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Unicyclic Graphs with a Perfect Matching Having Signless Laplacian Eigenvalue Two

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Abstract In this paper, a necessary and sufficient condition for a unicyclic graph with a perfect matching having signless Laplacian eigenvalue 2 is deduced.

Keywords Signless Laplacian matrix; unicyclic graph; multiplicity

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Its order is $|V(G)|$, denoted by n, and its size is $|E(G)|$, denoted by m. In this paper, all graphs are simple connected of order $n \geq 2$. For $v \in V(G)$, let $d(v)$ and $N(v)$ be the degree and the set of neighbors of v, respectively. For any $e \in E(G)$, we use $G - e$ to denote the graph obtained by deleting e from G. Similarly, for any set W of vertices (edges), $G - W$ and $G + W$ are the graphs obtained by deleting the vertices (edges) in W from G and by adding the vertices (edges) in W to G, respectively.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of G, respectively. The Laplacian and signless Laplacian matrices of G are respectively defined as

$$L(G) = D(G) - A(G) \quad \text{and} \quad Q(G) = D(G) + A(G).$$

The eigenvalues of $L(G)$ and $Q(G)$ (also called the Laplacian and signless Laplacian eigenvalues of G, respectively) are respectively denoted by

$$\mu_1(G) \geq \cdots \geq \mu_n(G) = 0 \quad \text{and} \quad q_1(G) \geq \cdots \geq q_n(G).$$

Also, the multiplicities of an eigenvalue λ for $L(G)$ and $Q(G)$ are denoted by $m_G(\lambda)$ and $m_G^+(\lambda)$, respectively.

Recall that the nullity of a graph G, denoted by $\eta(G)$, is the multiplicity of eigenvalue 0 for $A(G)$. Let $X = X(G)$ be the 0, 1 vertex-edge incidence matrix of G and $\mathcal{L}(G)$ be the line graph

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of G. Note that $X^TX = A(\mathcal{L}(G)) + 2I$ and $XX^T = Q(G)$ have the same nonzero eigenvalues with the same multiplicities. In particular, the multiplicity of eigenvalue 2 for $Q(G)$ is the same as the nullity of $\mathcal{L}(G)$, i.e., $m_G^+(2) = \eta(\mathcal{L}(G))$. Hence, results obtained for the nullity of the line graph of a graph in [1–3] can be immediately re-stated for the multiplicity of its signless Laplacian eigenvalue 2.

Let $\mathscr{U}_n$ be the set of unicyclic graphs of order n, and $\mathscr{U}_n^g$ be the set of unicyclic graphs of order n with girth g ($3 \le g \le n$). Clearly, if $n = g$, then there is only one graph C_n in $\mathscr{U}_n^n$. For each $U \in \mathscr{U}_n^g$, U consists of the (unique) cycle (say C_g) of length g and a certain number of trees attached at vertices of C_g having (in total) $n-g$ edges. Throughout this paper, we assume that the vertices of C_g are $v_1, v_2, \ldots, v_g$ (in a natural order around C_g, say in the clockwise direction). Then U can be written as $C(T_1, \ldots, T_g)$, which is obtained from a cycle C_g on vertices $v_1, v_2, \ldots, v_g$ by identifying v_i with the root of a tree T_i for each $i = 1, 2, \ldots, g$, where $\sum_{i=1}^g |V(T_i)| = n$. The sun graph of order $2n$ is a cycle C_n with an edge terminating in a pendent vertex attached to each vertex. A broken sun graph is a uncyclic subgraph of a sun graph. Let $n_i(G)$ be the number of vertices of degree i in G.

Recently, Akbari et al. [4] studied the multiplicity of Laplacian eigenvalue 2 in unicyclic graphs, some of their results are listed as follows:

Theorem 1.1 ([4]) *Let $U \in \mathscr{U}_n^g$ be a broken sun graph containing a perfect matching. Then $n_2(U) \equiv 0 \pmod 4$ if and only if $L(U)$ has an eigenvalue 2.*

Theorem 1.2 ([4]) *Let $U = C(T_1, \ldots, T_g)$ be a unicyclic graph containing a perfect matching and let s be the number of T_i of odd orders. Then*

(i) $s \equiv 0 \pmod 4$ if and only if $L(U)$ has an eigenvalue 2;

(ii) $m_U(2) = 2$ if and only if $s = g$ and $g \equiv 0 \pmod 4$.

Theorem 1.3 ([4]) *Let $U \in \mathscr{U}_n^g$ be a broken sun graph. Then $m_U(2) = 2$ if and only if $U \cong C_n$ and $n \equiv 0 \pmod 4$.*

Theorem 1.4 ([4]) *Let $U \in \mathscr{U}_n^g$ be a broken sun graph of order n without perfect matching. Then $g \equiv 0 \pmod 4$ and there are odd number of vertices of degree 2 between any pair of consecutive vertices of degree 3, if and only if $L(U)$ has an eigenvalue 2.*

In this paper, we further study the multiplicity of signless Laplacian eigenvalue 2 in a unicyclic graph. A necessary and sufficient condition for a unicyclic graph with a perfect matching having signless Laplacian eigenvalue 2 is deduced. Moreover, a necessary and sufficient condition for a unicyclic graph U with a perfect matching having $m_U^+(2) = 2$ is also deduced. Some of those results are similar to that in the Laplacian matrix, but some of them are somewhat different.

2. Preliminaries

In this section, we present some lemmas which will be used in the subsequent sections.

Lemma 2.1 ([5]) *If G is a bipartite graph, then the matrices $L(G)$ and $Q(G)$ are similar, i.e., the Laplacian and signless Laplacian eigenvalues of G are the same.*

Lemma 2.2 ([5]) *Let G be a graph of order n and $e \in E(G)$. Then the following holds*

$$q_1(G) \geq q_1(G-e) \geq q_2(G) \geq q_2(G-e) \geq \cdots \geq q_n(G) \geq q_n(G-e).$$

Lemma 2.3 ([6]) *Let T be a tree of order n. If $\lambda > 1$ is an integral eigenvalue of $L(T)$ with corresponding eigenvector $\boldsymbol{u}$, then*

(i) $\lambda | n$ (i.e., λ exactly divides n),

(ii) The multiplicity of λ is equal to 1,

(iii) No coordinate of $\boldsymbol{u}$ is zero.

Note that $\mathcal{L}(C_n) \cong C_n$, $\eta(\mathcal{L}(C_n)) = m^+_{C_n}(2)$ and $\eta(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4; \\ 0 & \text{otherwise.} \end{cases}$ (see [7]). Hence we have

Lemma 2.4 *For the cycle C_n of length n, $m^+_{C_n}(2) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4; \\ 0 & \text{otherwise.} \end{cases}$*

Recall that for a connected graph G with pendent vertex v and $N(v) = \{u\}$, we have $\eta(G) = \eta(G - \{u, v\})$ (see [7]). This implies that $\eta(\mathcal{L}(G)) = \eta(\mathcal{L}(G\#K_2))$, where $G\#K_2$ is a connected sum of G and K_2, which is obtained by joining some vertex of $V(G)$ to one of vertices of K_2 with a single edge. Hence we have

Lemma 2.5 *For any graph G, $m^+_G(2) = m^+_{G\#K_2}(2)$.*

3. Main results

In this section, we deduce a necessary and sufficient condition for a unicyclic graph with a perfect matching having signless Laplacian eigenvalue 2.

Theorem 3.1 *For $U \in \mathscr{U}_n$, if $\lambda > 1$ is an integral eigenvalue of $Q(U)$, then $m^+_U(\lambda) \leq 2$.*

Proof Suppose for a contradiction that $m^+_U(\lambda) \geq 3$. Note that there exists at least one edge $e \in E(U)$ such that $U - e$ is a tree since $U \in \mathscr{U}_n$. Then Lemmas 2.1 and 2.2 imply that $m^+_{U-e}(\lambda) = m_{U-e}(\lambda) \geq 2$. This contradicts Lemma 2.3. Hence $m^+_U(\lambda) \leq 2$. □

Remark 3.2 For $U \in \mathscr{U}_n$ and integer $\lambda > 1$, a similar result as that in Theorem 3.1 has been presented for $m_U(\lambda)$ in [4]. Moreover, in fact, Theorem 3.1 also gives a characterization on the multiplicity of signless Laplacian eigenvalue 2 for uncyclic graphs, that is, $m^+_U(2) \leq 2$ for any $U \in \mathscr{U}_n$.

Theorem 3.3 *Let $U \in \mathscr{U}_n$. If n is odd, then $m^+_U(2) \leq 1$.*

Proof Suppose for a contradiction that $m^+_U(\lambda) \geq 2$. Since $U \in \mathscr{U}_n$, there exists at least one edge $e \in E(U)$ such that $U - e$ is a tree. Then Lemmas 2.1 and 2.2 imply that $m^+_{U-e}(\lambda) = m_{U-e}(\lambda) \geq 1$. But Lemma 2.3 implies that $2|n$, which is a contradiction. Therefore, $m^+_U(2) \leq 1$. □

Remark 3.4 If $U \in \mathscr{U}_n$ with odd n, then $\eta(\mathcal{L}(U)) \leq 1$.

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $Q(G)$ corresponding to the eigenvalue 2, and x_i denote the entry of $\boldsymbol{x}$ corresponding to the vertex v_i of G. For any vertex $u \in V(G)$, if $d(u) = 1$ and $uu' \in E(G)$, then $x_u + x_{u'} = 2x_u$. So

$$x_{u'} = x_u. \tag{1}$$

Moreover, if $d(u) = 2$, then $2x_u + \sum_{u' \in N(u)} x_{u'} = 2x_u$. This implies that

$$\sum_{u' \in N(u)} x_{u'} = 0. \tag{2}$$

Also, if $d(u) = 3$, then $3x_u + \sum_{u' \in N(u)} x_{u'} = 2x_u$, which yields that

$$\sum_{u' \in N(u)} x_{u'} = -x_u. \tag{3}$$

Theorem 3.5 *Let $U \in \mathscr{U}_n^g$ be a broken sun graph containing a perfect matching.*

(i) *If g is even, then $n_2(U) \equiv 0 \pmod 4$ if and only if $Q(U)$ has an eigenvalue 2;*

(ii) *If g is odd, then $n_2(U) \equiv 2 \pmod 4$ if and only if $Q(U)$ has an eigenvalue 2.*

Proof The case when g is even follows from Lemma 2.1 and Theorem 1.1. In what follows, we assume that g is odd. Note that U is a broken graph contains a perfect matching. Then $g - n_3(U) = n_2(U)$ is even.

Firstly, suppose that $n_2(U) \equiv 2 \pmod 4$. Suppose $n_2(U) = 2$. Without loss of generality, let v_1 and v_2 be vertices of degree 2. Then $v_1v_2 \in E(U)$ since U has a perfect matching. We label the vertices of U in the form as shown in Figure 1. Assign 1 to the vertices $v_1, v_2, v_4, v_6, \ldots, v_{2k}, \ldots, v_{g-1}$, -1 to the vertices $v_3, v_5, \ldots, v_{2k+1}, \ldots, v_g$, also assign to each pendent vertex the same value of its neighbor. Then we obtain an eigenvector of $Q(U)$ for eigenvalue 2.

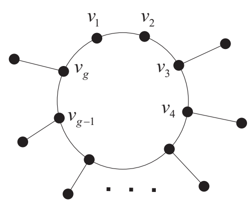

Figure 1 The graph U with $n_2(U) = 2$

Suppose $n_2(U) \geq 6$. By induction on g, we shall show that $Q(U)$ has an eigenvalue 2. For $g = 3$, 5, the result follows from Appendix A. Now, we assume that $g \geq 7$. Note that there exists even number of vertices of degree 2 between any pair of consecutive vertices of degree 3 since U has a perfect matching and $n_2(U) \geq 6$. We now consider the following two cases:

Case 1 U has at least four consecutive vertices of degree 2.

Without loss of generality, we assume that $d(v_k) = d(v_{k+1}) = d(v_{k+2}) = d(v_{k+3}) = 2$. Let $U' = U - \{v_k, v_{k+1}, v_{k+2}, v_{k+3}\} + v_{k-1}v_{k+4}$. Obviously, U' is a broken sun graph with a perfect matching whose girth is $g - 4$ and $n_2(U') \equiv 2 \pmod 4$. Thus, by induction hypothesis,

$Q(U')$ has an eigenvalue 2 with eigenvector $\boldsymbol{x} = (x_1, x_2, \ldots, x_{k-1}, x_{k+4}, \ldots, x_n)^T \in \mathbb{R}^{n-4}$. Let $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$ be a vector as

$$y_i = \begin{cases} x_i, & \text{if } 1 \le i \le k-1; \\ x_{k+4}, & \text{if } i = k; \\ -x_{k-1}, & \text{if } i = k+1; \\ -x_{k+4}, & \text{if } i = k+2; \\ x_{k-1}, & \text{if } i = k+3; \\ x_i, & \text{if } k+4 \le i \le n. \end{cases}$$

Case 2 U has at most two consecutive vertices of degree 2 between any pair of consecutive vertices of degree 3.

Recall that U has a perfect matching and $g \ge 7$ is odd. Then $n_3(U)$ is also odd and there exists odd number of vertices of degree 3 between at least one pair of consecutive vertices of degree 2, say v_k and v_{k+l+1}, where l is odd, $d(v_{k-1}) = d(v_{k+l+2}) = 2$, $d(v_{k-2}) = d(v_{k+l+3}) = 3$ and $d(v_{k+1}) = \cdots = d(v_{k+l}) = 3$ (shown in Figure 2). Let $U' = U - \{v_{k-1}, v_k, v_{k+l+1}, v_{k+l+2}\} + v_{k-2}v_{k+1} + v_{k+l}v_{k+l+3}$. Obviously, U' is a broken sun graph with a perfect matching whose girth is $g-4$ and $n_2(U') \equiv 2 \pmod 4$. Thus, by induction hypothesis, $Q(U')$ has an eigenvalue 2 with eigenvector $\boldsymbol{x} = (x_1, x_2, \ldots, x_{k-2}, x_{k+1}, \ldots, x_{k+l}, x_{k+l+3}, \ldots, x_n)^T \in \mathbb{R}^{n-4}$. Let $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$ be a vector as

$$y_i = \begin{cases} x_i, & \text{if } 1 \le i \le k-2; \\ x_{k+1}, & \text{if } i = k-1; \\ -x_{k-2}, & \text{if } i = k; \\ -x_i, & \text{if } k+1 \le i \le k+l; \\ -x_{k+l+3}, & \text{if } i = k+l+1; \\ x_{k+l}, & \text{if } i = k+l+2; \\ x_i, & \text{if } k+l+3 \le i \le g \end{cases}$$

and assign to each pendent vertex the same value of its neighbor.

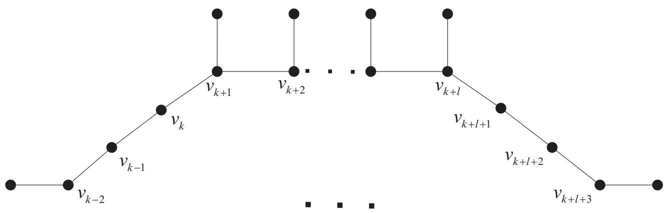

Figure 2 The graph U in Case 2

In both cases, one may check that the vector $\boldsymbol{y}$ satisfies Eqs. (1), (2) and (3). Hence $\boldsymbol{y}$ is an eigenvector of $Q(U)$ corresponding to the eigenvalue 2.

Conversely, assume that $Q(U)$ has an eigenvalue 2. Using induction on g, we shall show that $n_2(U) \equiv 2 \pmod 4$. By Appendix A, one may check that the result holds for $g = 3$, 5. Now we assume that $g \ge 7$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$ is an eigenvector of $Q(U)$ corresponding to the eigenvalue 2. Suppose for a contradiction that $n_2(U) \not\equiv 2 \pmod 4$. Then $n_2(U) \equiv 0 \pmod 4$ since U has a perfect matching. If $n_2(U) = 0$, then U is a sun graph (shown in Figure 3).

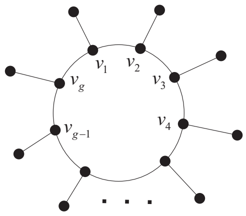

Figure 3 The sun graph U

If $x_1 = a$ and $x_2 = b$, then by Eqs. (1) and (3), we have $x_3 = -b-(a+b), x_4 = b+2(a+b), \ldots, x_{g-1} = b+(g-3)(a+b), x_g = -b-(g-2)(a+b)$. Similarly, on the other hand, we have $x_g = -a-(a+b), x_{g-1} = a+2(a+b), \ldots, x_4 = a+(g-3)(a+b), x_3 = -a-(g-2)(a+b)$. Those imply that

$$-b-(a+b) = -a-(g-2)(a+b) \quad \text{and} \quad -a-(a+b) = -b-(g-2)(a+b),$$

i.e.,

$$a-b = -(g-3)(a+b) \quad \text{and} \quad b-a = -(g-3)(a+b).$$

That is $a = b = -a$. Hence $x_i = 0$ for $1 \le i \le n$. This is a contradiction. Thus $n_2(U) \ge 4$. Note that there exists even number of vertices of degree 2 between any pair of consecutive vertices of degree 3 since U has a perfect matching.

Suppose U has at least four consecutive vertices of degree 2, say v_k, v_{k+1}, v_{k+2} and v_{k+3}. Let $U' = U - \{v_k, v_{k+1}, v_{k+2}, v_{k+3}\} + v_{k-1}v_{k+4}$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_{k-1}, y_{k+4}, \ldots, y_n)^T \in \mathbb{R}^{n-4}$ be a vector such that

$$y_i = \begin{cases} x_i, & \text{if } 1 \le i \le k-1; \\ x_i, & \text{if } k+4 \le i \le n. \end{cases}$$

Suppose U has at most two consecutive vertices of degree 2 between any pair of consecutive vertices of degree 3. Recall that U has a perfect matching and $g \ge 7$ is odd. Then $n_3(U)$ is also odd and there exists odd number of vertices of degree 3 between at least one pair of consecutive vertices of degree 2, say v_k and v_{k+l+1}, where l is odd, $d(v_{k-1}) = d(v_{k+l+2}) = 2$, $d(v_{k-2}) = d(v_{k+l+3}) = 3$ and $d(v_{k+1}) = \cdots = d(v_{k+l}) = 3$ (shown in Figure 2). Let $U' = \{v_{k-1}, v_k, v_{k+l+1}, v_{k+l+2}\} + v_{k-2}v_{k+1} + v_{k+l}v_{k+l+3}$. We define the vector $\boldsymbol{y} = (y_1, y_2, \ldots, y_{k-1}, y_{k+4}, \ldots, y_n)^T \in \mathbb{R}^{n-4}$ as below:

$$y_i = \begin{cases} x_i, & \text{if } 1 \le i \le k-2; \\ -x_i, & \text{if } k+1 \le i \le k+l; \\ x_i, & \text{if } k+l+3 \le i \le g \end{cases}$$

also assign to each pendent vertex the same value of its neighbor.

In both cases, one may check that the vector $\boldsymbol{y}$ satisfies Eqs. (1), (2) and (3), and so $Q(U')$ has an eigenvalue 2. Obviously, U' is a broken sun graph with a perfect matching whose girth is $g-4$ and $n_2(U') = n_2(U) - 4 \equiv 0 \pmod 4$. Repeating those process, we may obtain a sun graph U^* with a perfect matching such that $Q(U^*)$ has an eigenvalue 2. This is a contradiction. Hence, the proof is completed. □

Theorem 3.6 *Let $U = C(T_1, \dots, T_g)$ be a unicyclic graph containing a perfect matching and let s be the number of T_i of odd orders.*

(i) If g is even, then $s \equiv 0 \pmod 4$ if and only if $Q(G)$ has an eigenvalue 2;

(ii) If g is odd, then $s \equiv 2 \pmod 4$ if and only if $Q(G)$ has an eigenvalue 2.

Proof The case when g is even follows from Lemma 2.1 and Theorem 1.2. We now consider the case when g is odd.

Firstly, assume that $s \equiv 2 \pmod 4$. If U is a broken sun graph, then Theorem 3.5 implies that $Q(U)$ has an eigenvalue 2. Otherwise, there is an i, such that $|V(T_i)| \geq 3$. Let $u \in V(T_i)$ such that $d(u, v_i) = \max_{v \in V(T_i)} d(v, v_i)$, where v_i is the root of T_i. Since U has a perfect matching, u is a pendent vertex and its neighbor, say w, has degree 2. Thus, $U = (U - \{u, w\})\#S_2$. Clearly, U and $U - \{u, w\}$ have the same value s and $U - \{u, w\}$ also has a perfect matching. Repeating this process, we may obtain a new graph U^* such that U^* and U have the same value s and U^* is a broken sun graph with a perfect matching. Then Theorem 3.5 implies that $Q(U^*)$ has an eigenvalue 2. Hence $Q(U)$ has an eigenvalue 2 since Lemma 2.5 implies that $m_U^+(2) = m_{U-\{u,w\}}^+(2) = \cdots = m_{U^*}^+(2)$.

Nextly, assume that $Q(U)$ has an eigenvalue 2. If U is a broken sun graph, then the result follows since Theorem 3.5 implies that $s \equiv 2 \pmod 4$. Otherwise, there is an i, such that $|V(T_i)| \geq 3$. In the same way as previous part, one may obtained a broken sun graph U^* with a perfect matching such that U^* and U have the same value s. Then Lemma 2.5 implies that $Q(U^*)$ has an eigenvalue 2. Moreover, by Theorem 3.5, we have $s \equiv 2 \pmod 4$. This completes the proof. □

Remark 3.7 In fact, Theorem 3.6 also gives a necessary and sufficient condition for a unicyclic graph U with a perfect matching having $\eta(\mathcal{L}(U)) > 0$.

The proof of the following theorem is similar to the one given in [4, Theorem 10] or Laplacian matrix. For completeness, we now rewrite it for signless Laplacian matrix.

Theorem 3.8 *Let $U \in \mathscr{U}_n^g$ be a broken sun graph without perfect matching. Then $g \equiv 0 \pmod 4$ and there are odd number of vertices of degree 2 between any pair of consecutive vertices of degree 3, if and only if $Q(U)$ has an eigenvalue 2.*

Proof Firstly, assume that $g \equiv 0 \pmod 4$ and there are odd number of vertices of degree 2 between any pair of consecutive vertices of degree 3. Then U is a bipartite graph. Therefore Lemma 2.1 and Theorem 1.4 imply that $Q(U)$ has an eigenvalue 2.

Nextly, we assume that $Q(U)$ has an eigenvalue 2. We apply induction on g. If $3 \leq g \leq 6$, the result follows from Appendix A. In what follows, we assume that $g \geq 7$ and $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ is an eigenvector of $Q(U)$ corresponding to the eigenvalue 2. If U has at least four consecutive vertices of degree 2, say v_1, v_2, v_3, and v_4, then let $U' = U - \{v_1, v_2, v_3, v_4\} + v_5 v_g$ and $\boldsymbol{x'}$ be a vector obtained from $\boldsymbol{x}$ by deleting the components corresponding to v_1, v_2, v_3, v_4. By checking

Eqs. (1), (2), and (3), we have that $\boldsymbol{x}'$ is an eigenvector of $Q(U')$ corresponding to the eigenvalue 2. Thus, the assertion follows by induction hypothesis.

We now assume that there are at most three vertices of degree 2 between any pair of consecutive vertices of degree 3. Let $|x_k| = \max_{v_i \in V(G)} |x_i|$. Without loss of generality, we assume that $x_k = b > 0$ otherwise we consider $-\boldsymbol{x}$. Then we have the following claim.

Claim $d(v_k) = 2$.

Proof of Claim Suppose for a contradiction that $d(v_k) = 3$. Regarding Figure 3, let $N(v_k) = \{v_{k-1}, v_{k+1}, v_j\}$, where $d(v_j) = 1$. If $x_{k-1} = a$. By Eqs. (1) and (3), we have that $x_j = b$ and $x_{k+1} = -2b - a$. Then $a \leq b$ and $|-2b-a| \leq b$ implies that $a = -b$.

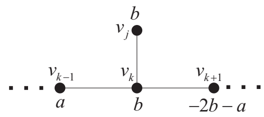

Figure 3 Values of the vertex v_k and its neighbors

Moreover, recall that U is not a sun graph and there exists odd number of vertices of degree 2 between at least one pair of consecutive vertices of degree 3 since U has no perfect matching. Without loss of generality, we assume that $d(v_{k+1}) = 2$. Then one of the following cases holds:

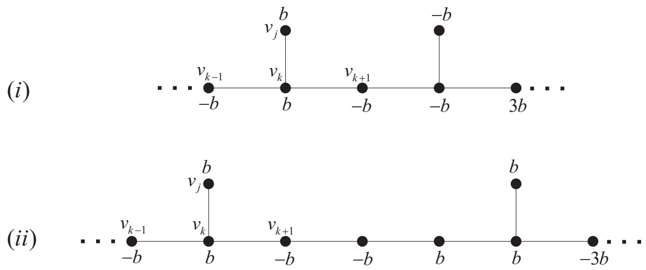

Figure 4 Two possible cases for the graph U when $d(v_k) = 3$

For each case, we find that $b = |-3b| = 3b$, i.e., $b = 0$, which contradicts $\boldsymbol{x} \neq 0$. Thus, the claim is proved.

If $x_{k-1} = a$, then by Claim and Eqs. (1), (2) and (3), one of the following cases occurs:

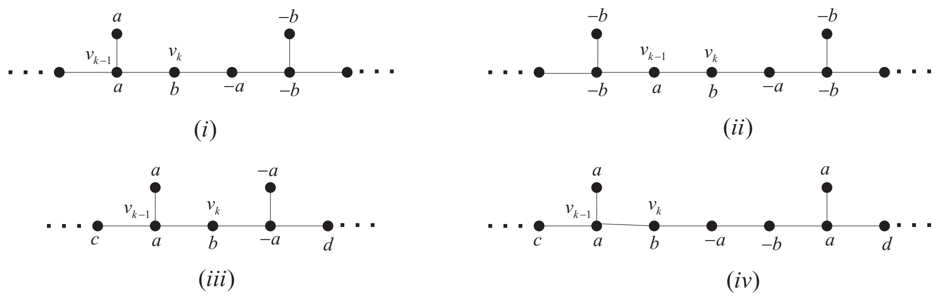

Figure 5 The possible cases for the graph U

The first two cases do not occur since the value of the component corresponding to a vertex of degree 3 is $-b$. For the graph Part (iii), we have $c = -2a - b$ and $d = 2a - b$. Then $\max\{|-2a-b|, |2a-b|\} \leq b$ implies that $a = 0$. Also, the similar result holds for the graph Part (iv).

Accordingly, for the graph Parts (iii) and (iv), using the same argument as that in [4, Theorem 10], one may prove that the result holds. Hence, the proof is completed. □

Remark 3.9 In fact, combining Theorems 1.4 and 3.8, and Lemma 2.1, we conclude that if $U \in \mathscr{U}_n^g$ is a non-bipartite broken sun graph without perfect matching, then $m_U^+(2) = 0$.

Corollary 3.10 *Let $U \in \mathscr{U}_n^g$ be a broken sun graph without perfect matching. Then $m_U^+(2) \leq 1$.*

Proof If $U \cong C_n$, then n is odd since U has no perfect matching. Therefore, the result follows from Lemma 2.4. In what follows, we assume that $U \neq C_n$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ is an eigenvector of $Q(U)$ corresponding to the eigenvalue 2. Without loss of generality, assume that $d(v_1) = 3$, $x_1 = a$ and $x_2 = b$. First, Theorem 3.8 implies that $g \equiv 0 \pmod 4$ and there are odd number of vertices of degree 2 between any pair of consecutive vertices of degree 3. Those together with Eqs. (2) and (3) imply that for $1 \leq i \leq g$,

$$x_i = \begin{cases} a, & \text{if } i = 4k+1; \\ -2(c_i - 1)a + b, & \text{if } i = 4k+2; \\ -a, & \text{if } i = 4k+3; \\ 2(c_i - 1)a - b, & \text{if } i = 4k, \end{cases}$$

where $c_i = |\{v_j \in V(G) : d(v_j) = 3, 1 \leq j \leq i\}|$. That is $x_g = 2(c_g - 1)a - b$. On the other hand, Eq. (3) implies that $x_g = -2a - b$. Thus $a = 0$. Moreover, from Eq. (1), each pendent vertex has the same value of its neighbor. Hence $m_U^+(2) \leq 1$. □

Theorem 3.11 *Let $U \in \mathscr{U}_n^g$ be a broken sun graph. Then $m_U^+(2) = 2$ if and only if $g = n \equiv 0 \pmod 4$, i.e., $U \cong C_n$ and $n \equiv 0 \pmod 4$.*

Proof By Lemma 2.1 and Theorem 1.3, to verify the statement, it suffices to show the following claim:

Claim For odd g, let $U \in \mathscr{U}_n^g$ be a broken sun graph. Then $m_U^+(2) \leq 1$.

Proof of Claim If U has no perfect matching, then the claim follows from Corollary 3.10. We now consider U has a perfect matching. Suppose for a contradiction that $m_U^+(2) \geq 2$. Then Theorem 3.5 implies that $n_2(U) \equiv 2 \pmod 4$. Hence U contains at least one vertex of degree 3. Without loss of generality, assume that $d(v_1) = 3$, $d(v_n) = 1$ and $v_1 v_n \in E(U)$. Then Lemma 2.2 implies $Q(U - v_n)$ has eigenvalue 2 since $m_U^+(2) \geq 2$. This contradicts Theorem 3.5 since $Q(U - v_n)$ has no perfect matching and g is odd. Hence $m_U^+(2) \leq 1$, which completes the proof of claim. □

Theorem 3.12 *Let $U = C(T_1, \ldots, T_g)$ be a unicyclic graph containing a perfect matching and let s be the number of T_i of odd orders. Then $m_U^+(2) = 2$ if and only if $s = g$ and $g \equiv 0 \pmod 4$.*

Proof For even g, the statement follows from Lemma 2.1 and Theorem 1.2. Hence, to complete the proof, it suffices to show the following claim:

Claim For odd g, if $U = C(T_1, \ldots, T_g)$ has a perfect matching, then $m_U^+(2) \leq 1$.

Proof of Claim If $|V(T_i)| \leq 1$ for $1 \leq i \leq g$, i.e., U is a broken sun graph, then the result follows from Theorems 3.1 and 3.11 since g is odd. If there is an i such that $|V(T_i)| \geq 3$, then there exists an edge $uv \in E(T_i)$ such that $U = (U - \{u, v\}) \# S_2$. Clearly, $U - \{u, v\}$ also has a perfect matching. Hence by Lemma 2.5, we have $m_U^+(2) = m_{U-\{u,v\}}^+(2)$ and the $|V(T_i)|$ in $U - \{u, v\}$ decreases by 2. One may repeat such process if some $|V(T_i)| \geq 3$ for $i = 1, 2, \ldots, g$, to obtain the resulting graph, denoted by U^*. Clearly, U^* is a broken sun graph with a perfect matching and $m_U^+(2) = m_{U-\{u,v\}}^+(2) = \cdots = m_{U^*}^+(2)$. Hence the result follows from Theorems 3.1 and 3.11 since g is odd. This completes the proof of claim. □

Remark 3.13 As mentioned before, for any graph G, $m_G^+(2) = \eta(\mathcal{L}(G))$. Hence, Theorem 3.12 also gives a characterization on $\eta(\mathcal{L}(U)) = 2$ for any unicyclic graph U with a perfect matching.

Appendix A

The signless Laplacian spectra of broken sun graphs with girth up to 6 are listed as follows, where the graphs with black vertices have a perfect matching and the others have no perfect matching.

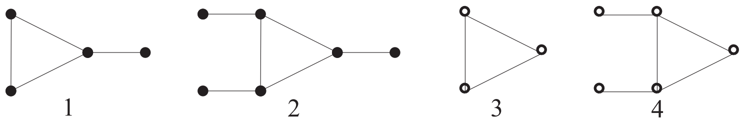

Figure 6 Broken sun graphs with girth 3

1) 4.56, 2.00, 1.00, 0.44,

2) 5.24, 2.62, 2.62, 0.76, 0.38, 0.38,

3) 4.00, 1.00, 1.00,

4) 4.94, 2.62, 1.54, 0.53, 0.38.

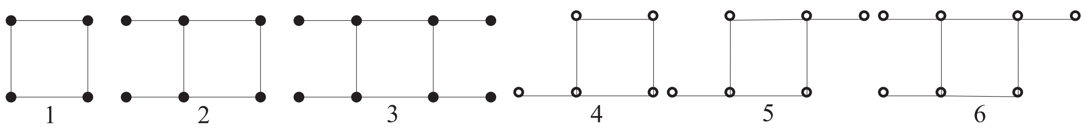

Figure 7 Broken sun graphs with girth 4

1) 4.00, 2.00, 2.00, 0,

2) 4.81, 3.00, 2.53, 1.00, 0.66, 0,

3) 5.24, 3.41, 3.41, 2.00, 0.76, 0.59, 0.59, 0,

4) 4.48, 2.69, 2.00, 0.83, 0,

5) 4.73, 3.41, 2.00, 1.27, 0.59, 0,

6) 5.03, 3.41, 2.87, 1.42, 0.68, 0.59, 0.

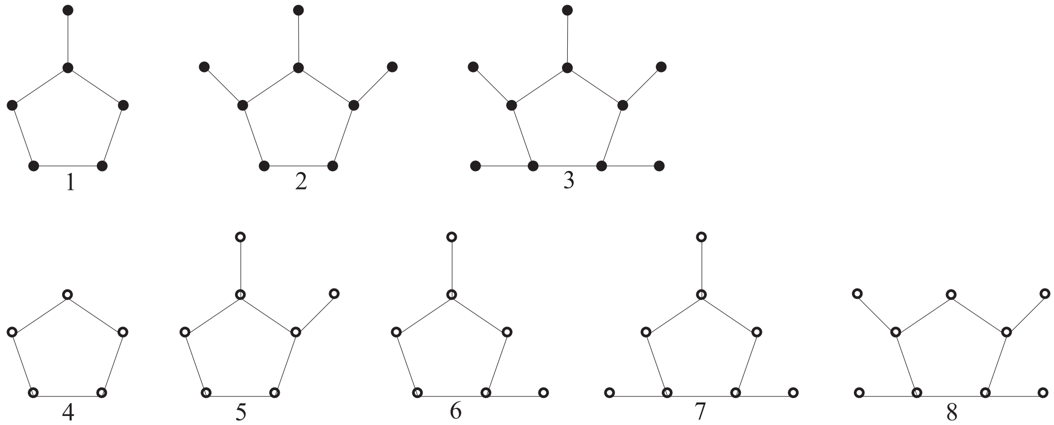

Figure 8 Broken sun graphs with girth 5

1) 4.44, 3.14, 2.62, 1.18, 0.38, 0.24,

2) 4.95, 3.73, 3.16, 2.00, 1.00, 0.71, 0.27, 0.18,

3) 5.24, 3.96, 3.96, 2.21, 2.21, 0.76, 0.66, 0.66, 0.17, 0.17,

4) 4.00, 2.62, 2.62, 0.38, 0.38,

5) 4.76, 3.25, 3.10, 1.55, 0.81, 0.34, 0.20,

6) 4.64, 3.73, 2.72, 1.41, 1.00, 0.27, 0.22,

7) 4.87, 3.86, 3.25, 1.55, 1.36, 0.67, 0.23, 0.20,

8) 5.08, 3.96, 3.53, 2.21, 1.44, 0.72, 0.66, 0.21, 0.17.

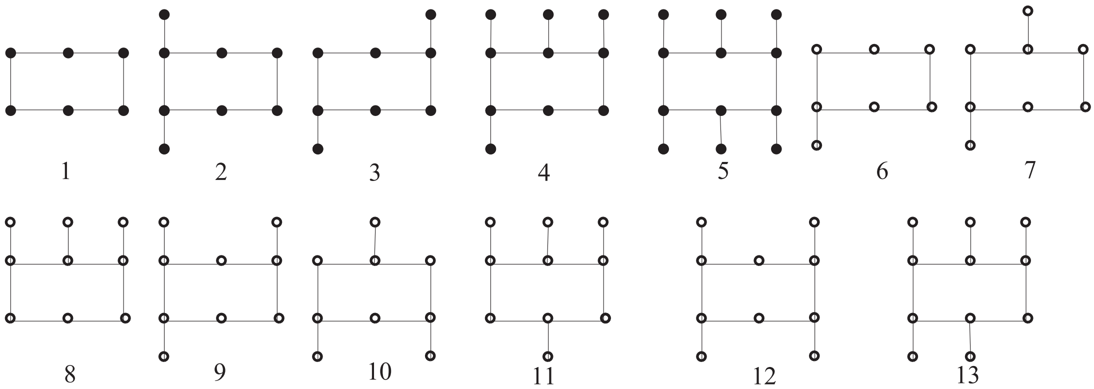

Figure 9 Broken sun graphs with girth 6

1) 4.00, 3.00, 3.00, 1.00, 1.00, 0,

2) 4.73, 3.41, 3.41, 2.00, 1.27, 0.59, 0.59, 0,

3) 4.56, 4.00, 3.00, 2.00, 1.00, 1.00, 0.44, 0,

4) 5.03, 4.17, 3.63, 2.31, 2.20, 1.00, 0.74, 0.52, 0.41, 0,

5) 5.24, 4.30, 4.30, 2.62, 2.62, 2.00, 0.76, 0.70, 0.70, 0.38, 0.38, 0,

6) 4.41, 3.41, 3.00, 1.59, 1.00, 0.59, 0,

7) 4.60, 3.88, 3.18, 1.65, 1.49, 0.74, 0.47, 0,

8) 4.91, 3.88, 3.41, 2.27, 1.65, 0.80, 0.59, 0.47, 0,

9) 4.81, 4.08, 3.41, 2.21, 1.49, 1.00, 0.59, 0.42, 0,

10) 4.73, 3.88, 3.88, 1.65, 1.65, 1.27, 0.47, 0.47, 0,

11) 4.97, 4.09, 3.88, 2.48, 1.65, 1.33, 0.72, 0.47, 0.42, 0,

12) 4.94, 4.30, 3.62, 2.62, 1.54, 1.38, 0.70, 0.53, 0.38, 0,

13) 5.12, 4.30, 3.96, 2.62, 2.30, 1.44, 0.74, 0.70, 0.44, 0.38, 0.

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