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Complete Manifolds with Harmonic Curvature and Finite L^p-Norm Curvature

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Abstract Let (M^n, g) $(n \geq 3)$ be an n-dimensional complete Riemannian manifold with harmonic curvature and positive Yamabe constant. Denote by R and $\mathring{Rm}$ the scalar curvature and the trace-free Riemannian curvature tensor of M, respectively. The main result of this paper states that $\mathring{Rm}$ goes to zero uniformly at infinity if for $p \geq n$, the L^p-norm of $\mathring{Rm}$ is finite.

As applications, we prove that (M^n, g) is compact if the L^p-norm of $\mathring{Rm}$ is finite and R is positive, and (M^n, g) is scalar flat if (M^n, g) is a complete noncompact manifold with nonnegative scalar curvature and finite L^p-norm of $\mathring{Rm}$. We prove that (M^n, g) is isometric to a spherical space form if for $p \geq \frac{n}{2}$, the L^p-norm of $\mathring{Rm}$ is sufficiently small and R is positive. In particular, we prove that (M^n, g) is isometric to a spherical space form if for $p \geq n$, R is positive and the L^p-norm of $\mathring{Rm}$ is pinched in $[0, C)$, where C is an explicit positive constant depending only on n, p, R and the Yamabe constant.

Keywords Harmonic curvature; trace-free curvature tensor; constant curvature space

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1. Introduction and main results

Recall that an n-dimensional Riemannian manifold (M^n, g) is said to be a manifold with harmonic curvature if the divergence of its Riemannian curvature tensor Rm vanishes, i.e., $\delta Rm = 0$. In view of the second Bianchi identity, we know that M has harmonic curvature if and only if the Ricci tensor of M is a Codazzi tensor. When $n \geq 3$, by the Bianchi identity, the scalar curvature is constant. Thus, every Riemannian manifold with parallel Ricci tensor has harmonic curvature. Moreover, the constant curvature spaces, Einstein manifolds and the locally conformally flat manifolds with constant scalar curvature are also important examples of manifolds with harmonic curvature, however, the converse does not hold [1]. According to the decomposition of the Riemannian curvature tensor, the metric with harmonic curvature is a natural candidate for this study since one of the important problems in Riemannian geometry is to understand classes of metrics that are, in some sense, close to being Einstein or having constant curvature. The

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another reason for this study on the metric with harmonic curvature is the fact that a Riemannian manifold has harmonic curvature if and only if the Riemannian connection is a solution of the Yang-Mills equations on the tangent bundle [2]. In recent years, the complete manifolds with harmonic curvature have been studied in literature [3–11]. Recently, Tian and Viaclovsky [11], Chen and Weber [12] have obtained ϵ-rigidity results for critical metric which relies on a Sobolev inequality and integral bounds on the curvature in dimension 4 and in higher dimension, respectively. The curvature pinching phenomenon plays an important role in global differential geometry. We are interested in L^p pinching problems for complete Riemannian manifold with harmonic curvature.

Throughout this paper, we always assume that M is an n-dimensional complete Riemannian manifold with $n \geq 3$. We now introduce the definition of the Yamabe constant. Given a complete Riemannian n-manifold M, the Yamabe constant $Q(M)$ is defined by

$$Q(M) = \inf_{0 \neq u \in C_0^\infty(M)} \frac{\int_M \left(|\nabla u|^2 + \frac{(n-2)}{4(n-1)} R u^2\right)}{\left(\int_M |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}},$$

where R is the scalar curvature of M. The important works of Schoen, Trudinger and Yamabe showed that the infimum in the above is always achieved [13,14]. There are complete noncompact Riemannian manifolds of negative scalar curvature with positive Yamabe constant. For example, any simply connected complete locally conformally flat manifold has positive Yamabe constant [15], and $Q(M)$ is always positive if R vanishes [16]. In contrast with the noncompact case, the Yamabe constant of a given compact manifold is determined by the sign of scalar curvature [13].

In this note, we extend in some sense some results due to [4,6,7,9,10] to obtain the following rigidity theorems.

Theorem 1.1 *Let M be a complete Riemannian n-manifold with harmonic curvature. Assume that M has the positive Yamabe constant or satisfies the Sobolev inequality*

$$\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_S \int_M |\nabla f|^2, \quad \forall f \in C_0^\infty(M).$$

For $p \geq n$, if $\int_M |\mathring{Rm}|^p < +\infty$, then, given any $\epsilon > 0$ and any $x_0 \in M$ there exists a geodesic ball $B_r(x_0)$ with center x_0 and radius r such that $|\mathring{Rm}|(x) < \epsilon$ for all $x \in M \setminus B_r(x_0)$.

Theorem 1.2 *Let M be a complete Riemannian n-manifold with harmonic curvature and positive scalar curvature. Assume that M has the positive Yamabe constant. For $p \geq n$, if $\int_M |\mathring{Rm}|^p < +\infty$, then M must be compact.*

Corollary 1.3 *Let M be a complete noncompact Riemannian n-manifold with harmonic curvature and nonnegative scalar curvature. Assume that M has the positive Yamabe constant. For $p \geq n$, if $\int_M |\mathring{Rm}|^p < +\infty$, then M must be scalar flat.*

Theorem 1.4 *Let M be a complete Riemannian n-manifold with harmonic curvature and* *positive scalar curvature. Assume that M has the positive Yamabe constant. For $p \geq n$, if*

$$\Big(\int_{M^n} |\mathring{Rm}|^p\Big)^{\frac{1}{p}} < C,$$

where

$$C = \begin{cases} \dfrac{3pR}{4(2p-n)c(n)}[\dfrac{8(2p-n)Q(M)}{3nR}]^{\frac{n}{2p}}, & n = 3 \text{ and } 3 \leq p < 6 \\ \dfrac{R}{(n-1)c(n)}[\dfrac{4(n-1)Q(M)}{(n-2)R}]^{\frac{n}{2p}}, & n = 3 \text{ and } p \geq 6, \text{and } n \geq 4, \end{cases}$$

then M is isometric to a spherical space form.

Theorem 1.5 *Let M be a complete Riemannian n-manifold with harmonic curvature, positive scalar curvature and positive Yamabe constant. Then there exists a small number C such that if*

$$\Big(\int_{M^n} |\mathring{Rm}|^p\Big)^{\frac{1}{p}} < C, \quad p \geq \frac{n}{2},$$

then M is isometric to a spherical space form. In particular, when $p = \frac{n}{2}$, there exists an explicit positive constant $C = \frac{4Q(M)}{(n-2)c(n)}$.

Remark 1.6 When $R \geq 0$, some $L^{\frac{n}{2}}$ trace-free Riemannian curvature pinching theorems have been shown by Kim [7] and Chu [3], in which the constant C is not explicit.

Theorem 1.7 *Let M^n $(n \geq 10)$ be a complete Riemannian n-manifold with harmonic curvature and negative scalar curvature. Assume that M has positive Sobolev constant. For $\gamma \in (1, \frac{n(n-2)+\sqrt{n(n-2)(n^2-10n+8)}}{4(n-1)})$, if $\int_M |\mathring{Rm}|^\gamma < \infty$, then there exists a small number C such that if*

$$\int_M |\mathring{Rm}|^p < C, \quad p \geq \frac{n}{2},$$

then M is a hyperbolic space form.

Remark 1.8 Theorem 1.7 can be considered as generalization of some result in [12]. When $p \geq n$ in Theorem 1.7, $\gamma \in (0, \frac{n(n-2)+\sqrt{n(n-2)(n^2-10n+8)}}{4(n-1)})$. In the case of $\gamma \in (0,1]$. Since $\int_M |\mathring{Rm}|^p < C$, by Theorem 1.1, $|\mathring{Rm}|$ is bounded. Hence $\int_M |\mathring{Rm}|^{\gamma+1} < \infty$ for $\int_M |\mathring{Rm}|^\gamma < \infty$. For $\gamma + 1 \in (1, \frac{n(n-2)+\sqrt{n(n-2)(n^2-10n+8)}}{4(n-1)})$, we apply Theorem 1.7 to prove the above result.

Remark 1.9 Let M be a complete, simply connected, locally conformally flat Riemannian n-manifold. Using the same argument as in this note, Peng and the first author obtain some analog of Theorems in this note and generalize the result due to [10] (see [17]).

2. Proof of Lemma

In what follows, we adopt, without further comment, the moving frame notation with respect to a chosen local orthonormal frame.

Let M be a Riemannian manifold with harmonic curvature. The decomposition of the

Riemannian curvature tensor into irreducible components yields

$$\begin{aligned}R_{ijkl} =&W_{ijkl} + \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) - \\ &\frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\ =&W_{ijkl} + \frac{1}{n-2}(\mathring{R}_{ik}\delta_{jl} - \mathring{R}_{il}\delta_{jk} + \mathring{R}_{jl}\delta_{ik} - \mathring{R}_{jk}\delta_{il}) + \\ &\frac{R}{n(n-1)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),\end{aligned}$$

where R_{ijkl}, W_{ijkl}, R_{ij} and $\mathring{R}_{ij}$ denote the components of Rm, the Weyl curvature tensor W, the Ricci tensor Ric and the trace-free Ricci tensor $\mathring{Ric} = Ric - \frac{R}{n}g$, respectively, and R is the scalar curvature.

The trace-free Riemannian curvature tensor $\mathring{Rm}$ is

$$\mathring{R}_{ijkl} = R_{ijkl} - \frac{R}{n(n-1)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{1}$$

Then the following equalities are easily obtained from the properties of curvature tensor:

$$g^{ik}\mathring{R}_{ijkl} = \mathring{R}_{jl}, \tag{2}$$

$$\mathring{R}_{ijkl} + \mathring{R}_{iljk} + \mathring{R}_{iklj} = 0, \tag{3}$$

$$\mathring{R}_{ijkl} = \mathring{R}_{klij} = -\mathring{R}_{jikl} = -\mathring{R}_{ijlk}, \tag{4}$$

$$|\mathring{Rm}|^2 = |W|^2 + \frac{4}{n-2}|\mathring{Ric}|^2. \tag{5}$$

Moreover, by the assumption of harmonic curvature, we compute

$$\mathring{R}_{ijkl,m} + \mathring{R}_{ijmk,l} + \mathring{R}_{ijlm,k} = 0, \tag{6}$$

and

$$\mathring{R}_{ijkl,l} = 0. \tag{7}$$

Now, we compute the Laplacian of $|\mathring{Rm}|^2$.

Lemma 2.1 *Let M be a complete Riemannian n-manifold with harmonic curvature. Then*

$$\triangle|\mathring{Rm}|^2 \geq 2|\nabla\mathring{Rm}|^2 - 2c(n)|\mathring{Rm}|^3 - \frac{8R}{n(n-1)}|\mathring{Ric}|^2 + \frac{4R}{n}|\mathring{Rm}|^2, \tag{8}$$

where $c(n) = 5 + \sqrt{\frac{(n-1)(n-2)}{n}}$.

Remark 2.2 Lemma 2.1 has been proved in [4], in which the constant $c(n)$ is not explicit. For completeness, we also write it out.

Proof By the Ricci identities, we obtain from (1)–(7)

$$\begin{aligned}\triangle|\mathring{Rm}|^2 =&2|\nabla\mathring{Rm}|^2 + 2\langle\mathring{Rm}, \triangle\mathring{Rm}\rangle = 2|\nabla\mathring{Rm}|^2 + 2\mathring{R}_{ijkl}\mathring{R}_{ijkl,mm} \\ =&2|\nabla\mathring{Rm}|^2 + 2\mathring{R}_{ijkl}(\mathring{R}_{ijkm,lm} + \mathring{R}_{ijml,km}) \\ =&2|\nabla\mathring{Rm}|^2 + 4\mathring{R}_{ijkl}\mathring{R}_{ijkm,lm}\end{aligned}$$

$$
\begin{aligned}
&=2|\nabla \mathring{R}m|^2 + 4\mathring{R}_{ijkl}(\mathring{R}_{ijkm,ml} + \mathring{R}_{hjkm}R_{hilm}+\\
&\mathring{R}_{ihkm}R_{hjlm} + \mathring{R}_{ijhm}R_{hklm} + \mathring{R}_{ijkh}R_{hmlm})\\
&=2|\nabla \mathring{R}m|^2 + 4\mathring{R}_{ijkl}(\mathring{R}_{hjkm}R_{hilm} + \mathring{R}_{ihkm}R_{hjlm}+\\
&\mathring{R}_{ijhm}R_{hklm} + \mathring{R}_{ijkh}R_{hmlm})\\
&=2|\nabla \mathring{R}m|^2 + 4\mathring{R}_{ijkl}(\mathring{R}_{hjkm}\mathring{R}_{hilm} + \mathring{R}_{ihkm}\mathring{R}_{hjlm} + \mathring{R}_{ijhm}\mathring{R}_{hklm}+\\
&\mathring{R}_{ijkh}\mathring{R}_{hmlm}) + \frac{4R}{n(n-1)}\mathring{R}_{ijkl}(\mathring{R}_{ljki} + \mathring{R}_{ilkj} + \mathring{R}_{ijlk}+\\
&\mathring{R}_{jk}\delta_{il} - \mathring{R}_{ik}\delta_{jl}) + \frac{4R}{n}|\mathring{R}m|^2\\
&=2|\nabla \mathring{R}m|^2 - 4\mathring{R}_{ijkl}(2\mathring{R}_{ihkm}\mathring{R}_{hjml} + \frac{1}{2}\mathring{R}_{hmij}\mathring{R}_{klhm} + \mathring{R}_{ijkh}\mathring{R}_{hl})-\\
&\frac{8R}{n(n-1)}|\mathring{R}ic|^2 + \frac{4R}{n}|\mathring{R}m|^2\\
&\geq 2|\nabla \mathring{R}m|^2 - 2c(n)|\mathring{R}m|^3 - \frac{8R}{n(n-1)}|\mathring{R}ic|^2 + \frac{4R}{n}|\mathring{R}m|^2,
\end{aligned}
$$

where the algebraic inequality $|\lambda_i| \leq \sqrt{\frac{n-1}{n}}|T|$ for the eigenvalues λ_i of trace-free symmetric n-matrices T is used in the above. This completes the proof of this Lemma. □

From (5), we have $|\mathring{R}ic|^2 \leq \frac{n-2}{4}|\mathring{R}m|^2$. Combining the above with (8), we obtain

$$\triangle|\mathring{R}m|^2 \geq 2|\nabla \mathring{R}m|^2 - 2c(n)|\mathring{R}m|^3 + 2AR|\mathring{R}m|^2, \tag{9}$$

where

$$A = \begin{cases} \dfrac{1}{n-1}, & R \geq 0 \\ \dfrac{2}{n}, & R < 0. \end{cases}$$

3. Proof of Theorems

Now we can prove Theorem 1.1 based on (9).

Proof of Theorem 1.1 From (9), by the Kato inequality $|\nabla \mathring{R}m|^2 \geq |\nabla|\mathring{R}m||^2$, we obtain

$$|\mathring{R}m|\triangle|\mathring{R}m| = \frac{1}{2}\triangle|\mathring{R}m|^2 - |\nabla|\mathring{R}m||^2 \geq -c(n)|\mathring{R}m|^3 + AR|\mathring{R}m|^2. \tag{10}$$

Let $u = |\mathring{R}m|$. By (10), we compute

$$
\begin{aligned}
u^\alpha\triangle u^\alpha =& u^\alpha\left(\alpha(\alpha-1)u^{\alpha-2}|\nabla u|^2 + \alpha u^{\alpha-1}\triangle u\right)\\
=&\frac{\alpha-1}{\alpha}|\nabla u^\alpha|^2 + \alpha u^{2\alpha-2}u\triangle u\\
\geq&\frac{\alpha-1}{\alpha}|\nabla u^\alpha|^2 - c(n)\alpha u^{2\alpha+1} + \alpha ARu^{2\alpha},
\end{aligned}
\tag{11}
$$

where α is a positive constant. When $\alpha \geq 1$, using the Young's inequality, from (11) we obtain

$$u^\alpha\triangle u^\alpha \geq -au^{4\alpha} - bu^{2\alpha}, \tag{12}$$

where a and b are positive constants depending only on n, α and R. Setting $w = u^\alpha$, we can rewrite (12) as

$$-\triangle w \leq aw^3 + bw. \tag{13}$$

Since M has the positive Yamabe constant or satisfies the Sobolev inequality, combining with (13), we can carry out the proof of this Theorem by using the same argument as in the proof of [18, Theorem 1.1]. $\Box$

Proof of Theorem 1.2 By (1), we have

$$R_{ijij} = \overset{\circ}{R}_{ijij} + \frac{R}{n(n-1)}. \tag{14}$$

Note that R is positive. From (14), we see from Theorem 1.1 that there is a positive constant δ such that $R_{ijij} > \delta$ in $M \setminus \Omega$ for some compact set Ω. This implies that the Ricci curvature is bounded from below by a positive constant outside some geodesic sphere, hence the manifold is compact (for detail, see [10, Lemma 3.5]). $\Box$

Proof of Theorem 1.4 When $R > 0$, we see from Theorem 1.2 that M is compact. Taking $\alpha = \frac{p}{n}$. By (11), using the Young's inequality, we have

$$u^\alpha \triangle u^\alpha \geq \frac{\alpha - 1}{\alpha}|\nabla u^\alpha|^2 - \frac{c(n)}{2}\epsilon^{1-2\alpha}u^{4\alpha} - [\frac{c(n)}{2}(2\alpha - 1)\epsilon - AR\alpha]u^{2\alpha}. \tag{15}$$

Setting $w = u^\alpha$, we can rewrite (15) as

$$w\triangle w \geq \frac{\alpha - 1}{\alpha}|\nabla w|^2 - \frac{c(n)}{2}\epsilon^{1-2\alpha}w^4 - [\frac{c(n)}{2}(2\alpha - 1)\epsilon - AR\alpha]w^2. \tag{16}$$

From (16), we obtain

$$w^\beta \triangle w^\beta \geq (1 - \frac{1}{\alpha\beta})|\nabla w^\beta|^2 - \frac{c(n)}{2}\beta\epsilon^{1-2\alpha}w^{2(\beta+1)} - \beta[\frac{c(n)}{2}(2\alpha - 1)\epsilon - AR\alpha]w^{2\beta}, \tag{17}$$

where β is a positive constant. From (17), integrating by parts, we get

$$(2 - \frac{1}{\alpha\beta})\int_M |\nabla w^\beta|^2 - \frac{c(n)}{2}\beta\epsilon^{1-2\alpha}\int_M w^{2(\beta+1)} - \beta[\frac{c(n)}{2}(2\alpha - 1)\epsilon - \frac{R\alpha}{n-1}]\int_M w^{2\beta} \leq 0. \tag{18}$$

By the Hölder inequality and (18), we have

$$\begin{aligned}(2 - \frac{1}{\alpha\beta})\int_M |\nabla w^\beta|^2 - \frac{c(n)}{2}\beta\epsilon^{1-2\alpha}(\int_M w^{\frac{2n\beta}{n-2}})^{\frac{n-2}{n}}(\int_M w^n)^{\frac{2}{n}} - \\ \beta[\frac{c(n)}{2}(2\alpha - 1)\epsilon - \frac{R\alpha}{n-1}]\int_M w^{2\beta} \leq 0.\end{aligned} \tag{19}$$

Case 1 When $n = 3$ and $1 \leq \alpha < 2$, set $\epsilon = \frac{3\alpha R}{4(2\alpha-1)c(n)}$ and $\beta = \frac{1}{\alpha}$. By the definition of Yamabe constant $Q(M)$, from (19) we get

$$\Big[Q(M) - \frac{c(n)\epsilon^{1-2\alpha}}{2\alpha}\Big(\int_M |\overset{\circ}{Rm}|^p\Big)^{\frac{2}{n}}\Big]\Big(\int_M w^{\frac{2n\beta}{n-2}}\Big)^{\frac{n-2}{n}} \leq 0. \tag{20}$$

We choose $(\int_M |\overset{\circ}{Rm}|^p)^{\frac{1}{p}} < \frac{3pR}{4(2p-n)c(n)}[\frac{8(2p-n)Q(M)}{3nR}]^{\frac{n}{2p}}$ such that (20) implies $(\int_M w^{\frac{2n\beta}{n-2}})^{\frac{n-2}{n}} = 0$, that is, $|\overset{\circ}{Rm}| = 0$, i.e., M is Einstein manifold and locally conformally flat manifold. Hence M is isometric to a spherical space form.

Case 2 When $n = 3$ and $\alpha \geq 2$, and $n \geq 4$, set $\epsilon = \frac{R}{(n-1)c(n)}$ and $\frac{1}{\alpha\beta} = 1 + \sqrt{1 - \frac{2}{(n-2)\alpha}}$. We also get

$$\Big[(2 - \frac{1}{\alpha\beta})Q(M) - \frac{c(n)}{2}\beta\epsilon^{1-2\alpha}\Big(\int_M |\overset{\circ}{Rm}|^p\Big)^{\frac{2}{n}}\Big]\Big(\int_M w^{\frac{2n\beta}{n-2}}\Big)^{\frac{n-2}{n}} \leq 0. \tag{21}$$

We choose $(\int_M |\overset{\circ}{Rm}|^p)^{\frac{1}{p}} < \frac{R}{(n-1)c(n)}[\frac{4(n-1)Q(M)}{(n-2)R}]^{\frac{n}{2p}}$ such that (21) implies $(\int_M w^{\frac{2n\beta}{n-2}})^{\frac{n-2}{n}} = 0$, that is, $|\overset{\circ}{Rm}| = 0$, i.e., M is Einstein manifold and locally conformally flat manifold. Hence M is isometric to a spherical space form. $\square$

Proof of Theorem 1.5 Let ϕ be a smooth compactly supported function on M. Taking $\alpha = \frac{p}{n} \geq \frac{1}{2}$. First choosing $\beta = \frac{n}{2}$ in (17), multiplying (17) by ϕ^2 and integrating over M, we obtain

$$\begin{aligned}(1 - \frac{2}{n\alpha})\int_M |\nabla w^{\frac{n}{2}}|^2\phi^2 \leq& \frac{nc(n)\epsilon^{1-2\alpha}}{4}\int_M w^{n+2}\phi^2 + \int_M w^{\frac{n}{2}}\phi^2\triangle w^{\frac{n}{2}} + \\ & \frac{n}{4}[c(n)(2\alpha-1)\epsilon - \frac{2R\alpha}{n-1}]\int_M w^n\phi^2 \\ =& \frac{nc(n)\epsilon^{1-2\alpha}}{4}\int_M w^{n+2}\phi^2 - 2\int_M w^{\frac{n}{2}}\phi\langle\nabla\phi, \nabla w^{\frac{n}{2}}\rangle - \\ & \int_M |\nabla w^{\frac{n}{2}}|^2\phi^2 + \frac{n}{4}[c(n)(2\alpha-1)\epsilon - \frac{2R\alpha}{n-1}]\int_M w^n\phi^2,\end{aligned}$$

which gives

$$\begin{aligned}(2 - \frac{2}{n\alpha})\int_M |\nabla w^{\frac{n}{2}}|^2\phi^2 \leq& \frac{nc(n)\epsilon^{1-2\alpha}}{4}\int_M w^{n+2}\phi^2 - 2\int_M w^{\frac{n}{2}}\phi\langle\nabla\phi, \nabla w^{\frac{n}{2}}\rangle + \\ & \frac{n}{4}[c(n)(2\alpha-1)\epsilon - \frac{2R\alpha}{n-1}]\int_M w^n\phi^2.\end{aligned} \tag{22}$$

Using the Cauchy-Schwarz inequality, we can rewrite (22) as

$$\begin{aligned}(2 - \frac{2}{n\alpha} - \varepsilon)\int_M |\nabla w^{\frac{n}{2}}|^2\phi^2 \leq& \frac{nc(n)\epsilon^{1-2\alpha}}{4}\int_M w^{n+2}\phi^2 + \frac{1}{\varepsilon}\int_M w^n|\nabla\phi|^2 + \\ & \frac{n}{4}[c(n)(2\alpha-1)\epsilon - \frac{2R\alpha}{n-1}]\int_M w^n\phi^2,\end{aligned} \tag{23}$$

for the positive constant ε. By the definition of Yamabe constant $Q(M)$ and (23), we have

$$\begin{aligned}&Q(M)\Big(\int_M (\phi w^n)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} \leq \int_M \big(|\nabla(\phi w^{\frac{n}{2}})|^2 + \frac{(n-2)Rw^n\phi^2}{4(n-1)}\big) \\ &= \int_M (w^n|\nabla\phi|^2 + \phi^2|\nabla w^{\frac{n}{2}}|^2 + 2\phi w^{\frac{n}{2}}\langle\nabla\phi, \nabla w^{\frac{n}{2}}\rangle + \frac{(n-2)Rw^n\phi^2}{4(n-1)}) \\ &\leq (1 + \frac{1}{\eta})\int_M w^n|\nabla\phi|^2 + (1+\eta)\int_M \phi^2|\nabla w^{\frac{n}{2}}|^2 + \int_M \frac{(n-2)Rw^n\phi^2}{4(n-1)} \\ &\leq B\int_M w^n|\nabla\phi|^2 + E\int_M w^{n+2}\phi^2 + D\int_M w^n\phi^2,\end{aligned} \tag{24}$$

where

$$B = 1 + \frac{1}{\eta} + \frac{1+\eta}{\varepsilon(2 - \frac{2}{n\alpha} - \varepsilon)}, \quad E = \frac{(1+\eta)nc(n)\epsilon^{1-2\alpha}}{4(2 - \frac{2}{n\alpha} - \varepsilon)},$$

$$D = \frac{(n-2)R}{4(n-1)} - \frac{(1+\eta)n\alpha R}{2(n-1)(2-\frac{2}{n\alpha}-\varepsilon)} + \frac{(1+\eta)nc(n)(2\alpha-1)\epsilon}{4(2-\frac{2}{n\alpha}-\varepsilon)}.$$

Noting that ϵ and ε are sufficiently small, we choose $\eta > \max\{0, \frac{(n-2)(n\alpha-1)}{n^2\alpha^2} - 1\}$ such that $D \le 0$. Thus from (24) we have

$$\begin{aligned} Q(M)\Big(\int_M (\phi w^n)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} &\le B\int_M w^n|\nabla\phi|^2 + E\int_M w^{n+2}\phi^2 \\ &\le B\int_M w^n|\nabla\phi|^2 + E\Big(\int_M (\phi w^n)^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}}\Big(\int_M w^n\Big)^{\frac{2}{n}}. \end{aligned}$$

Since $\int_M u^{n\alpha}$ is sufficiently small, the second term on the right-hand side of the above can be absorbed in the left-hand side. Therefore, there exists a constant $F > 0$, such that

$$F\Big(\int_M (\phi u^{n\alpha})^{\frac{n}{n-2}}\Big)^{\frac{n-2}{n}} \le B\int_M u^{n\alpha}|\nabla\phi|^2. \tag{25}$$

Let us choose a cutoff function ϕ satisfying the properties that

$$\phi(x) = \begin{cases} 1 \text{ on } B(r) \\ 0 \text{ on } M\setminus B(2r), \end{cases}$$

and $|\nabla\phi| \le \frac{2}{r}$. In particular, if M is compact, and if $r > d$, where d is the diameter of M, then $\phi = 1$ on M. From (25), we get

$$F\Big(\int_{B_r} u^{\frac{n^2}{n-2}\alpha}\Big)^{\frac{n-2}{n}} \le \frac{4}{r^2}B\int_M u^{n\alpha}. \tag{26}$$

Let $r \to +\infty$. By assumption that $\int_M u^{n\alpha} < \infty$, from (26), we have $u = 0$, i.e., M is Einstein manifold and locally conformally flat manifold. Hence M is isometric to a spherical space form.

When $p = \frac{n}{2}$, we choose η such that $D = 0$, i.e., $(2 - \frac{2}{n\alpha} - \varepsilon) = \frac{(1+\eta)n}{(n-2)}$. Thus we have

$$\frac{Q(M)}{E} = \frac{4Q(M)\Lambda_0(2-\frac{2}{n\alpha}-\epsilon)}{nc(n)(1+\eta)} = \frac{4Q(M)}{(n-2)C(n)}.$$

So we choose $(\int_M |\mathring{Rm}|^{\frac{n}{2}})^{\frac{2}{n}} < \frac{4Q(M)}{(n-2)C(n)}$ such that $Q(M) - E(\int_M u^{\frac{n}{2}})^{\frac{2}{n}} > 0$. The rest of the proof runs as before. $\square$

Remark 3.1 Taking $\alpha = \frac{1}{2}$ in the proof of Theorem 1.5, we obtain some trace-free Riemannian curvature pinching theorems for complete Riemannian manifolds with harmonic curvature, zero scalar curvature and positive Yamabe constant, which were proved in [4,7].

Proof of Theorem 1.7 Multiplying (17) by ϕ^2 and integrating over M, we obtain

$$\begin{aligned} (1-\frac{1}{\alpha\beta})\int_M |\nabla w^\beta|^2\phi^2 \le& \frac{c(n)}{2}\beta\epsilon^{1-2\alpha}\int_M w^{2(\beta+1)}\phi^2 + \int_M w^\beta\phi^2\triangle w^\beta + \\ & \beta[\frac{c(n)}{2}(2\alpha-1)\epsilon - \frac{2R\alpha}{n}]\int_M w^{2\beta}\phi^2 \\ =& \frac{c(n)}{2}\beta\epsilon^{1-2\alpha}\int_M w^{2(\beta+1)}\phi^2 - 2\int_M w^\beta\phi\langle\nabla\phi, \nabla w^\beta\rangle - \\ & \int_M |\nabla w^\beta|^2\phi^2 + \beta[\frac{c(n)}{2}(2\alpha-1)\epsilon - \frac{2R\alpha}{n}]\int_M w^{2\beta}\phi^2, \end{aligned}$$

which gives

$$(2-\frac{1}{\alpha\beta})\int_M |\nabla w^\beta|^2\phi^2 \leq \frac{c(n)}{2}\beta\epsilon^{1-2\alpha}\int_M w^{2(\beta+1)}\phi^2 - 2\int_M w^\beta\phi\langle\nabla\phi,\nabla w^\beta\rangle + \beta[\frac{c(n)}{2}(2\alpha-1)\epsilon - \frac{2R\alpha}{n}]\int_M w^{2\beta}\phi^2. \tag{27}$$

Using the Cauchy-Schwarz inequality, we can rewrite (27) as

$$(2-\frac{1}{\alpha\beta}-\varepsilon)\int_M |\nabla w^\beta|^2\phi^2 \leq \frac{c(n)}{2}\beta\epsilon^{1-2\alpha}\int_M w^{2(\beta+1)}\phi^2 + \frac{1}{\varepsilon}\int_M w^{2\beta}|\nabla\phi|^2 + \beta[\frac{c(n)}{2}(2\alpha-1)\epsilon - \frac{2R\alpha}{n}]\int_M w^{2\beta}\phi^2, \tag{28}$$

for the positive constant ε. By the definition of Yamabe constant $Q(M)$ and (28), we have

$$\begin{aligned}
Q(M)\Big(\int_M (\phi w^\beta)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} &\leq \int_M \Big(|\nabla(\phi w^\beta)|^2 + \frac{(n-2)Rw^{2\beta}\phi^2}{4(n-1)}\Big)\\
&= \int_M (w^{2\beta}|\nabla\phi|^2 + \phi^2|\nabla w^\beta|^2 + 2\phi w^\beta\langle\nabla\phi,\nabla w^\beta\rangle + \frac{(n-2)Rw^{2\beta}\phi^2}{4(n-1)})\\
&\leq (1+\frac{1}{\eta})\int_M w^{2\beta}|\nabla\phi|^2 + (1+\eta)\int_M \phi^2|\nabla w^\beta|^2 + \int_M \frac{(n-2)Rw^{2\beta}\phi^2}{4(n-1)}\\
&\leq G\int_M w^{2\beta}|\nabla\phi|^2 + H\int_M w^{2(\beta+1)}\phi^2 + I\int_M w^{2\beta}\phi^2,
\end{aligned} \tag{29}$$

where

$$G = 1+\frac{1}{\eta}+\frac{1+\eta}{\varepsilon(2-\frac{1}{\alpha\beta}-\varepsilon)},\quad H = \frac{(1+\eta)c(n)\beta\epsilon^{1-\alpha}}{2(2-\frac{1}{\alpha\beta}-\varepsilon)},$$
$$I = \frac{(n-2)R}{4(n-1)} - \frac{2(1+\eta)\alpha\beta R}{n(2-\frac{1}{\alpha\beta}-\varepsilon)} + \frac{(1+\eta)c(n)(2\alpha-1)\beta\epsilon}{2(2-\frac{1}{\alpha\beta}-\varepsilon)}.$$

We first consider the case of $\gamma \in (1, \frac{n(n-2)+\sqrt{n(n-2)(n^2-10n+8)}}{4(n-1)})$. When $n \geq 10$, noting that ϵ, ε and η are sufficiently small, we choose $\frac{1}{2} < \alpha\beta < \frac{n(n-2)+\sqrt{n(n-2)(n^2-10n+8)}}{8(n-1)}$ such that $I \leq 0$. Thus from (29) we have

$$\begin{aligned}
\Lambda_0\Big(\int_M (\phi w^\beta)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} &\leq G\int_M w^{2\beta}|\nabla\phi|^2 + H\int_M w^{2(\beta+1)}\phi^2\\
&\leq G\int_M w^{2\beta}|\nabla\phi|^2 + H\Big(\int_M (\phi w^\beta)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}}\Big(\int_M w^n\Big)^{\frac{2}{n}}.
\end{aligned}$$

Since $\int_M w^n = \int_M u^{n\alpha}$ is sufficiently small, the second term on the right-hand side of the above can be absorbed in the left-hand side. Therefore, there exists a constant $J > 0$, such that

$$J\Big(\int_M (\phi u^{\alpha\beta})^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \leq G\int_M u^{2\alpha\beta}|\nabla\phi|^2. \tag{30}$$

Let us choose a cutoff function ϕ satisfying the properties that

$$\phi(x) = \begin{cases} 1 \text{ on } B(r)\\ 0 \text{ on } M\setminus B(2r), \end{cases}$$

and $|\nabla\phi| \leq \frac{2}{r}$. In particular, if M is compact, and if $r > d$, where d is the diameter of M, then

$\phi = 1$ on M. From (30), we get

$$J\Big(\int_{B_r} u^{\frac{2n}{n-2}\alpha\beta}\Big)^{\frac{n-2}{n}} \leq \frac{4}{r^2} B \int_M u^{2\alpha\beta}. \tag{31}$$

Let $r \to +\infty$. By assumption that $\int_M u^{2\alpha\beta} < \infty$, from (31), we have $u = 0$, i.e., M is Einstein manifold and locally conformally flat manifold. Hence M is isometric to a hyperbolic space form. $\square$

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