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A Class of Multivariate Hermite Interpolation of Total Degree

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Abstract In this paper we study a class of multivariate Hermite interpolation problem on 2^d nodes with dimension $d \geq 2$ which can be seen as a generalization of two classical Hermite interpolation problems of $d = 2$. Two combinatorial identities are firstly given and then the regularity of the proposed interpolation problem is proved.

Keywords multivariate Hermite interpolation; regularity; total degree

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1. Introduction

Let n, d be nonnegative integers and Π_n^d be the space of polynomials of total degree at most n. Let $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$ be a set of pairwise distinct points in $\mathbb{R}^d$ and $\mathbf{p} = \{t_1, t_2, \ldots, t_m\}$ be a set of m nonnegative integers. The Hermite interpolation problem to be considered in this paper is described as follows: For given real values $\{c_{i,\boldsymbol{\alpha}}, 1 \leq i \leq m, 0 \leq |\boldsymbol{\alpha}| \leq t_i\}$, find a polynomial $f \in \Pi_n^d$ satisfying

$$D^{\boldsymbol{\alpha}} f(X_i) = c_{i,\boldsymbol{\alpha}}, \ \ 1 \leq i \leq m, \ 0 \leq |\boldsymbol{\alpha}| \leq t_i, \tag{1}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d), |\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_d$,

$$D^{\boldsymbol{\alpha}} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$$

and the numbers t_i and n are assumed to satisfy

$$\binom{n+d}{d} = \sum_{i=1}^{m} \binom{t_i + d}{d}. \tag{2}$$

The interpolation problem $(\mathbf{p}, \mathscr{X})$ is called regular if the above equation has a unique solution for each choice of values $\{c_{i,\boldsymbol{\alpha}}\}$. Otherwise, the interpolation problem is singular. If $(\mathbf{p}, \mathscr{X})$ is regular for almost all $\mathscr{X} \subset \mathbb{R}^d$, then we say that $(\mathbf{p}, \mathscr{X})$ is almost regular. In fact, if the interpolation problem is regular for some $\mathscr{X} \subset \mathbb{R}^d$, then it is also regular for almost all $\mathscr{X} \subset \mathbb{R}^d$ (see [1]). Thus we also say that $(\mathbf{p}, \mathscr{X})$ is almost regular if it is regular for some $\mathscr{X} \subset \mathbb{R}^d$.

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The research on the regularity of multivariate Hermite interpolation is very difficult because Eq. (2) does not hold in many cases. For $m \leq d+1$, all interpolation schemes are singular [1,2]. For $m \leq d+3$, a complete description for the regularity of the interpolation problem was given in [3]. For $m \leq d(d+3)/2$, authors [4] presented many regular interpolation schemes. For more results of multivariate Hermite interpolation, one can refer to [5–10] and the references therein. Especially, the following theorem is classical in the theory of multivariate Hermite interpolation, which has been widely used in multivariate splines and finite element theory.

Theorem 1.1 ([1]) *Let $d = 2$. Interpolating the value of a function and all of its partial derivatives of order up to p at each of the three vertices of a triangle as well as the value of the function and all of its derivatives of order up to $p+1/p-1$ at a fourth point lying anywhere in the interior of the triangle by polynomials from $\Pi^2_{2p+2}/\Pi^2_{2p+1}$ is regular.*

In [4], the result in Theorem 1.1 was extended to $d = 3$.

Theorem 1.2 ([4]) *Let $\mathscr{X} = \{X_1 = (1,1,1)^T, X_2 = (1,0,0)^T, X_3 = (0,1,0)^T, X_4 = (0,0,1)^T, X_5 = (1,1,0)^T, X_6 = (1,0,1)^T, X_7 = (0,1,1)^T, X_8 = (0,0,0)^T\}$, and $\mathbf{p} = \{t-1,t,t,t,t,t,t,t+1\}$ or $\mathbf{p} = \{t-1,t-1,t-1,t-1,t,t,t,t\}$ $(t \geq 1)$. Then $(\mathbf{p}, \mathscr{X})$ is regular.*

The purpose of this paper is to generalize Theorems 1.1 and 1.2 to higher dimension.

2. Main results

Eq. (2) is a necessary condition to study whether the interpolation problem is regular. So, we first present a lemma with n, d and $\mathbf{p}$. For convenience, let $\binom{N}{M} = 0$ if $N < M$.

Lemma 2.1 *Suppose $t \in Z$. Then*

$$\binom{t+1+d}{d} + \sum_{i=0}^{[(d-1)/2]} \binom{d+1}{2i+2}\binom{t-i+d}{d} = \binom{2t+2+d}{d}, \tag{3}$$

$$\sum_{i=0}^{[d/2]} \binom{d+1}{2i+1}\binom{t-i+d}{d} = \binom{2t+1+d}{d}, \tag{4}$$

where $[\cdot]$ denotes the integral part.

Proof The proof is by induction. For $d = 2$, it is easy to get

$$\binom{t+1+2}{2} + 3\binom{t+2}{2} = \binom{2t+2+2}{2},$$

$$3\binom{t+2}{2} + \binom{t+1}{2} = \binom{2t+1+2}{2},$$

which implies that Eqs. (3) and (4) hold.

Suppose that Eqs. (3) and (4) hold for $d = m$, and we will show that they also hold for

$d = m+1$. If m is an even number, adding Eq. (3) to Eq. (4) will give

$$\begin{aligned}&\binom{t+1+m}{m} + \sum_{i=0}^{m/2-1}\left[\binom{m+1}{2i+2}+\binom{m+1}{2i+1}\right]\binom{t-i+m}{m} + \binom{t-m/2+m}{m}\\&= \binom{t+1+m}{m} + \sum_{i=0}^{m/2-1}\binom{m+2}{2i+2}\binom{t-i+m}{m} + \binom{t-m/2+m}{m}\\&= \binom{2t+2+m}{m} + \binom{2t+1+m}{m},\end{aligned}$$

which holds for any $t \in Z$. Substituting j for t and taking the summation with respect to j on the both sides of the equation, we can get

$$\begin{aligned}&\sum_{j=-1}^{t}\binom{j+1+m}{m} + \sum_{j=-1}^{t}\sum_{i=0}^{m/2-1}\binom{m+2}{2i+2}\binom{j-i+m}{m} + \sum_{j=-1}^{t}\binom{j-m/2+m}{m}\\&= \sum_{j=-1}^{t}\binom{2j+2+m}{m} + \sum_{j=-1}^{t}\binom{2j+1+m}{m}.\end{aligned} \tag{5}$$

By simple computation, we have

$$\begin{aligned}&\sum_{j=-1}^{t}\binom{j+1+m}{m} = \binom{t+2+m}{m+1},\\&\sum_{j=-1}^{t}\binom{j-i+m}{m} = \binom{t-i+m+1}{m+1},\\&\sum_{j=-1}^{t}\binom{j-m/2+m}{m} = \binom{t-m/2+m+1}{m+1},\\&\sum_{j=-1}^{t}\left[\binom{2j+1+m}{m}+\binom{2j+2+m}{m}\right] = \binom{2t+2+m+1}{m+1}.\end{aligned}$$

Substituting these equalities into Eq. (5) yields

$$\begin{aligned}&\binom{t+2+m}{m+1} + \sum_{i=0}^{m/2-1}\binom{m+2}{2i+2}\binom{t-i+m+1}{m+1} + \binom{t-m/2+m+1}{m+1}\\&= \binom{t+1+m+1}{m+1} + \sum_{i=0}^{m/2}\binom{m+1+1}{2i+2}\binom{t-i+m+1}{m+1}\\&= \binom{2t+2+m+1}{m+1},\end{aligned}$$

which implies that Eq. (3) holds for $d = m+1$.

Replacing t with $t-1$ in Eq. (3), we have

$$\binom{t+m}{m} + \sum_{i=0}^{m/2-1}\binom{m+1}{2i+2}\binom{t-1-i+m}{m} = \binom{2t+m}{m}$$

i.e.,

$$\binom{t+m}{m}+\sum_{i=1}^{m/2}\binom{m+1}{2i}\binom{t-i+m}{m}=\binom{2t+m}{m}.$$

Adding this equation to (4) yields

$$\sum_{i=0}^{m/2}\left[\binom{m+1}{2i}+\binom{m+1}{2i+1}\right]\binom{t-i+m}{m}=\binom{2t+m}{m}+\binom{2t+1+m}{m},$$

i.e.,

$$\sum_{i=0}^{m/2}\binom{m+2}{2i+1}\binom{t-i+m}{m}=\binom{2t+m}{m}+\binom{2t+1+m}{m}.$$

It holds for any $t\in Z$. Substituting j for t and taking the summation with respect to j on the both sides of the equation, we can get

$$\sum_{j=0}^{t}\sum_{i=0}^{m/2}\binom{m+2}{2i+1}\binom{j-i+m}{m}=\sum_{j=0}^{t}\left[\binom{2j+m}{m}+\binom{2j+1+m}{m}\right].$$

By further calculation, it is easy to obtain

$$\sum_{i=0}^{m/2}\binom{m+2}{2i+1}\binom{t-i+m+1}{m+1}=\binom{2t+1+m+1}{m+1}.$$

Since m is an even number, the above equality can be rewritten as

$$\sum_{i=0}^{[(m+1)/2]}\binom{m+1+1}{2i+1}\binom{t-i+m+1}{m+1}=\binom{2t+1+m+1}{m+1}.$$

Therefore, Eq. (4) holds for $d=m+1$.

Similarly, if m is an odd number, Eqs.(3) and (4) also hold for $d=m+1$. By induction, we complete the proof. $\Box$

Next, we consider the corresponding interpolation problems when Eqs. (3) and (4) hold. Noting that

$$1+\sum_{i=0}^{[(d-1)/2]}\binom{d+1}{2i+2}=1+\sum_{i=0}^{[(d-1)/2]}\left[\binom{d}{2i+1}+\binom{d}{2i+2}\right]=\sum_{i=0}^{d}\binom{d}{i}=2^d,$$
$$\sum_{i=0}^{[d/2]}\binom{d+1}{2i+1}=\sum_{i=0}^{[d/2]}\left[\binom{d}{2i}+\binom{d}{2i+1}\right]=2^d,$$

the number of interpolation nodes is 2^d and the interpolation nodes can be selected as follows. Suppose that $K=[0,1]^d$ denotes the d dimensional hypercube. Obviously, it denotes unit rectangle or unit cube for $d=2$ or 3, respectively. Define

$$\mathscr{X}_k=\{X\in\mathscr{X}:|X|=k\},\quad k=0,1,2,\ldots,d.$$

Then the number of the points in $\mathscr{X}_k$ is

$$|\mathscr{X}_k|=\binom{d}{k},\quad k=0,1,2,\ldots,d,$$

and obviously $\mathscr{X} = \cup_{k=0}^{d} \mathscr{X}_k$. And let

$$p_0 = t+1, p_1 = t, p_2 = t, \ldots, p_i = t - [(i-1)/2], \ldots, p_d = t - [(d-1)/2]$$

$$q_0 = t, q_1 = t, q_2 = t-1, \ldots, q_k = t - [k/2], \ldots, q_d = t - [d/2].$$

Then we have the following theorem.

Theorem 2.2 *(i) If $f \in \Pi_{2t+2}^d$ satisfies*

$$D^{\boldsymbol{\alpha}} f(X_i) = 0, \quad X_i \in \mathscr{X}_k, \quad 0 \le |\boldsymbol{\alpha}| \le p_k, \ k = 0, 1, \ldots, d, \tag{6}$$

then $f \equiv 0$.

(ii) If $f \in \Pi_{2t+1}^d$ satisfies

$$D^{\boldsymbol{\alpha}} f(X_i) = 0, \quad X_i \in \mathscr{X}_k, \quad 0 \le |\boldsymbol{\alpha}| \le q_k, \ k = 0, 1, \ldots, d, \tag{7}$$

then $f \equiv 0$.

Proof The proof is by induction with respect to the dimension d. For $d = 2, 3$, the results are true and given by [1,3]. Suppose two statements hold for any dimension less than d and we will show that they also hold for d. We prove the first result firstly.

Again we prove it by induction about t. For $t = 0$, the result is the same as [3, Theorem 15] and is correct. Assume that the result is correct for any integer less than t.

Any polynomial of order $2t+2$ can be written as

$$f(x_1, x_2, \ldots, x_d) = x_1 f_1(x_1, x_2, \ldots, x_d) + r_1(x_2, x_3, \ldots, x_d),$$

where f_1 is a polynomial with $\deg(f_1) \le 2t+1$ and r_1 is a polynomial with $\deg(r_1) \le 2t+2$.

Consider the interpolation conditions on the hyperplane $x_1 = 0$

$$\begin{aligned} &D^{\boldsymbol{\alpha}} f(X_i) = 0, \quad X_i \in \mathscr{X}_k|_{x_1=0}, \quad \boldsymbol{\alpha} = (0, \alpha_2, \ldots, \alpha_d), \\ &0 \le |\boldsymbol{\alpha}| \le p_k, \quad k = 0, 1, \ldots, d, \end{aligned} \tag{8}$$

where $\mathscr{X}_k|_{x_1=0}$ denotes the set of the points in $\mathscr{X}_k$ with $x_1 = 0$.

Substituting f into (8) yields

$$\begin{aligned} &D^{\boldsymbol{\alpha}} r_1(X_i) = 0, \quad X_i \in \mathscr{X}_k|_{x_1=0}, \quad \boldsymbol{\alpha} = (0, \alpha_2, \ldots, \alpha_d), \\ &0 \le |\boldsymbol{\alpha}| \le p_k, \quad k = 0, 1, \ldots, d. \end{aligned} \tag{9}$$

Since r_1 is a polynomial with respect to $x_2, \ldots, x_d$ and of $\deg(r_1) \le 2t+2$, interpolation problem (9) can be seen as a $d-1$ dimensional interpolation problem. According to the inductive hypothesis for d, we have $r_1 \equiv 0$ and $f = x_1 f_1$. Similarly, f can be divided by $x_i, i = 2, \ldots, d$. Hence f can be written as

$$f(x_1, x_2, \ldots, x_d) = x_1 x_2 x_3 \ldots x_d g(x_1, x_2, \ldots, x_d), \quad \deg(g) \le 2t + 2 - d.$$

If $2t + 2 < d$, we have $f \equiv 0$. Otherwise, we consider the interpolation conditions on the hyperplane $x_1 = 1$. The following conditions should be satisfied

$$D^{\boldsymbol{\alpha}} g(X_i) = 0, \quad X_i \in \mathscr{X}_k|_{x_1=1}, \quad 0 \le |\boldsymbol{\alpha}| \le p_k - d + k, \ k = 0, 1, \ldots, d, \tag{10}$$

where $p_k - d + k = t + [k/2] + 1 - d$. Again let

$$g(x_1, x_2, \ldots, x_d) = (x_1 - 1)g_1(x_1, \ldots, x_2) + r_2(x_2, \ldots, x_d),$$

where $r_2(x_2, \ldots, x_d)$ is a polynomial of degree no more than $2t + 2 - d$. Substituting g into (10) yields

$$D^{\boldsymbol{\alpha}} r_2(X_i) = 0, \quad X_i \in \mathscr{X}_k|_{x_1=1}, \quad \boldsymbol{\alpha} = (0, \alpha_2, \ldots, \alpha_d),$$
$$0 \le |\boldsymbol{\alpha}| \le t + [k/2] + 1 - d, \quad k = 0, 1, \ldots, d.$$

Thus we have $r_2 \equiv 0$ by inductive hypothesis of the second statement for odd d and the first statement for even d, which means that g can be divided by $x_1 - 1$. Similarly, g can be divided by $x_i - 1, i = 2, 3, \ldots, d$. If $2t + 2 - 2d < 0$, the result is true. Otherwise, together with the previous conclusion, we have

$$f(x_1, x_2, \ldots, x_d) = x_1 x_2 \ldots x_d (x_1 - 1)(x_2 - 1) \ldots (x_d - 1) h(x_1, x_2, \ldots, x_d),$$

where $\deg(h) \le 2t + 2 - 2d$ and

$$D^{\boldsymbol{\alpha}} h(X_i) = 0, \quad X_i \in \mathscr{X}_k, \quad 0 \le |\boldsymbol{\alpha}| \le p_k - d, \ k = 0, 1, \ldots, d. \tag{11}$$

Clearly, (11) is the same interpolation problem as (6), but substituting $t - d$ for t. By the inductive hypothesis for t, we have $h \equiv 0$ and hence $f \equiv 0$. Therefore, the first statement holds for d and is proved by inductive method.

The proof of the second statement is similar and omitted. We complete the proof. □

Remark 2.3 In Theorem 2.2, only $t \ge 0$ is required for (6) and $t \ge 1$ is required for (7). If $p_k < 0$, it implies that no interpolation happens at this point and the results are also true.

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