors with harmonic symbols that have local continuous extension property up to the boundary. At the same paper, they also solved the problem for multiple products with number of factors depending on the dimension in the case where symbols have additional (global or local) Lipschitz continuity up to the boundary.

For the problem of finite rank Perturbation of the Brown-Halmos type results involving products of Toeplitz operators, Ding and Zheng [11] obtained a complete description for finite rank commutators of two Toeplitz operators on the Hardy space. Recently, Choe et al. [12] proved that an arbitrary positive integer can be the rank of a commutator of two Toeplitz operators in the course of study of the same problem on the higher dimensional Hardy space.

In the Bergman space setting, however, there are a lot of nontrivial compact Toeplitz operators. Given a complex Borel measure μ with compact support in the complex plane $\mathbb{C}$, Luecking, Daniel [13] showed that T_μ has finite rank if and only if μ is a finite linear combination of point masses. Čučkovič [14] characterized the finite rank perturbations of the Brown-Halmos type results involving products of Toeplitz operators acting on the Bergman space. By using Berezin transform, Guo, Sun and Zheng [15] characterized finite rank (semi-) commutators of two Toeplitz operators with harmonic symbols and as a consequence, there is no nonzero finite rank commutators of Toeplitz operators with harmonic symbols. Unlike the harmonic case, Čučković and Louhichi [16] later showed that commutators of Toeplitz operators with two quasihomogeneous symbols can induce nonzero finite rank operators. More explicitly, they constructed two quasihomogeneous symbols for which the corresponding Toeplitz operators induce a commutator with rank 1. As an extension, Choe et al. gave characterizations of sums of finitely many Toeplitz products with harmonic symbols having finite rank or being compact, see [17] for detail. The corresponding problems have been characterized in the higher dimension cases of Bergman spaces and Dirichlet space [12,18,19].

In this paper, we study zero product of two Toeplitz operators with analytic and co-analytic symbols and finite rank of the Brown-Halmos type results involving products of Toeplitz operators acting on the harmonic Bergman space. We obtain the conclusions as follows:

Theorem 1.1 *Suppose for every integer $1 \le l \le M$, f_l and g_l, are bounded analytic functions on $\mathbb{D}$. Then the following are equivalent:*

(a) $\sum_{l=1}^{M} \bar{f}_l g_l = 0$;

(b) $\sum_{l=1}^{M} T_{f_l} T_{\bar{g}_l} = 0$;

(c) $\sum_{l=1}^{M} T_{\bar{g}_l} T_{f_l} = 0$;

(d) $\sum_{l=1}^{M} T_{g_l} T_{\bar{f}_l} = 0$;

(e) $\sum_{l=1}^{M} T_{\bar{f}_l} T_{g_l} = 0$.

Theorem 1.2 *Suppose for every integer $1 \le l \le M$, f_l and g_l, are bounded analytic functions on $\mathbb{D}$. Then the following are equivalent:*

(a) $\sum_{l=1}^{M} \bar{f}_l g_l = 0$ *and* $F = 0$;

(b) $\sum_{l=1}^{M} T_{\bar{g}_l} T_{f_l} = F$;

(c) $\sum_{l=1}^{M} T_{f_l} T_{\bar{g}_l} = F$;

(d) $\sum_{l=1}^{M} T_{g_l} T_{\bar{f}_l} = F$;

(e) $\sum_{l=1}^{M} T_{\bar{f}_l} T_{g_l} = F$.

2. Preliminary

Berezin transform is one of the basic tools in the study of operators on any reproducing kernel Hilbert space. For any function $f \in L^1(\mathbb{D}, \mathrm{d}A)$ on $\mathbb{D}$, the Berezin transform of f is defined by

$$Bf(z) = \langle f k_z, k_z \rangle,$$

where $z \in \mathbb{D}$, $f \in b^2$ and k_z is the normalized reproducing kernel of Bergman space. Recall that

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \overline{z}}$$

is the usual Laplacian on the complex plane. When dealing with the Berezin transform, it will be convenient for us to use the invariant Laplacian

$$\widetilde{\triangle} f = (1 - |z|^2) \triangle f(z),$$

where f is any twice differentiable function on $\mathbb{D}$.

A direct computation gives the following lemma.

Lemma 2.1 *Suppose that f, g are bounded harmonic functions on $\mathbb{D}$, and $f = f_1 + \overline{f_2}$, $g = g_1 + \overline{g_2}$, where f_1, f_2, g_1, g_2 are analytic functions. Then*

$$\begin{aligned}\langle T_f T_g k_z, k_z \rangle =& f_1(z) g_1(z) + B(\overline{f_2} g_1) + \overline{f_2(z) g_2(z)} + f_1(z)\overline{g_2(z)} + \\ & (1 - |z|^2)^2 P(f_1 \overline{P(g_2 \overline{K_z})}) - (1 - |z|^2)^2 f_1(z) \overline{g_2(z)}.\end{aligned}$$

Proof A direct computation gives that

$$\begin{aligned}\langle T_f T_g k_z, k_z \rangle &= \langle Q(f Q(g k_z)), k_z \rangle = \langle f Q(g_1 k_z + \overline{g_2} k_z), k_z \rangle \\ &= \langle (f_1 + \overline{f_2}) g_1 k_z, k_z \rangle + \langle (f_1 + \overline{f_2}) Q(\overline{g_2} k_z), k_z \rangle \\ &= f_1(z) g_z(z) + B(\overline{f_2} g_1) + \langle f_1 Q(\overline{g_2} k_z), k_z \rangle + \langle Q(\overline{g_2} k_z), f_2 k_z \rangle.\end{aligned}$$

Since

$$\langle Q(\overline{g_2} k_z), f_2 k_z \rangle = \langle \overline{g_2} k_z, f_2 k_z \rangle = \langle P(\overline{g_2} k_z), f_2 k_z \rangle = \overline{g_2(z)} \langle k_z, f_2 k_z \rangle = \overline{g_2(z) f_2(z)},$$

and

$$\langle f_1 Q(\overline{g_2} k_z), k_z \rangle = \langle f_1 [P(\overline{g_2} k_z) + \overline{P(g_2 \overline{k_z})} - P(\overline{g_2} k_z)(0)], k_z \rangle$$

$$= \langle f_1\overline{g_2(z)}k_z + f_1\overline{P(g_2\overline{k_z})} - (1-|z|^2)f_1\overline{g_2(z)}, k_z\rangle$$
$$= f_1(z)\overline{g_2(z)} + (1-|z|^2)^2 P(f_1\overline{P(g_2\overline{k_z})}) - (1-|z|^2)^2 f_1(z)\overline{g_2(z)},$$

we have

$$\langle T_fT_gk_z, k_z\rangle = f_1(z)g_1(z) + B(\overline{f_2}g_1) + \overline{f_2(z)g_2(z)} + f_1(z)\overline{g_2(z)} + (1-|z|^2)^2 P(f_1\overline{P(g_2\overline{K_z})}) - (1-|z|^2)^2 f_1(z)\overline{g_2(z)}. \quad \square$$

The Bloch space $\mathfrak{B}$ of $\mathbb{D}$ is defined to be the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathfrak{B}} = \sup_{z\in\mathbb{D}}\{(1-|z|^2)|f'(z)|\} < \infty.$$

It is easy to check that $\| \ \|_{\mathfrak{B}}$ is a complete semi-norm on $\mathfrak{B}$, and $\mathfrak{B}$ can be made into a Banach space by introducing the norm $\|f\| = |f(0)| + \|f\|_{\mathfrak{B}}$.

For each bounded harmonic function f on the unit disk, f can be written uniquely as a sum of an analytic function and a co-analytic function on the unit disk $\mathbb{D}$ up to a constant. Let f_1 denote the analytic part and $\overline{f_2}$ the co-analytic part with $\overline{f_2(0)} = 0$. In fact, f_1 and f_2 are in both the Hardy space H^2 and the Bloch space.

Lemma 2.2 *Suppose that f and g are bounded functions on $\mathbb{D}$. Then*

$$\langle T_fT_g\overline{k_z}, \overline{k_z}\rangle = \langle T_gT_fk_z, k_z\rangle.$$

Proof A direct computation gives

$$\langle T_fT_g\overline{k_z}, \overline{k_z}\rangle = \langle \overline{k_z}, T_{\overline{g}}T_{\overline{f}}\overline{k_z}\rangle = \overline{\langle T_{\overline{g}}T_{\overline{f}}\overline{k_z}, \overline{k_z}\rangle} = \overline{\langle \overline{g}T_{\overline{f}}\overline{k_z}, \overline{k_z}\rangle} = \langle g\overline{Q(\overline{fk_z})}, k_z\rangle$$
$$= \langle gQ(fk_z), k_z\rangle = \langle T_gT_fk_z, k_z.\rangle \quad \square$$

The rank one operator $x \otimes y$ is defined by

$$(x\otimes y)h = \langle h, y\rangle x,$$

where $x, y, h \in b^2$. Let N be an arbitrary number in $\mathbb{N}^+$ (the set of positive integers). If an operator F has finite rank, it can be written as

$$F = \sum_{j=1}^{N}(x_j \otimes y_j),$$

where $x_j, y_j \in b^2$, $N \in \mathbb{N}^+$. Notice that

$$\langle (x_j\otimes y_j)k_z, k_z\rangle = (1-|z|^2)^2\langle x_j, K_z\rangle\langle K_z, y_j\rangle = (1-|z|^2)^2 P(x_j)\overline{P(y_j)}.$$

In the following, we will denote

$$x_{j_1} = P(x_j), \quad x_{j_2} = x_j - x_{j_1} + x(0),$$
$$y_{j_1} = P(y_j), \quad y_{j_2} = y_j - y_{j_1} + y(0).$$

In this section we will prove a lemma that will be needed in later section.

Lemma 2.3 *Suppose for every integer $1 \le l \le M$, g_l is analytic, $f_l = f_{l,1} + \overline{f_{l,2}}$ and $h = h_1 + \overline{h_2}$*

are bounded harmonic functions on $\mathbb{D}$ with $f_{l,2}(0)=0$ and $\tilde{\Delta}h^n$ is bounded. Suppose

$$\sum_{l=1}^{M} T_{f_l}T_{g_l} = T_{h^n} + F,$$

where $n>1$ and $F=\sum_{j=1}^{N} x_j\otimes y_j$ is a finite rank operator with $x_j,\, y_j\in b^2$ and $N\in\mathbb{N}^+$. Then we have

(a) $\sum_{l=1}^{M}\overline{f_{l,2}}g_{l,1}-h^n$ *is harmonic;*

(b) $\sum_{l=1}^{M} f_lg_l-h^n=(1-|z|^2)^2\sum_{j=1}^{N}x_{j_1}\bar{y}_{j_1}$, *for $z\in\mathbb{D}$.*

Proof Suppose that $\sum_{l=1}^{M}T_{f_l}T_{g_l}=T_{h^n}+F$. We obtain that

$$\Big\langle \sum_{l=1}^{M} T_{f_l}T_{g_l}k_z,k_z\Big\rangle = \langle T_{h^n}k_z,k_z\rangle + \langle Fk_z,k_z\rangle.$$

By Lemma 2.1,

$$\Big\langle \sum_{l=1}^{M} T_{f_l}T_{g_l}k_z,k_z\Big\rangle = \sum_{l=1}^{M}[f_{l,1}(z)g_{l,1}(z)+B(\overline{f_{l,2}}g_{l,1})] = B(h^n)+\langle Fk_z,k_z\rangle.$$

Notice that

$$\langle Fk_z,k_z\rangle=(1-|z|^2)^2\sum_{j=1}^{N}x_{j_1}\bar{y}_{j_1}.$$

By the above two equations, we obtain that

$$B\Big(\sum_{l=1}^{M}\overline{f_{l,2}}g_{l,1}-h^n\Big)(z)=-\sum_{l=1}^{M}f_{l,1}g_{l,1}+(1-|z|^2)^2\sum_{j=1}^{N}x_{j_1}\overline{y_{j_1}}. \tag{2.1}$$

Applying the invariant Laplacian $\tilde{\Delta}$ to both sides of Eq. (2.1) and using the commutativity of Berezin transform and Laplacian operator [7, Lemma 1], we have

$$\begin{aligned} B\Big(\tilde{\Delta}\Big(\sum_{l=1}^{M}\overline{f_{l,2}}g_{l,1}-h^n\Big)\Big)(z) &= \tilde{\Delta}B\Big(\sum_{l=1}^{M}\overline{f_{l,2}}g_{l,1}-h^n\Big)(z)\\ &=\tilde{\Delta}\Big[-\sum_{l=1}^{M}f_{l,1}g_{l,1}+(1-|z|^2)^2\sum_{j=1}^{N}x_{j_1}\overline{y_{j_1}}\Big]. \end{aligned}\tag{2.2}$$

We denote the sequence by

$$\hat{x_i}=\begin{cases} x_{i_1}, & 1\le i\le N,\\ -zx_{(N-i)_1}, & N+1\le i\le 2N,\\ z^2x_{(2N-i)_1}, & 2N+1\le i\le 3N,\end{cases}$$

$$\hat{y_i}=\begin{cases} y_{i_1}, & 1\le i\le N,\\ 2y_{(N-i)_1}, & N+1\le i\le 2N,\\ z^2y_{(2N-i)_1}, & 2N+1\le i\le 3N,\end{cases}$$

where $i=1,\dots,3N$. Then $(1-|z|^2)^2\sum_{j=1}^{N}x_{j_1}\bar{y}_{j_1}$ in (2.2) can be written as $\sum_{i=1}^{3N}\hat{x}_i\bar{\hat{y}}_i$. Let

$\sigma = \tilde{\Delta}(\sum_{l=1}^{M} \overline{f_{l,2}} g_{l,1} - h^n)$. Then we can rewrite (2.2) as follows

$$\begin{aligned} B(\sigma)(z) &= (1-|z|^2)^2 \int_{\mathbb{D}} \frac{\sigma(\xi)}{(1-\bar{z}\xi)^2(1-\bar{\xi}z)^2} \mathrm{d}A(\xi) \\ &= (1-|z|^2)^2 \Big[-\sum_{l=1}^{M} f'_{l,1}(z) g'_{l,1}(z) + \sum_{i=1}^{3N} \hat{x}'_i(z) \overline{\hat{y}'_i(z)} \Big]. \end{aligned} \tag{2.3}$$

Cancelling $(1-|z|^2)^2$ on both sides of (2.3) and complexifying this equation as in [7], we can obtain that

$$\int_{\mathbb{D}} \frac{\sigma(\xi)}{(1-\bar{w}\xi)^2(1-\bar{\xi}z)^2} \mathrm{d}A(\xi) = -\sum_{l=1}^{M} f'_{l,1}(z) g'_{l,1}(z) + \sum_{i=1}^{3N} \hat{x}'_i(z) \overline{\hat{y}'_i(w)}, \tag{2.4}$$

for all $z, w \in \mathbb{D}$. Differentiating both sides of (2.4) k times with respect to $\bar{w}$ and then let $w = 0$ gives

$$\int_{\mathbb{D}} \frac{\sigma(\xi)\xi^k}{(1-\bar{\xi}z)^2} \mathrm{d}A(\xi) = \sum_{i=1}^{3N} a_{k,i} \hat{x}'_i(z) = \tilde{T}_\sigma(\xi^k),$$

for some constants $a_{k,i}$. From the argument of [15, Proposition 4], one can see that $\tilde{T}_\sigma$ has finite rank. Notice that

$$\tilde{\Delta}\Big(\sum_{l=1}^{M} \overline{f_{l,2}} g_{l,1} - h^n \Big) = \sum_{l=1}^{M} \tilde{\Delta}(\overline{f_{l,2}} g_{l,1}) - \tilde{\Delta}(h^n).$$

Using the fact that $f_{l,2}, g_{l,1} \in \mathfrak{B}$ for every $1 \le l \le M$, we have

$$\sum_{l=1}^{M} \tilde{\Delta}(\overline{f_{l,2}} g_{l,1}) = \sum_{l=1}^{M} (1-|z|^2) \overline{f'_{l,2}} (1-|z|^2) g'_{l,1} \le \sum_{l=1}^{M} \|f_{l,2}\|_{\mathfrak{B}} \|g_{l,1}\|_{\mathfrak{B}} < \infty.$$

With the fact that $\tilde{\Delta}(h^n)$ is bounded, thus $\sigma(z)$ is in $L^\infty(\mathbb{D})$. By [15, Theorem 2], $\sigma \equiv 0$. Hence $\sum_{l=1}^{M} \overline{f_{l,2}} g_{l,1} - h^n$ is a harmonic function. Thus (a) holds. Because the harmonic function is the fixed point of the Berezin transform, using $B(\sum_{l=1}^{M} \overline{f_{l,2}} g_{l,1} - h^n)(z) = \sum_{l=1}^{M} \overline{f_{l,2}} g_{l,1} - h^n$, we get that

$$\sum_{l=1}^{M} f_l g_l - h^n = (1-|z|^2)^2 \sum_{j=1}^{N} x_{j_1} \bar{y}_{j_1}$$

for all $z \in \mathbb{D}$, hence (b) holds. This completes the proof. □

Lemma 2.4 *Suppose for every integer $1 \le l \le M$, f_l and g_l, are bounded analytic functions on $\mathbb{D}$. Then*

$$\sum_{l=1}^{M} f_l(z) \bar{g}_l(z) = 0,$$

if and only if

$$\sum_{l=1}^{M} P(f_l \overline{P(g_l \bar{K}_z)})(z) = 0.$$

Proof Assume that $\sum_{l=1}^{M} P(f_l \overline{P(g_l \bar{K}_z)})(z) = 0$. Then

$$\sum_{l=1}^{M} P(f_l \overline{P(g_l \bar{K}_z)})(z) = \sum_{l=1}^{M} \langle P(f_l \bar{K}_z), P(g_l \bar{K}_z) \rangle = \sum_{l=1}^{M} \langle h_f K_z, h_g K_z \rangle$$

$$= \|K_z\|^2 \sum_{l=1}^{M} B(h_g^* h_f)(z) = 0.$$

Since h_f is the small Hankel operator on the Bergman space, and the Berezin transform on this space is one to one, we have $h_g^* h_f = h_{g^*} h_f = 0$. For $g^* = U\bar{g}(z) = \overline{g(\bar{z})}$, then g^* is analytic and $g^*(\bar{z}) = \overline{g(z)}$. By [20, Theorem 2.4], $h_{g^*} h_f = 0$ if and only if $\sum_{l=1}^{M} f_l(z) g_l^*(\bar{z}) = \sum_{l=1}^{M} f_l(z)\overline{g_l(z)} = 0$. This completes the proof. □

3. Main results

In this section, we give characterization for the finite rank of Toeplitz operators with the analytic and co-analytic symbols. In the following theorem, we consider the so-called "zero-product" problem of characterizing zero products of finite sums of Toeplitz operators with analytic and co-analytic symbols on the harmonic Bergman space.

Theorem 3.1 *Suppose for every integer $1 \le l \le M$, f_l and g_l, are bounded analytic functions on $\mathbb{D}$. Then the following are equivalent:*

(a) $\sum_{l=1}^{M} \bar{f}_l g_l = 0$;

(b) $\sum_{l=1}^{M} T_{f_l} T_{\bar{g}_l} = 0$;

(c) $\sum_{l=1}^{M} T_{\bar{g}_l} T_{f_l} = 0$;

(d) $\sum_{l=1}^{M} T_{g_l} T_{\bar{f}_l} = 0$;

(e) $\sum_{l=1}^{M} T_{\bar{f}_l} T_{g_l} = 0$.

Proof If (d) holds, we notice that it is the spacial case of (b) in Lemma 2.3 when $F = 0$, $h = 0$, we have $\sum_{l=1}^{M} g_l \bar{f}_l = 0$. (a) holds.

Conversely, if $\sum_{l=1}^{M} \bar{g}_l f_l = 0$ holds, by Lemma 2.4, $\sum_{l=1}^{M} P(f_l \overline{P(g_l \bar{K}_z)}) = 0$. Complexify these equations as in [7], we have

$$\sum_{l=1}^{M} P(f_l \overline{P(g_l \bar{K}_z)})(a) = 0, \tag{3.1}$$

$$\sum_{l=1}^{M} g_l(z)\overline{f_l(a)} = 0, \tag{3.2}$$

where $a, z \in \mathbb{D}$.

We will prove $\sum_{l=1}^{M} T_{\bar{f}_l} T_{g_l} R_z = 0$ because $\{R_z : z \in \mathbb{D}\}$ spans a dense subset of b^2.

By (3.1) and (3.2), we have

$$\sum_{l=1}^{M} T_{\bar{f}_l} T_{g_l} K_z(a) = \sum_{l=1}^{M} \langle \bar{f}_l Q(g_l K_z), R_a\rangle = \sum_{l=1}^{M} Q(\bar{f}_l g_l K_z)(a) = 0$$

$$\sum_{l=1}^{M} T_{\bar{f}_l} T_{g_l} \bar{K}_z(a) = \sum_{l=1}^{M} \langle \bar{f}_l Q(g_l \bar{K}_z), R_a\rangle = \sum_{l=1}^{M} \langle \bar{f}_l p(g_l \bar{K}_z), R_a\rangle + \sum_{l=1}^{M} \langle \bar{f}_l \overline{p(\bar{g}_l K_z)}, R_a\rangle - \sum_{l=1}^{M} \langle \bar{f}_l p(g_l \bar{K}_z)(0), R_a\rangle.$$

For each part of above, we have

$$\sum_{l=1}^{M}\langle \bar{f}_l p(g_l\bar{K}_z), R_a\rangle = \sum_{l=1}^{M}\langle \bar{f}_l g_l\bar{K}_z, R_a\rangle + \sum_{l=1}^{M}\overline{P(f_l\overline{P(g_l\bar{K}_z)})(a)} + \sum_{l=1}^{M}\langle \bar{f}_l g_l, K_z\rangle = 0,$$

and

$$\sum_{l=1}^{M}\langle \bar{f}_l\overline{p(\bar{g}_l K_z)}, R_a\rangle = \sum_{l=1}^{M} g_l(z)\langle \bar{f}_l\bar{K}_z, R_a\rangle = \sum_{l=1}^{M} g_l(z)\overline{f_l(a)K_z(a)} = 0.$$

Then $\sum_{l=1}^{M}\langle \bar{f}_l p(g_l\bar{K}_z)(0), R_a\rangle = 0$. We obtain that $\sum_{l=1}^{M} T_{\bar{f}_l}T_{g_l} = 0$. (a) and (e) are equivalent.

If (d) holds, by Lemma 2.2, we have $\sum_{l=1}^{M}\langle T_{g_l}T_{\bar{f}_l}\bar{k}_z, \bar{k}_z\rangle = \sum_{l=1}^{M}\langle T_{\bar{f}_l}T_{g_l}k_z, k_z\rangle$, then we have (a) and (d) are equivalent. This completes the proof. □

Theorem 3.2 *Suppose for every integer $1 \le l \le M$, f_l and g_l, are bounded analytic functions on $\mathbb{D}$. Then the following are equivalent:*

(a) $\sum_{l=1}^{M}\bar{f}_l g_l = 0$ *and* $F = 0$;
(b) $\sum_{l=1}^{M} T_{\bar{g}_l}T_{f_l} = F$;
(c) $\sum_{l=1}^{M} T_{f_l}T_{\bar{g}_l} = F$;
(d) $\sum_{l=1}^{M} T_{g_l}T_{\bar{f}_l} = F$;
(e) $\sum_{l=1}^{M} T_{\bar{f}_l}T_{g_l} = F$.

Proof If (a) holds, by Theorem 3.1, (b) and (c) hold. Conversely, if (b) holds, we have

$$\Big\langle \sum_{l=1}^{M} T_{\bar{g}_l}T_{f_l}k_z, k_z\Big\rangle = B\Big(\sum_{l=1}^{M}\bar{g}_l f_l\Big) = \langle Fk_z, k_z\rangle.$$

Let $\sigma = \sum_{l=1}^{M}\bar{g}f$. We have

$$\begin{aligned} B(\sigma)(z) &= (1-|z|^2)^2\int_{\mathbb{D}}\frac{\sigma(\xi)}{(1-\bar{z}\xi)^2(1-\bar{\xi}z)^2}\mathrm{d}A(\xi) \\ &= (1-|z|^2)^2\sum_{j=1}^{N} x_{j_1}\bar{y}_{j_1}. \end{aligned} \tag{3.3}$$

Cancel $(1-|z|^2)^2$ on both sides of (3.3) and complexify this equation as in [7], we obtain that

$$\int_{\mathbb{D}}\frac{\sigma(\xi)}{(1-\bar{w}\xi)^2(1-\bar{\xi}z)^2}\mathrm{d}A(\xi) = \sum_{j=1}^{N} x_{j_1}\bar{y}_{j_1}, \tag{3.4}$$

for all $z, w \in \mathbb{D}$. Differentiating both sides of (3.4) k times with respect to $\bar{w}$ and then let $w = 0$ gives

$$\int_{\mathbb{D}}\frac{\sigma(\xi)\xi^k}{(1-\bar{\xi}z)^2}\mathrm{d}A(\xi) = \sum_{j=1}^{N} a_{k,j}x_{j_1} = \tilde{T}_\sigma(\xi^k),$$

for some constants $a_{k,i}$. From the argument of [15, Proposition 4], one can see that $\tilde{T}_\sigma$ has finite rank. $\sigma(z)$ is in $L^\infty(\mathbb{D})$. By [15, Theorem 2], $\sigma \equiv 0$. Thus, $\sum_{l=1}^{M}\bar{f}_l g_l = 0$. By Theorem 3.1, $F = 0$. (a) holds.

Using Theorem 3.1 again, one can see (a)$\Longleftrightarrow$ (c), (a)$\Longleftrightarrow$ (d) and (a)$\Longleftrightarrow$ (e). This completes the proof. □

Corollary 3.3 *Suppose that f is bounded harmonic function and g is analytic, then $T_fT_g = 0$ if and only if $f = 0$ or $g = 0$.*

Proof $\langle T_fT_gk_z, k_z\rangle = f_1g_1 + B(\bar{f}_2g_1) = 0$. Thus, we have $B(\bar{f}_2g_1) = -f_1g_1$. f_1g_1 is analytic, then $\bar{f}_2g_1 + f_1g_1 = fg = 0$. This completes the proof. □

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