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Positive Periodic Solutions of Second-Order Singular Coupled Systems with Damping Terms

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Abstract We establish the existence of positive periodic solutions of the second-order singular coupled systems

$$\begin{cases} x'' + p_1(t)x' + q_1(t)x = f_1(t, y(t)) + c_1(t), \\ y'' + p_2(t)y' + q_2(t)y = f_2(t, x(t)) + c_2(t), \end{cases}$$

where $p_i,\ q_i,\ c_i \in C(\mathbb{R}/T\mathbb{Z};\mathbb{R}),\ i = 1, 2;\ \ f_1,\ f_2 \in C(\mathbb{R}/T\mathbb{Z} \times (0,\infty), \mathbb{R})$ and may be singular near the zero. The proof relies on Schauder's fixed point theorem and anti-maximum principle. Our main results generalize and improve those available in the literature.

Keywords positive periodic solutions; singular coupled systems; Schauder's fixed point theorem; weak singularities

MR(2010) Subject Classification 34B15; 34B18

1. Introduction

This paper studies the existence of positive periodic solutions of the second-order non-autonomous singular coupled systems

$$\begin{cases} x'' + p_1(t)x' + q_1(t)x = f_1(t, y(t)) + c_1(t), \\ y'' + p_2(t)y' + q_2(t)y = f_2(t, x(t)) + c_2(t), \end{cases} \tag{1.1}$$

where $p_i,\ q_i,\ c_i \in C(\mathbb{R}/T\mathbb{Z};\mathbb{R}),\ i = 1, 2;\ \ f_1,\ f_2 \in C(\mathbb{R}/T\mathbb{Z} \times (0,\infty), \mathbb{R})$ and may be singular near the zero.

In the past few decades, the periodic problem for the semilinear singular equation

$$x'' + a(t)x = \frac{b(t)}{x^\alpha} + c(t), \tag{1.2}$$

where $a,\ b,\ c \in L^1[0, T]$ and $\alpha > 0$, has deserved the attention of many specialists in differential equations. The interest in scalar equations with singularity began with some works of Forbat and Huaux [1,2], where the singular nonlinearity models the restoring force caused by a compressed perfect gas (see [3] for a more complete list of references). Later, the interest in this problem increased with the paper of Lazer and Solimini [4]. They obtained for (1.2) with $a(t) \equiv 0$, $b(t) \equiv 1$, $\alpha \geq 1$ (called strong force condition in a terminology first introduced by Gordon [5,6]),

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a necessary and sufficient condition to ensure the existence of positive periodic solutions that the mean value of c is negative, i.e., $\overline{c} := \frac{1}{T}\int_0^T c(t)\mathrm{d}t < 0$. Moreover, if $0 < \alpha < 1$ (weak force condition), they found examples of c with negative mean value such that periodic solutions do not exist. This work is a hallmark for the problem, since its publication many researches have focused their attention on the study of singular equations.

Since then, the strong force condition became standard in the related works, see for example [7–22] and the references therein. Here we must mention the results in [15]. It is proved

$$x'' + \mu x = \frac{b}{x^\alpha} + p(t) \tag{1.3}$$

possesses a T-periodic solution for $\alpha \geq 1$, $b > 0$, $p \in L^1[0,T]$ and $\mu \neq (\frac{k\pi}{T})^2$ for all $k \in \mathbb{Z}$. Moreover, the open problem of finding additional conditions to ensure the existence of periodic solutions in the resonant case $\mu = (\frac{k\pi}{T})^2$, is explicitly quoted. From this point of view, the results of [4] correspond to some conditions on p to deal with the resonant case $\mu = 0$ in (1.3). In [16], for the first time, the authors proved if $\mu = 0$, (1.3) has at least one positive periodic solution, provided that the mean value of p is negative and has a uniform lower bound; if $\mu = (\frac{\pi}{T})^2$, (1.3) has at least one positive periodic solution when p is positive, which does not require the strong force condition $\alpha \geq 1$. These conclusions had been improved in [7].

Compared with the literature available for strong singularities, the study of the existence of periodic solutions under the presence of a weak singularity ($0 < \alpha < 1$) is much more recent and the number of references is considerably smaller. The likely reason may be that with a weak singularity, the energy near the origin becomes finite, and this fact is not helpful for obtaining a priori bound needed for a classical application of the degree theory, and also not helpful for the fast rotation needed in recent versions of the Poincaré-Birkhoff theorem. Fortunately, some results in the literature show in some situations weak singularities may help to create periodic solutions [10,16,23–25]. In addition, many researchers have focused on the existence of positive periodic solutions of singular systems composed of the first and second-order differential equations, see for instance, [10,25–27] and the references therein. It has been shown that many results of nonsingular systems are still valid for singular cases.

For convenience, we denote by ξ^* and ξ_* the essential supremum and infimum of a given function $\xi \in L^1[0,T]$, if they exist. We write $\xi \succ 0$ if $\xi \geq 0$ for a.e., $t \in [0,T]$ and it is positive in a set of positive measure. Very recently, Cao and Jiang [25] studied the coupled system

$$\begin{cases} x'' + a_1(t)x = f_1(t, y(t)) + c_1(t), \\ y'' + a_2(t)y = f_2(t, x(t)) + c_2(t), \end{cases} \tag{1.4}$$

where $a_1,\ a_2,\ c_1,\ c_2 \in C[0,T]$, $f_1, f_2 \in C([0,T] \times (0,+\infty), (0,+\infty))$ and may be singular near the zero. Under the basic assumption

(H1) The Green's function $G_i(t,s)$, associated with

$$x'' + a_i(t)x = 0, \quad x(0) = x(T), \quad x'(0) = x'(T),$$

is nonnegative for every $(t,s) \in [0,T] \times [0,T]$, $i = 1,2$.

They proved a series of excellent results as below.

Theorem 1.1 ([25]) *Let (H1) hold and define*

$$\gamma_i(t) = \int_0^T G_i(t,s)c_i(s)\mathrm{d}s, \quad i = 1, 2. \tag{1.5}$$

Assume that

(H2) There exist b_i, $\hat{b}_i \in L^1(0,T)$ *with* $b_i \succ 0$, $\hat{b}_i \succ 0$ *and* $0 < \alpha_i < 1$ *such that*

$$0 \le \frac{\hat{b}_i(t)}{x^{\alpha_i}} \le f_i(t,x) \le \frac{b_i(t)}{x^{\alpha_i}}, \quad \text{for all } \ x > 0, \ \text{a.e. } t \in [0,T], \ i = 1, 2.$$

If $\gamma_{1*} \ge 0$, $\gamma_{2*} \ge 0$, *then (1.4) has a positive* T*-periodic solution.*

Theorem 1.2 ([25]) *Let (H1) and (H2) hold. Set*

$$\hat{\beta}_i = \int_0^T G_i(t,s)\hat{b}_i(s)\mathrm{d}s, \ \beta_i = \int_0^T G_i(t,s)b_i(s)\mathrm{d}s, \quad i = 1, 2.$$

If $\gamma_1^* \le 0$, $\gamma_2^* \le 0$ *and*

$$\gamma_{1*} \ge \big[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{1*}}{(\beta_2^*)^{\alpha_1}}\big]^{\frac{1}{1-\alpha_1\alpha_2}} \cdot \big(1 - \frac{1}{\alpha_1\alpha_2}\big), \tag{1.6}$$

$$\gamma_{2*} \ge \big[\alpha_1\alpha_2 \cdot \frac{\hat{\beta}_{2*}}{(\beta_1^*)^{\alpha_2}}\big]^{\frac{1}{1-\alpha_1\alpha_2}} \cdot \big(1 - \frac{1}{\alpha_1\alpha_2}\big), \tag{1.7}$$

then (1.4) has a positive T*-periodic solution.*

Theorem 1.3 ([25]) *Let (H1) and (H2) hold. If* $\gamma_{1*} \ge 0$, $\gamma_2^* \le 0$ *and*

$$\gamma_{2*} \ge r_{21} - \hat{\beta}_{2*} \cdot \frac{r_{21}^{\alpha_1\alpha_2}}{(\beta_1^* + \gamma_1^* r_{21}^{\alpha_1})^{\alpha_2}}, \tag{1.8}$$

where r_{21} *is the unique positive solution of the equation*

$$r_2^{1-\alpha_1\alpha_2}(\beta_1^* + \gamma_1^* r_2^{\alpha_1})^{1+\alpha_2} = \alpha_1\alpha_2\beta_1^*\hat{\beta}_{2*}, \tag{1.9}$$

then (1.4) has a positive T*-periodic solution.*

Theorem 1.4 ([25]) *Let (H1) and (H2) hold. If* $\gamma_1^* \le 0$, $\gamma_{2*} \ge 0$ *and*

$$\gamma_{1*} \ge r_{11} - \hat{\beta}_{1*} \cdot \frac{r_{11}^{\alpha_1\alpha_2}}{(\beta_2^* + \gamma_2^* r_{11}^{\alpha_2})^{\alpha_1}}, \tag{1.10}$$

where r_{11} *is the unique positive solution of the equation*

$$r_1^{1-\alpha_1\alpha_2}(\beta_2^* + \gamma_2^* r_1^{\alpha_2})^{1+\alpha_1} = \alpha_1\alpha_2\beta_2^*\hat{\beta}_{1*}, \tag{1.11}$$

then (1.4) has a positive T*-periodic solution.*

Obviously, (H2) extensively used in [25] is so restrictive that above results are only applicable to (1.1) with nonlinearity f_i which is bounded in origin and infinity by functions of the form $\frac{1}{x^{\lambda_i}}$. Of course the natural question is what would happen if we allow the nonlinearity f_i is bounded by two different functions $\frac{1}{x^{\alpha_i}}$ and $\frac{1}{x^{\beta_i}}$, $i = 1, 2$.

The purpose of this paper is to study the existence of positive periodic solutions of (1.1) under more general assumptions

(A1) $p_i,\ q_i,\ c_i \in C(\mathbb{R}/T\mathbb{Z};\mathbb{R})$, $i=1,2$; $f_1,\ f_2 \in C(\mathbb{R}/T\mathbb{Z}\times(0,\infty),\mathbb{R})$ and may be singular near the zero.

(A2) There exist $\hat{b}_i, b_i, e_i \in L^1(0,T)$ with $\hat{b}_i, b_i, e_i \succ 0$ and positive constants $\alpha_i,\beta_i,\mu_i,\nu_i \in (0,1)$, such that

$$0 \le \frac{\hat{b}_i(t)}{x^{\alpha_i}} \le f_i(t,x) \le \frac{b_i(t)}{x^{\beta_i}},\quad x\in[1,\infty),\ \text{a.e. } t\in[0,T],\quad i=1,2,$$

$$0 \le \frac{\hat{b}_i(t)}{x^{\mu_i}} \le f_i(t,x) \le \frac{e_i(t)}{x^{\nu_i}},\quad x\in(0,1),\ \text{a.e. } t\in[0,T],\quad i=1,2.$$

(A3) There exist $\hat{b}_i, b_i, e_i \in L^1(0,T)$ with $\hat{b}_i, b_i, e_i \succ 0$ and constants $\alpha_1,\beta_1,\beta_2,\mu_1,\mu_2,\nu_2 \in (0,1)$, such that

$$0 \le \frac{\hat{b}_1(t)}{x^{\alpha_1}} \le f_1(t,x) \le \frac{b_1(t)}{x^{\beta_1}},\quad x\in[1,\infty),\ \text{a.e. } t\in[0,T],$$

$$0 \le \frac{\hat{b}_1(t)}{x^{\mu_1}} \le f_1(t,x) \le \frac{e_1(t)}{x^{\beta_1}},\quad x\in(0,1),\ \text{a.e. } t\in[0,T];$$

Moreover,

$$0 \le \frac{\hat{b}_2(t)}{x^{\mu_2}} \le f_2(t,x) \le \frac{b_2(t)}{x^{\beta_2}},\quad x\in[1,\infty),\ \text{a.e. } t\in[0,T],$$

$$0 \le \frac{\hat{b}_2(t)}{x^{\mu_2}} \le f_2(t,x) \le \frac{e_2(t)}{x^{\nu_2}},\quad x\in(0,1),\ \text{a.e. } t\in[0,T].$$

Remark 1.5 It is worth remarking that the singular coupled system (1.1) with damping terms, has not attracted much attention in the literature. To the best of our knowledge, the existence results are relatively little even for the single second-order damped differential equations. We refer the readers to [27–30] for several existence results.

Remark 1.6 Let us consider the function

$$f_i(t,u) = \begin{cases} \dfrac{1}{u^{\varepsilon_i}}, & u\in[1,\infty),\\ \dfrac{1}{u^{\eta_i}}, & u\in(0,1), \end{cases} \tag{1.6}$$

where $\varepsilon_i,\ \eta_i \in (0,1)$. Clearly, f_i is continuous and satisfies (A2) with

$$\alpha_i=\beta_i=\varepsilon_i,\ \mu_i=\nu_i=\eta_i;\quad \hat{b}_i(t)=b_i(t)=e_i(t)\equiv 1,\quad i=1,2.$$

However, it does not satisfy (H2) since it cannot be bounded by a single function $\frac{h_i(t)}{u^{\gamma_i}}$ for any $\gamma_i\in(0,1)$ and any $h_i\succ 0$. Similarly, our condition (A3) is also more general than (H2).

2. Preliminaries

We say that the linear equation

$$x'' + p(t)x' + q(t)x = 0, \tag{2.1}$$

associated to periodic boundary conditions

$$x(0)=x(T),\quad x'(0)=x'(T) \tag{2.2}$$

is non-resonant if its unique T-periodic solution is the trivial one. When (2.1), (2.2) is non-resonant, as a consequence of Fredholm's alternative, the nonhomogeneous equation

$$x'' + p(t)x' + q(t)x = l(t) \tag{2.3}$$

admits a unique T-periodic solution, which can be written as $x(t) = \int_0^T G(t,s)l(s)\mathrm{d}s$, where $G(t,s)$ is Green's function of (2.1), (2.2). Moreover, (2.1) admits the anti-maximum principle if (2.3) has a unique T-periodic solution x_l for all $l \in C(\mathbb{R}/T\mathbb{Z})$, and $x_l(t) > 0$ for all t if $l \succ 0$. Recently, an explicit criterion, which guarantees (2.1) admits the anti-maximum principle, had been proved in [28]. For simplicity of statement, define

$$\sigma(p)(t) := e^{\int_0^t p(s)\mathrm{d}s}, \quad \sigma_1(p)(t) := \sigma(p)(T)\int_0^t \sigma(p)(s)\mathrm{d}s + \int_t^T \sigma(p)(s)\mathrm{d}s.$$

Lemma 2.1 ([28]) *Assume $q \not\equiv 0$ and*

$$\int_0^T q(s)\sigma(p)(s)\sigma_1(-p)(s) \geq 0, \tag{2.4}$$

$$\sup_{0\leq t\leq T}\Big\{\int_t^{t+T}\sigma(-p)(s)\mathrm{d}s \cdot \int_t^{t+T}[q(s)]_+\sigma(p)(s)\mathrm{d}s\Big\} \leq 4, \tag{2.5}$$

where $[q(s)]_+ = \max\{q(s), 0\}$. Then the anti-maximum principle for (2.1) holds.

In the recent paper [29], the authors proved if (2.1) admits the anti-maximum principle, then $G(t,s)$ is nonnegative for all $(t,s) \in [0,T] \times [0,T]$. Moreover, they obtained

Lemma 2.2 ([29]) *Assume $q \not\equiv 0$ and (2.5) holds. Then the distance between two consecutive zeroes of a nontrivial solution of (2.1) is always strictly greater than T.*

Note that Lemma 2.2 implies Green's function $G(t,s)$ does not vanish. As a consequence of two previous Lemmas, Chu et al. [29] proved the following

Lemma 2.3 ([29]) *Suppose $q \not\equiv 0$ and (2.4)-(2.5) are satisfied. Then $G(t,s)$ is positive for all $(t,s) \in [0,T] \times [0,T]$.*

Remark 2.4 In the special case $p \equiv 0$ (there is no damping terms), the inequalities (2.4) and (2.5) reduce to $\int_0^T q(s)\mathrm{d}s > 0$ and $\|[q(s)]_+\|_1 < \frac{4}{T}$, respectively, which are conditions used to ensure the positivity of Green's function of

$$x'' + q(t)x = 0, \quad x(0) = x(T), \quad x'(0) = x'(T).$$

See Cabada and Cid [31] for more details.

In the following, we always assume

(A0) The Green's function $G_i(t,s)$, associated with the linear problem

$$x'' + a_i(t)x' + b_i(t)x = 0, \quad x(0) = x(T), \quad x'(0) = x'(T),$$

is positive for all $(t,s) \in [0,T] \times [0,T]$.

To state and prove our main results, we need some notations as bellow.

$$\hat{B}_i(t) := \int_0^T G_i(t,s)\hat{b}_i(s)\mathrm{d}s, \; E_i(t) := \int_0^T G_i(t,s)e_i(s)\mathrm{d}s, \quad i = 1, 2; \tag{2.6}$$

$$B_i(t) := \int_0^T G_i(t,s)b_i(s)\mathrm{d}s, \quad i=1,2; \tag{2.7}$$

$$\rho_i^* := E_i^* + B_i^*, \ \sigma_i := \max\{\mu_i, \alpha_i\}, \ \delta_i := \max\{\nu_i, \beta_i\}, \quad i=1,2. \tag{2.8}$$

3. The case $\gamma_{1*} \geq 0$, $\gamma_{2*} \geq 0$

Theorem 3.1 *Let (A0), (A1) and (A2) hold. If $\gamma_{1*} \geq 0$, $\gamma_{2*} \geq 0$, then (1.1) has a positive T-periodic solution.*

Proof Let us denote the set of continuous T-periodic functions as C_T. Then a T-periodic solution of (1.1) is just a fixed point of the completely continuous map

$$A(x,y) = (A_1x, A_2y): \ C_T \times C_T \to C_T \times C_T$$

defined as

$$(A_1x)(t) := \int_0^T G_1(t,s)[f_1(s,y(s)) + c_1(s)]\mathrm{d}s = \int_0^T G_1(t,s)f_1(s,y(s))\mathrm{d}s + \gamma_1(t),$$

$$(A_2y)(t) := \int_0^T G_2(t,s)[f_2(s,x(s)) + c_2(s)]\mathrm{d}s = \int_0^T G_2(t,s)f_2(s,x(s))\mathrm{d}s + \gamma_2(t).$$

By a direct application of Schauder's fixed point theorem, the proof is finished if we prove A maps the closed convex set

$$K = \big\{(x,y) \in C_T \times C_T : r_1 \leq x(t) \leq R_1, \ r_2 \leq y(t) \leq R_2, \ \text{for all } t \in [0,T], \ R_i > 1, \ i=1,2\big\}$$

into itself, where $R_1 > r_1 > 0$, $R_2 > r_2 > 0$ are positive constants to be fixed properly.

For given $u \in K$, setting

$$J_{i1} := \{t \in [0,T] : \ r_i \leq u(t) < 1\}, \ J_{i2} := \{t \in [0,T] : R_i \geq u(t) \geq 1\}, \quad i=1,2.$$

Then by (A0), (A2) and $R_2 > 1$, we have for given $(x,y) \in K$,

$$\begin{aligned}(A_1x)(t) &= \int_0^T G_1(t,s)f_1(s,y(s))\mathrm{d}s + \gamma_1(t)\\ &\geq \int_{J_{11}} G_1(t,s)f_1(s,y(s))\mathrm{d}s + \int_{J_{12}} G_1(t,s)f_1(s,y(s))\mathrm{d}s + \gamma_{1*}\\ &\geq \int_{J_{11}} G_1(t,s)\frac{\hat{b}_1(t)}{y^{\mu_1}}\mathrm{d}s + \int_{J_{12}} G_1(t,s)\frac{\hat{b}_1(t)}{y^{\alpha_1}}\mathrm{d}s\\ &\geq \int_0^T G_1(t,s)\frac{\hat{b}_1(t)}{R_2^{\sigma_1}}\mathrm{d}s \geq \hat{B}_{1*}\cdot\frac{1}{R_2^{\sigma_1}},\end{aligned}$$

where σ_1 is given by (2.8). Note that for every $(x,y) \in K$,

$$\begin{aligned}(A_1x)(t) &= \int_{J_{11}} G_1(t,s)f_1(s,y(s))\mathrm{d}s + \int_{J_{12}} G_1(t,s)f_1(s,y(s))\mathrm{d}s + \gamma_1^*\\ &\leq \int_{J_{11}} G_1(t,s)\frac{e_1(s)}{y^{\nu_1}}\mathrm{d}s + \int_{J_{12}} G_1(t,s)\frac{b_1(s)}{y^{\beta_1}}\mathrm{d}s + \gamma_1^*\\ &\leq \int_0^T G_1(t,s)\frac{e_1(s)}{r_2^{\nu_1}}\mathrm{d}s + \int_0^T G_1(t,s)b_1(s)\mathrm{d}s + \gamma_1^*\end{aligned}$$

$$\leq \frac{1}{r_2^{\nu_1}} \cdot E_1^* + (B_1^* + \gamma_1^*).$$

By the same strategy, we get

$$\begin{aligned}(A_2y)(t) &\geq \int_{J_{21}} G_2(t,s) f_2(s,x(s))\mathrm{d}s + \int_{J_{22}} G_2(t,s) f_2(s,x(s))\mathrm{d}s + \gamma_{2*}\\ &\geq \int_{J_{21}} G_2(t,s)\frac{\hat{b}_2(t)}{R_1^{\mu_2}}\mathrm{d}s + \int_{J_{22}} G_2(t,s)\frac{\hat{b}_2(t)}{R_1^{\alpha_2}}\mathrm{d}s\\ &\geq \int_0^T G_2(t,s)\frac{\hat{b}_2(t)}{R_1^{\sigma_2}}\mathrm{d}s \geq \hat{B}_{2*}\cdot\frac{1}{R_1^{\sigma_2}},\end{aligned}$$

where σ_2 is defined as in (2.8). Moreover,

$$\begin{aligned}(A_2y)(t) &\leq \int_{J_{21}} G_2(t,s) f_2(s,x(s))\mathrm{d}s + \int_{J_{22}} G_2(t,s) f_2(s,x(s))\mathrm{d}s + \gamma_2^*\\ &\leq \int_{J_{21}} G_2(t,s)\frac{e_2(s)}{x^{\nu_2}}\mathrm{d}s + \int_{J_{22}} G_2(t,s)\frac{b_2(s)}{x^{\beta_2}}\mathrm{d}s + \gamma_2^*\\ &\leq \int_0^T G_2(t,s)\frac{e_2(s)}{r_1^{\nu_2}}\mathrm{d}s + \int_0^T G_2(t,s) b_2(s)\mathrm{d}s + \gamma_2^*\\ &\leq \frac{1}{r_1^{\nu_2}}\cdot E_2^* + (B_2^* + \gamma_2^*).\end{aligned}$$

Therefore, $(A_1x, A_2y) \in K$ if r_1, r_2, R_1 and R_2 are chosen such that

$$\hat{B}_{1*}\cdot\frac{1}{R_2^{\sigma_1}} \geq r_1, \quad \frac{1}{r_2^{\nu_1}}\cdot E_1^* + (B_1^* + \gamma_1^*) \leq R_1;$$

$$\hat{B}_{2*}\cdot\frac{1}{R_1^{\sigma_2}} \geq r_2, \quad \frac{1}{r_1^{\nu_2}}\cdot E_2^* + (B_2^* + \gamma_2^*) \leq R_2,$$

and they should satisfy $R_i > r_i > 0$, $R_i > 1$, $i = 1, 2$.

Since $\hat{B}_{i*} > 0$, $E_{i*} > 0$, taking $R = R_1 = R_2$, $r = r_1 = r_2$, $r = \frac{1}{R}$, it is sufficient to find $R > 1$ such that

$$\hat{B}_{1*}\cdot R^{1-\sigma_1} \geq 1, \quad R^{\nu_1}\cdot E_1^* + (B_1^* + \gamma_1^*) \leq R;$$

$$\hat{B}_{2*}\cdot R^{1-\sigma_2} \geq 1, \quad R^{\nu_2}\cdot E_2^* + (B_2^* + \gamma_2^*) \leq R,$$

and these inequalities hold for R large enough because $\sigma_i < 1$, $\nu_i < 1$, $i = 1, 2$. □

Remark 3.2 It is not difficult to see even in the special case $\alpha_i = \beta_i = \mu_i = \nu_i$, our condition (A2) is more general than (H2). Hence Theorem 3.1 generalizes Theorem 1.1.

4. The case $\gamma_1^* \leq 0$, $\gamma_2^* \leq 0$

The aim of this section is to show that the presence of a weak singular nonlinearity makes it possible to find positive solutions when $\gamma_1^* \leq 0$, $\gamma_2^* \leq 0$.

Theorem 4.1 *Let (A0), (A1) and (A2) hold. Assume*

$$\rho_1^* > \max\left\{(\delta_1\sigma_2\hat{B}_{2*})^{\frac{1}{\sigma_2}},\ (\delta_1\sigma_2\hat{B}_{2*})^{\delta_1}\right\}, \tag{4.1}$$

$$\rho_2^* > \max\big\{(\delta_2\sigma_1\hat{B}_{1*})^{\frac{1}{\sigma_1}},\ (\delta_2\sigma_1\hat{B}_{1*})^{\delta_2}\big\}. \tag{4.2}$$

If $\gamma_1^* \le 0$, $\gamma_2^* \le 0$ *and*

$$\gamma_{2*} \ge \big(\delta_1\sigma_2\cdot\frac{\hat{B}_{2*}}{(\rho_1^*)^{\sigma_2}}\big)^{\frac{1}{1-\delta_1\sigma_2}}\cdot\big(1-\frac{1}{1-\delta_1\sigma_2}\big), \tag{4.3}$$

$$\gamma_{1*} \ge \big(\delta_2\sigma_1\cdot\frac{\hat{B}_{1*}}{(\rho_2^*)^{\sigma_1}}\big)^{\frac{1}{1-\delta_2\sigma_1}}\cdot\big(1-\frac{1}{1-\delta_2\sigma_1}\big), \tag{4.4}$$

then (1.1) has a positive T*-periodic solution, where* ρ_i^*, δ_i, σ_i, $i=1,2$ *are given by (2.8).*

Proof Define a closed convex set K as

$$K=\big\{(x,y)\in C_T\times C_T: r_1\le x(t)\le R_1,\ r_2\le y(t)\le R_2,\ t\in[0,T],\ R_i>1>r_i>0\big\}.$$

By Schauder's fixed point theorem, the proof is finished if we prove A maps K into itself. For given $(x,y)\in K$, it follows from (A0), (A2) and $R_2>1>r_2$ that

$$\begin{aligned}(A_1x)(t)&\ge\int_{J_{11}}G_1(t,s)f_1(s,y(s))\mathrm{d}s+\int_{J_{12}}G_1(t,s)f_1(s,y(s))\mathrm{d}s+\gamma_{1*}\\&\ge\int_{J_{11}}G_1(t,s)\frac{\hat{b}_1(t)}{R_2^{\mu_1}}\mathrm{d}s+\int_{J_{12}}G_1(t,s)\frac{\hat{b}_1(t)}{R_2^{\alpha_1}}\mathrm{d}s+\gamma_{1*}\\&\ge\int_0^T G_1(t,s)\frac{\hat{b}_1(t)}{R_2^{\sigma_1}}\mathrm{d}s+\gamma_{1*}\ge\hat{B}_{1*}\cdot\frac{1}{R_2^{\sigma_1}}+\gamma_{1*},\\(A_1x)(t)&\le\int_{J_{11}}G_1(t,s)\frac{e_1(s)}{y^{\nu_1}}\mathrm{d}s+\int_{J_{12}}G_1(t,s)\frac{b_1(s)}{y^{\beta_1}}\mathrm{d}s\\&\le\int_{J_{11}}G_1(t,s)\frac{e_1(s)}{r_2^{\nu_1}}\mathrm{d}s+\int_{J_{12}}G_1(t,s)\frac{b_1(s)}{r_2^{\beta_1}}\mathrm{d}s\\&\le\int_0^T G_1(t,s)\frac{e_1(s)}{r_2^{\delta_1}}\mathrm{d}s+\int_0^T G_1(t,s)\frac{b_1(s)}{r_2^{\delta_1}}\mathrm{d}s\le\frac{1}{r_2^{\delta_1}}\cdot\rho_1^*.\end{aligned}$$

By simple estimates, we can also obtain

$$(A_2y)(t)\ge\hat{B}_{2*}\cdot\frac{1}{R_1^{\sigma_2}}+\gamma_{2*},$$

$$(A_2y)(t)\le\frac{1}{r_1^{\delta_2}}\cdot\rho_2^*.$$

Clearly, $(A_1x,A_2y)\in K$ if r_1, r_2, R_1 and R_2 are chosen such that

$$\hat{B}_{1*}\cdot\frac{1}{R_2^{\sigma_1}}+\gamma_{1*}\ge r_1,\quad \frac{1}{r_2^{\delta_1}}\cdot\rho_1^*\le R_1; \tag{4.5}$$

$$\hat{B}_{2*}\cdot\frac{1}{R_1^{\sigma_2}}+\gamma_{2*}\ge r_2,\quad \frac{1}{r_1^{\delta_2}}\cdot\rho_2^*\le R_2, \tag{4.6}$$

and they should satisfy that $R_i>1>r_i>0$, $i=1,2$.

If we fix $R_1=\frac{1}{r_2^{\delta_1}}\cdot\rho_1^*$, $R_2=\frac{1}{r_1^{\delta_2}}\cdot\rho_2^*$, then the first inequality of (4.6) holds if r_2 satisfies $\hat{B}_{2*}\cdot r_2^{\delta_1\sigma_2}\cdot(\rho_1^*)^{-\sigma_2}+\gamma_{2*}\ge r_2$, or equivalently, $\gamma_{2*}\ge g(r_2):=r_2-\frac{\hat{B}_{2*}}{(\rho_1^*)^{\sigma_2}}r_2^{\delta_1\sigma_2}$. The function

$g(r_2)$ possesses a minimum at $r_{20} := \left(\delta_1\sigma_2 \cdot \frac{\hat{B}_{2*}}{(\rho_1^*)^{\sigma_2}}\right)^{\frac{1}{1-\delta_1\sigma_2}}$. Taking $r_2 = r_{20}$, then (4.6) holds if

$$\gamma_{2*} \geq g(r_{20}) := \big(\delta_1\sigma_2 \cdot \frac{\hat{B}_{2*}}{(\rho_1^*)^{\sigma_2}}\big)^{\frac{1}{1-\delta_1\sigma_2}} \cdot \big(1 - \frac{1}{1-\delta_1\sigma_2}\big),$$

which is just (4.3). Similarly, $\gamma_{1*} \geq h(r_1) := r_1 - \frac{\hat{B}_{1*}}{(\rho_2^*)^{\sigma_1}} r_1^{\delta_2\sigma_1}$. $h(r_1)$ possesses a minimum at $r_{10} := (\delta_2\sigma_1 \cdot \frac{\hat{B}_{1*}}{(\rho_2^*)^{\sigma_1}})^{\frac{1}{1-\delta_2\sigma_1}}$. Taking $r_1 = r_{10}$, then

$$\gamma_{1*} \geq h(r_{10}) := \big(\delta_2\sigma_1 \cdot \frac{\hat{B}_{1*}}{(\rho_2^*)^{\sigma_1}}\big)^{\frac{1}{1-\delta_2\sigma_1}} \cdot \big(1 - \frac{1}{1-\delta_2\sigma_1}\big),$$

which is condition (4.4). The second inequality holds directly from the choice of R_1 and R_2, so it remains to prove $R_i > 1 > r_i > 0$, $i = 1, 2$. This is easily verified by (4.1) and (4.2). □

Remark 4.2 (4.1) and (4.2) are crucial to ensure $R_i > 1 > r_i > 0$. In the proof of Theorem 4.1, we require $R_i > 1 > r_i > 0$ since the exponents in inequalities of (H2) are different, which makes it more difficult to estimate some inequalities. However, in the special case $\lambda_i := \alpha_i = \beta_i = \mu_i = \nu_i$, $i = 1, 2$, if we define

$$\omega_i(t) := \max\{b_i(t), e_i(t)\}, \text{ a.e. } t \in [0, T], \quad i = 1, 2,$$

then Theorem 4.1 reduces to Theorem 1.2. Moreover, (4.1) and (4.2) are not needed because $R_i > 1 > r_i > 0$ can be easily verified by $\hat{b}_i(t) \leq \omega_i(t)$. Finally, it is worth remarking that Theorem 4.1 applies to systems which cannot be treated by Theorem 1.2, see Example 4.3 as below.

Example 4.3 Let us consider the singular coupled system

$$\begin{cases} x'' + \dfrac{1}{4}x = \dfrac{4-t}{y^{\frac{1}{5}}} - c_1, \quad t \in (0, \pi), \\ y'' + \dfrac{1}{9}y = \dfrac{1+t}{x^{\frac{1}{4}}} - c_2, \quad t \in (0, \pi), \\ x(0) = x(\pi), \quad x'(0) = x'(\pi), \\ y(0) = y(\pi), \quad y'(0) = y'(\pi), \end{cases} \tag{4.7}$$

where $f_1(t, y) = \frac{4-t}{y^{\frac{1}{5}}}$, $f_2(t, x) = \frac{1+t}{x^{\frac{1}{4}}}$, $q_1 \equiv \frac{1}{4}$, $q_2 \equiv \frac{1}{9}$. We choose $p_i \equiv 0$ $(i = 1, 2)$ such that the calculation of Green's function is more convenient. c_1 and c_2 are positive constants with

$$c_1 \in \big(0, \ \frac{1}{20} \cdot (\frac{1}{3\sqrt{33}})^{\frac{6}{5}}\big], \quad c_2 \in \big(0, \ \frac{1}{15} \cdot (\frac{1}{2\sqrt[3]{40}})^{\frac{6}{5}}\big]. \tag{4.8}$$

It is not difficult to check

$$(A_1x)(t) := \int_0^\pi G_1(t, s)\frac{4-s}{y(s)^{\frac{1}{5}}}\mathrm{d}s + \int_0^\pi G_1(t, s)(-c_1)\mathrm{d}s,$$

$$(A_1y)(t) := \int_0^\pi G_2(t, s)\frac{1+s}{x(s)^{\frac{1}{4}}}\mathrm{d}s + \int_0^\pi G_1(t, s)(-c_2)\mathrm{d}s,$$

where

$$G_1(t,s)=\begin{cases}\sin\dfrac{\pi-t+s}{2}+\sin\dfrac{t-s}{2}, & 0\le s\le t\le\pi,\\ \sin\dfrac{\pi-s+t}{2}+\sin\dfrac{s-t}{2}, & 0\le t\le s\le\pi,\end{cases}$$

$$G_2(t,s)=\begin{cases}\sin\dfrac{\pi-t+s}{3}+\sin\dfrac{t-s}{3}, & 0\le s\le t\le\pi,\\ \sin\dfrac{\pi-s+t}{3}+\sin\dfrac{s-t}{3}, & 0\le t\le s\le\pi.\end{cases}$$

Obviously, $G_i(t,s)>0$ for $(t,s)\in[0,\pi]\times[0,\pi]$ and f_i satisfies (A1). Moreover, $\int_0^\pi G_1(t,s)\mathrm{d}s=4$, $\int_0^\pi G_2(t,s)\mathrm{d}s=3$.

Let

$$\hat{b}_1(t)\equiv\frac{1}{2},\ b_1(t)\equiv 4,\ e_1(t)\equiv 6;$$

$$\alpha_1=\frac{1}{2},\quad \beta_1=\frac{1}{6},\quad \mu_1=\frac{1}{7},\quad \nu_1=\frac{1}{2}.$$

Then $\sigma_1=\max\{\mu_1,\alpha_1\}=\frac{1}{2}$, $\delta_1=\max\{\beta_1,\nu_1\}=\frac{1}{2}$, and

$$0<\frac{\frac{1}{2}}{y^{\frac{1}{2}}}\le\frac{4-t}{y^{\frac{1}{5}}}\le\frac{4}{y^{\frac{1}{6}}},\quad y\in[1,\infty),\ t\in[0,\pi];$$

$$0<\frac{\frac{1}{2}}{y^{\frac{1}{7}}}\le\frac{4-t}{y^{\frac{1}{5}}}\le\frac{6}{y^{\frac{1}{2}}},\quad u\in(0,1),\ t\in[0,\pi].$$

On the other hand, let

$$\hat{b}_2(t)\equiv 1,\ b_2(t)\equiv 5,\ e_2(t)\equiv 6;$$

$$\alpha_2=\frac{1}{3},\quad \beta_2=\frac{1}{5},\quad \mu_2=\frac{1}{8},\quad \nu_2=\frac{1}{3},$$

we can then obtain $\sigma_2=\max\{\mu_2,\alpha_2\}=\frac{1}{3}$, $\delta_2=\max\{\beta_2,\nu_2\}=\frac{1}{3}$, and

$$0<\frac{1}{x^{\frac{1}{3}}}\le\frac{1+t}{x^{\frac{1}{4}}}\le\frac{5}{x^{\frac{1}{5}}},\quad x\in[1,\infty),\ t\in[0,\pi];$$

$$0<\frac{1}{x^{\frac{1}{8}}}\le\frac{1+t}{x^{\frac{1}{4}}}\le\frac{6}{x^{\frac{1}{3}}},\quad x\in(0,1),\ t\in[0,\pi].$$

Hence (A2) is also satisfied.

Simple computation gives

$$\hat{B}_{1*}=\hat{B}_1^*=2,\quad B_{1*}=B_1^*=16,\quad E_{1*}=E_1^*=24,$$

$$\hat{B}_{2*}=\hat{B}_2^*=3,\quad B_{2*}=B_2^*=15,\quad E_{2*}=E_2^*=18;$$

$$\rho_1^*=E_1^*+B_1^*=40,\quad \rho_2^*=E_2^*+B_2^*=33;$$

$$(\delta_1\sigma_2\hat{B}_{2*})^{\frac{1}{\sigma_2}}=\frac{1}{8},\quad (\delta_1\sigma_2\hat{B}_{2*})^{\delta_1}=\frac{\sqrt{2}}{2};\quad (\delta_2\sigma_1\hat{B}_{1*})^{\frac{1}{\sigma_1}}=\frac{1}{9},\quad (\delta_2\sigma_1\hat{B}_{1*})^{\delta_2}=\frac{1}{\sqrt[3]{3}},$$

and conditions (4.1) and (4.2) are also satisfied.

Finally, it follows from

$$\gamma_1(t)=\int_0^\pi G_1(t,s)(-c_1)\mathrm{d}s=-4c_1,\quad \gamma_2(t)=\int_0^\pi G_2(t,s)(-c_2)\mathrm{d}s=-3c_2$$

that $\gamma_{1*} = \gamma_1^* = -4c_1 < 0$, $\gamma_{2*} = \gamma_2^* = -3c_2 < 0$. (4.8) yields

$$\gamma_{1*} = -4c_1 \geq -4 \cdot \frac{1}{20}\big(\frac{1}{3\sqrt{33}}\big)^{\frac{6}{5}} = -\frac{1}{5}\big(\frac{1}{3\sqrt{33}}\big)^{\frac{6}{5}} = \big(\delta_2\sigma_1 \cdot \frac{\hat{B}_{1*}}{(\rho_2^*)^{\sigma_1}}\big)^{\frac{1}{1-\delta_2\sigma_1}} \cdot \big(1 - \frac{1}{1-\delta_2\sigma_1}\big),$$

$$\gamma_{2*} = -3c_2 \geq -3 \cdot \frac{1}{15}\big(\frac{1}{2\sqrt[3]{40}}\big)^{\frac{6}{5}} = -\frac{1}{5}\big(\frac{1}{2\sqrt[3]{40}}\big)^{\frac{6}{5}} = \big(\delta_1\sigma_2 \cdot \frac{\hat{B}_{2*}}{(\rho_1^*)^{\sigma_2}}\big)^{\frac{1}{1-\delta_1\sigma_2}} \cdot \big(1 - \frac{1}{1-\delta_1\sigma_2}\big).$$

Therefore, (4.3) and (4.4) are satisfied. Consequently, Theorem 4.1 implies system (4.7) has a positive periodic solution.

5. The case $\gamma_{1*} \geq 0$, $\gamma_2^* \leq 0$ ($\gamma_1^* \leq 0$, $\gamma_{2*} \geq 0$)

Theorem 5.1 *Let (A0), (A1) and (A3) hold. If* $\gamma_{1*} \geq 0$, $\gamma_2^* \leq 0$ *and*

$$\gamma_{2*} \geq r_{21} - \hat{B}_{2*} \cdot \frac{r_{21}^{\mu_2\beta_1}}{(\rho_1^* + \gamma_1^* r_{21}^{\beta_1})^{\mu_2}}, \tag{5.1}$$

where $0 < r_{21} < +\infty$ *is the unique positive solution of the equation*

$$r_2^{1-\mu_2\beta_1} \cdot (\rho_1^* + \gamma_1^* r_2^{\beta_1})^{1+\mu_2} = \mu_2\beta_1\hat{B}_{2*}\rho_1^*, \tag{5.2}$$

then (1.1) has a positive T-periodic solution.

Proof Let K be a closed convex set defined as

$$K = \big\{(x,y) \in C_T \times C_T : r_1 \leq x(t) \leq R_1,\ r_2 \leq y(t) \leq R_2,\ t \in [0,T],\ R_2 > 1,\ r_1 < 1\big\}.$$

To prove the theorem, we shall follow the same strategy as in the proofs of previous theorems.

For given $(x,y) \in K$, by (A0), (A3) and $R_2 > 1$, we have

$$\begin{aligned}(A_1x)(t) &\geq \int_{J_{11}} G_1(t,s)f_1(s,y(s))\mathrm{d}s + \int_{J_{12}} G_1(t,s)f_1(s,y(s))\mathrm{d}s + \gamma_{1*}\\
&\geq \int_{J_{11}} G_1(t,s)\frac{\hat{b}_1(s)}{y^{\mu_1}}\mathrm{d}s + \int_{J_{12}} G_1(t,s)\frac{\hat{b}_1(s)}{y^{\alpha_1}}\mathrm{d}s\\
&\geq \int_0^T G_1(t,s)\frac{\hat{b}_1(s)}{R_2^{\sigma_1}}\mathrm{d}s \geq \hat{B}_{1*} \cdot \frac{1}{R_2^{\sigma_1}},\\
(A_1x)(t) &\leq \int_{J_{11}} G_1(t,s)\frac{e_1(s)}{y^{\beta_1}}\mathrm{d}s + \int_{J_{12}} G_1(t,s)\frac{b_1(s)}{y^{\beta_1}}\mathrm{d}s + \gamma_1^*\\
&\leq \int_{J_{11}} G_1(t,s)\frac{e_1(s)}{r_2^{\beta_1}}\mathrm{d}s + \int_{J_{12}} G_1(t,s)\frac{b_1(s)}{r_2^{\beta_1}}\mathrm{d}s + \gamma_1^*\\
&\leq \int_0^T G_1(t,s)\frac{e_1(s)}{r_2^{\beta_1}}\mathrm{d}s + \int_0^T G_1(t,s)\frac{b_1(s)}{r_2^{\beta_1}}\mathrm{d}s + \gamma_1^*\\
&\leq \frac{1}{r_2^{\beta_1}} \cdot \rho_1^* + \gamma_1^*.\end{aligned}$$

Similarly, we can get

$$(A_2y)(t) \leq \frac{1}{r_1^{\delta_2}} \cdot \rho_2^*, \quad (A_2y)(t) \geq \hat{B}_{2*} \cdot \frac{1}{R_1^{\mu_2}} + \gamma_{2*}.$$

Now, $(A_1x, A_2y) \in K$ if r_1, r_2, R_1 and R_2 are chosen such that

$$\hat{B}_{1*} \cdot \frac{1}{R_2^{\sigma_1}} \geq r_1, \quad \frac{1}{r_1^{\delta_2}} \cdot \rho_2^* \leq R_2; \tag{5.3}$$

$$\frac{1}{r_2^{\beta_1}} \cdot \rho_1^* + \gamma_1^* \leq R_1, \quad \hat{B}_{2*} \cdot \frac{1}{R_1^{\mu_2}} + \gamma_{2*} \geq r_2, \tag{5.4}$$

and they should satisfy that $R_2 > 1$, $r_1 < 1$.

Let $R_2 = \frac{1}{r_1^{\delta_2}} \cdot \rho_2^*$ be fixed. The first inequality of (5.3) holds if r_1 satisfies

$$\frac{\hat{B}_{1*}}{(\rho_2^*)^{\sigma_1}} \cdot r_1^{\delta_2\sigma_1} \geq r_1, \tag{5.5}$$

or equivalently,

$$0 < r_1 \leq \Big(\frac{\hat{B}_{1*}}{(\rho_2^*)^{\sigma_1}}\Big)^{\frac{1}{1-\delta_2\sigma_1}}. \tag{5.6}$$

If we choose $0 < r_1 < 1$ small enough, then (5.6) holds, and $R_2 > 1$ is large enough.

If we fix $R_1 = \frac{1}{r_2^{\beta_1}} \cdot \rho_1^* + \gamma_1^*$, then the second inequality of (5.4) holds provided that r_2 verifies $\gamma_{2*} \geq r_2 - \hat{B}_{2*} \cdot \frac{1}{R_1^{\mu_2}} = r_2 - \hat{B}_{2*} \cdot \frac{r_2^{\mu_2\beta_1}}{(\rho_1^* + \gamma_1^* r_2^{\beta_1})^{\mu_2}}$, or equivalently,

$$\gamma_{2*} \geq f(r_2) := r_2 - \hat{B}_{2*} \cdot \frac{r_2^{\mu_2\beta_1}}{(\rho_1^* + \gamma_1^* r_2^{\beta_1})^{\mu_2}}. \tag{5.7}$$

It is not difficult to check

$$f'(r_2) = 1 - \mu_2\beta_1\hat{B}_{2*}\rho_1^* \cdot r_2^{\mu_2\beta_1-1} \cdot (\rho_1^* + \gamma_1^* r_2^{\beta_1})^{-1-\mu_2}, \tag{5.8}$$

and then $f'(0) = -\infty$, $f'(+\infty) = 1$, hence there exists r_{21} such that $f'(r_{21}) = 0$. Furthermore,

$$\begin{aligned} f''(r_2) =& \mu_2\beta_1\hat{B}_{2*}\rho_1^*(1-\mu_2\beta_1) \cdot r_2^{\mu_2\beta_1-1} \cdot (\rho_1^* + \gamma_1^* r_2^{\beta_1})^{-1-\mu_2} + \\ & \mu_2\beta_1\hat{B}_{2*}\rho_1^* \cdot r_2^{\mu_2\beta_1-1}(1+\mu_2)(\rho_1^* + \gamma_1^* r_2^{\beta_1})^{-2-\mu_2} \cdot \beta_1\gamma_1^* r_2^{\beta_1-1} > 0, \end{aligned} \tag{5.9}$$

and therefore $f(r_2)$ possesses a minimum at r_{21}, i.e., $f(r_{21}) = \min\limits_{r_2 \in (0,\infty)} f(r_2)$.

Since $f'(r_{21}) = 0$, we get $1 - \mu_2\beta_1\hat{B}_{2*}\rho_1^* \cdot r_{21}^{\mu_2\beta_1-1} \cdot (\rho_1^* + \gamma_1^* r_{21}^{\beta_1})^{-1-\mu_2} = 0$, or equivalently,

$$r_{21}^{1-\mu_2\beta_1} \cdot (\rho_1^* + \gamma_1^* r_{21}^{\beta_1})^{1+\mu_2} = \mu_2\beta_1\hat{B}_{2*}\rho_1^*. \tag{5.10}$$

Taking $r_2 = r_{21}$, the second inequality of (5.4) holds if $\gamma_{2*} \geq f(r_{21})$, which is just (5.1). The first inequality of (5.4) holds directly by the choice of R_1. $\Box$

Remark 5.2 Note that the right-hand side of (5.1) is always negative, which is equivalent to showing $f(r_{21}) < 0$. By (5.10), this is obviously satisfied because

$$\begin{aligned} f(r_{21}) &= r_{21} - \hat{B}_{2*} \cdot \frac{r_{21}^{\mu_2\beta_1}}{(\rho_1^* + \gamma_1^* r_{21}^{\beta_1})^{\mu_2}} \\ &= \frac{r_{21}^{\mu_2\beta_1} \cdot \hat{B}_{2*}}{(\rho_1^* + \gamma_1^* r_{21}^{\beta_1})^{1+\mu_2}} \cdot \big((\mu_2\beta_1 - 1)\rho_1^* - \gamma_1^* r_{21}^{\beta_1}\big) < 0. \end{aligned} \tag{5.11}$$

Moreover, Theorem 5.1 is still valid if we choose α_1, μ_1, β_2, $\nu_2 \in (0,1)$ and $\mu_2 > 0$, $\beta_1 > 0$ with $\mu_2\beta_1 < 1$, which implies f_1 satisfies weak force condition, f_2 satisfies either strong force condition or weak force condition.

Remark 5.3 In the special case $\alpha_1 = \beta_1 = \beta_2 = \mu_1 = \mu_2 = \nu_2$, our condition (A3) is also more general than (H2), so Theorem 5.1 improves Theorem 1.3.

Using the same methods as in the proof of Theorem 5.1 with obvious changes, we can prove the following

Theorem 5.4 *Let (A0), (A1) hold. Assume*

(A4) There are $\hat{b}_i,\ b_i,\ e_i \in L^1(0,T)$ *with* $\hat{b}_i,\ b_i,\ e_i \succ 0,\ \alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \mu_2,\ \nu_1 \in (0,1)$ *satisfying*

$$0 \le \frac{\hat{b}_1(t)}{x^{\alpha_1}} \le f_1(t,x) \le \frac{b_1(t)}{x^{\beta_1}},\quad x \in [1,\infty),\ \text{a.e. } t \in [0,T],$$

$$0 \le \frac{\hat{b}_1(t)}{x^{\alpha_1}} \le f_1(t,x) \le \frac{e_1(t)}{x^{\nu_1}},\quad x \in (0,1),\ \text{a.e. } t \in [0,T];$$

Moreover, suppose

$$0 \le \frac{\hat{b}_2(t)}{x^{\alpha_2}} \le f_2(t,x) \le \frac{b_2(t)}{x^{\beta_2}},\quad x \in [1,\infty),\ \text{a.e. } t \in [0,T],$$

$$0 \le \frac{\hat{b}_2(t)}{x^{\mu_2}} \le f_2(t,x) \le \frac{e_2(t)}{x^{\beta_2}},\quad x \in (0,1),\ \text{a.e. } t \in [0,T].$$

If $\gamma_1^* \le 0$, $\gamma_{2*} \ge 0$ *and*

$$\gamma_{1*} \ge r_{11} - \hat{B}_{1*} \cdot \frac{r_{11}^{\beta_2\alpha_1}}{(\rho_2^* + \gamma_2^* r_{11}^{\beta_2})^{\alpha_1}}, \tag{5.12}$$

(1.1) possesses a positive T-periodic solution, where r_{11} is the unique positive solution of

$$r_1^{1-\beta_2\alpha_1} \cdot (\rho_2^* + \gamma_2^* r_1^{\beta_2})^{1+\alpha_1} = \beta_2\alpha_1\hat{B}_{1*}\rho_2^*. \tag{5.13}$$

Remark 5.5 As Remark 5.2, we can show the right-hand side of (5.12) is always negative. Moreover, Theorem 5.4 is still valid if we choose $\alpha_2,\ \mu_2,\ \beta_1,\ \nu_1 \in (0,1)$ and $\beta_2 > 0$, $\alpha_1 > 0$ with $\beta_2\alpha_1 < 1$. This implies f_1 satisfies either strong or weak force condition, and f_2 satisfies weak force condition.

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