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Morrey Estimates for Schrödinger Type Elliptic Equations

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Abstract In this paper, by means of Olsen type inequalities related to the fractional integral operator, the authors establish the interior estimates in Morrey spaces for Schrödinger type elliptic equations with potentials satisfying a reverse Hölder condition.

Keywords Elliptic equations; fractional integral; potential

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1. Introduction

Let us consider the linear Schrödinger type elliptic equations

$$\mathcal{L}u(x) = -\sum_{i,j=1}^{n} a_{i,j}(x)u_{x_ix_j} + V(x)u = f(x),\ \text{a.e. in } \Omega, \tag{1}$$

where Ω is an open set of $\mathbb{R}^n$, $n \geq 3$, $a_{ij} = a_{ji}$,

$$\mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq \mu|\xi|^2,\quad \text{for some } \mu > 0,\ \text{a.e. } x \in \Omega,\ \xi \in \mathbb{R}^n, \tag{2}$$

$$a_{ij} \in L^\infty(\Omega) \cap \mathrm{VMO}(\Omega). \tag{3}$$

As to the potential $V(x)$, we assume that it is not identically zero, belongs to $RH_q(\Omega)$ for some exponent $q \geq n/2$, which means that $V \in L^q_{\rm loc}$, $V \geq 0$, and there exists a constant C such that the reverse Hölder inequality

$$\Big(\frac{1}{|B|}\int_B V(y)^q\mathrm{d}y\Big)^{1/q} \leq \frac{C}{|B|}\int_B V(y)\mathrm{d}y \tag{4}$$

holds for every ball $B \subset \Omega$. An important property of the RH_q class, proved in [1], assures that the condition $V \in RH_q$ also implies $V \in RH_{q+\varepsilon}$ for some $\varepsilon > 0$ and that the $RH_{q+\varepsilon}$ constant of V is controlled in terms of the one of RH_q membership.

For a nondivergence form elliptic equation with VMO coefficients but with null potential, the results were obtained in [2–5]; Inspired by [6], we study Morrey regularity of nondivergence elliptic equations with potentials satisfying a reverse Hölder condition.

The main purpose of the present work is to show that

$$\|D^2u\|_{L^{p,\lambda}(\Omega')} + \|Vu\|_{L^{p,\lambda}(\Omega')} \leq C(\|f\|_{L^{p,\lambda}(\Omega'')} + \|u\|_{L^{p,\lambda}(\Omega'')}) \tag{5}$$

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for any $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, and $1 < p < \frac{n}{2}$, $0 < \lambda < n - 2p$.

The paper is organized as follows. In Section 2 we give some basic notions. In this section we recall also continuity results regarding Olsen type inequalities related to the fractional integral operator and commutators that will appear in the interior representation formula of the Vu estimates, which replace on the duality methods used in [6]. Then a priori estimate is established in Section 3.

2. Some preliminary facts from real analysis

Let Ω be an open subset of $\mathbb{R}^n$. We take a ball $B_r(x)$ centered at x with radius $r > 0$, for $\Omega_r \equiv B_r(x) \cap \Omega$.

Definition 2.1 *We say that any $f \in L^1_{\rm loc}(\Omega)$ is in the spaces* BMO(Ω) *if*

$$\sup_{\gamma>0,x\in\Omega} \frac{1}{|\Omega_\gamma|} \int_{\Omega_\gamma} |f(y) - f_{\Omega_\gamma}|{\rm d}y \equiv \|f\|_* < \infty, \tag{6}$$

where f_{Ω_γ} is the average over Ω_γ of f. Moreover, for any $f \in$ BMO(Ω) and $r > 0$, we set

$$\sup_{\gamma\le r,x\in\Omega} \frac{1}{|\Omega_\gamma|} \int_{\Omega_\gamma} |f(y) - f_{\Omega_\gamma}|dy \equiv \theta(r). \tag{7}$$

We say that any $f \in$ BMO(Ω) is in the spaces VMO(Ω) *if $\theta(r) \to 0$ as $r \to 0$ and refer to $\theta(r)$ as the modulus of f.*

Definition 2.2 *For $1 < p < \infty$, $0 < \lambda < n$, $f \in L^1_{\rm loc}(\Omega)$, let*

$$\|f\|^p_{L^{p,\lambda}(\Omega)} = \sup_{\gamma>0,x\in\Omega} \frac{1}{\gamma^\lambda} \int_{\Omega_\gamma} |f(y)|^p{\rm d}y$$

and define the Morrey space $L^{p,\lambda}(\Omega)$ to be the set of those measurable f for which $\|f\|^p_{L^{p,\lambda}(\Omega)}$ is finite. f is said to belong to the Sobolev-Morrey space $W^2L^{p,\lambda}(\Omega)$ if and only if f and its distributional derivatives $f_{x_i}, f_{x_ix_j}$ $(i, j = 1, \ldots, n)$ are in $L^{p,\lambda}(\Omega)$. Let $\|f\|_{W^2L^{p,\lambda}(\Omega)} \equiv \|f\|_{L^{p,\lambda}(\Omega)} + \sum_{i=1}^n \|f_{x_i}\|_{L^{p,\lambda}(\Omega)} + \sum_{i,j=1}^n \|f_{x_ix_j}\|_{L^{p,\lambda}(\Omega)}$.

We also assume that $f \in W^2_{\rm loc}L^{p,\lambda}(\Omega)$ if $f \in W^2L^{p,\lambda}(\Omega')$ for every $\Omega' \subset\subset \Omega$.

Remark 2.3 As usual, BMO$(\mathbb{R}^n)$ or $L^{p,\lambda}(\mathbb{R}^n)$ means B ranges in the class of balls of $\mathbb{R}^n$.

Let $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the Riesz potential or the (classical) fractional integral operator, which is given by the formula

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}{\rm d}y, \quad 0 < \alpha < n.$$

It is well known that the Riesz potential I_α plays an important role in harmonic analysis. Below, we describe two main results on Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$. Assume the relevance for indexes satisfies

$$0 < \alpha < n,\ 1 < p < \frac{n}{\alpha},\ 0 < \lambda < n - \alpha p,\ \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}. \tag{8}$$

Adams [7] proved the boundedness of I_α from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, Fazio and Ragusa

[8] obtained commutators $[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x)$ from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, with $b(x) \in \mathrm{BMO}(\mathbb{R}^n)$. Here, we shall be interested in the boundedness of the multiplication operators $f \mapsto V \cdot I_\alpha f$ and $f \mapsto V \cdot [b, I_\alpha]f$ on Morrey spaces. For $1 < p < \frac{n}{\alpha}$, $0 \le \lambda < n - \alpha p$, Olsen [9] (particularly for $n = 3$) proved $W \cdot I_\alpha$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$, provided that $W \in L^{\frac{n-\lambda}{\alpha},\lambda}(\mathbb{R}^n)$.

Lemma 2.4 *Suppose that (8) and $V(x) \in L^{\frac{n}{\alpha}}(\mathbb{R}^n)$, $b(x) \in \mathrm{BMO}(\mathbb{R}^n)$, we have*

$$\|V \cdot I_\alpha f\|_{L^{p,\lambda}(\mathbb{R}^n)} \le C\|V\|_{L^{\frac{n}{\alpha}}(\mathbb{R}^n)}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} \tag{9}$$

and

$$\|V \cdot [b, I_\alpha]f\|_{L^{p,\lambda}(\mathbb{R}^n)} \le C\|b\|_*\|V\|_{L^{\frac{n}{\alpha}}(\mathbb{R}^n)}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}. \tag{10}$$

Proof Taking $B_r(x_0) \subset \mathbb{R}^n$, we observe that I_α is of type $(L^{p,\lambda}(\mathbb{R}^n), L^{q,\lambda}(\mathbb{R}^n))$. Then, by Hölder's inequality, we have

$$\begin{aligned}
\Big(\frac{1}{r^\lambda}\int_{B_r(x_0)}|V \cdot I_\alpha f|^p \mathrm{d}x\Big)^{\frac{1}{p}} &\le C\Big(\frac{1}{r^\lambda}\int_{B_r(x_0)}|V|^{\frac{n-\lambda}{\alpha}}\mathrm{d}x\Big)^{\frac{\alpha}{n-\lambda}} \cdot \Big(\frac{1}{r^\lambda}\int_{B_r(x_0)}|I_\alpha f|^q \mathrm{d}x\Big)^{\frac{1}{q}}\\
&\le C\Big(\frac{1}{r^\lambda}\int_{B_r(x_0)}|V|^{\frac{n-\lambda}{\alpha}}\mathrm{d}x\Big)^{\frac{\alpha}{n-\lambda}}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}\\
&\le C\frac{1}{r^{\frac{\alpha\lambda}{n-\lambda}}}\Big(\int_{B_r(x_0)}|V|^{\frac{n}{\alpha}}\mathrm{d}x\Big)^{\frac{\alpha}{n}}|B_r(x_0)|^{\frac{\alpha}{n-\lambda}\cdot\frac{\lambda}{n}}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}\\
&\le C\|V\|_{L^{\frac{n}{\alpha}}(\mathbb{R}^n)}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

The desired inequality [9] follows. Also, note that $(L^{p,\lambda}(\mathbb{R}^n), L^{q,\lambda}(\mathbb{R}^n))$ type of $[b, I_\alpha]f$, So we can obtain the inequality [10] by a similar process.

From Lemma 2.4 and the extension theorem of BMO(Ω)-functions in [10], by a procedure similar to Theorem 2.11 in [2] and Theorem 2.3 in [3], we can obtain the following corollary.

Corollary 2.5 *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $0 < \lambda < n - \alpha p$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ and $V(x) \in RH_q$, $q \ge n/\alpha$. Suppose B is an open ball of $\mathbb{R}^n$ and $b(x) \in \mathrm{BMO}(B)$. Then there exists a constant $C > 0$ such that for all $f \in L^{p,\lambda}(B)$,*

$$\|V \cdot I_\alpha f\|_{L^{p,\lambda}(B)} \le C\|V\|_{L^{\frac{n}{\alpha}}(B)}\|f\|_{L^{p,\lambda}(B)},$$

$$\|V \cdot [b, I_\alpha]f\|_{L^{p,\lambda}(B)} \le C\|b\|_*\|V\|_{L^{\frac{n}{\alpha}}(B)}\|f\|_{L^{p,\lambda}(B)}.$$

Furthermore, for any $\varepsilon > 0$, there exists positive $\gamma_0 = \gamma_0(\varepsilon, \theta)$ such that for any ball B_r with the radius $r \in (0, \gamma_0), B_r \cap \Omega \equiv \Omega_r \ne \emptyset$, and all $f \in L^{p,\lambda}(\Omega_r)$,

$$\|V \cdot [b, I_\alpha]f\|_{L^{p,\lambda}(\Omega_r)} \le C\varepsilon\|V\|_{L^{\frac{n}{\alpha}}(\Omega_r)}\|f\|_{L^{p,\lambda}(\Omega_r)}.$$

3. Priori estimates and some regularity results

We take a ball $B_r(x_0)$ with r chosen later. For $\Omega_r \equiv B_r(x_0) \cap \Omega$, freeze the coefficients of first term at x_0, and get the operator $\mathcal{L}_0 u = -\sum_{i,j}^n a_{i,j}(x_0)u_{x_i x_j} + V(x)u$, which allows us to apply the results proved by Dziubanski [11, Proposition 4.9] to deduce the following

Lemma 3.1 *The operator $\mathcal{L}_0$ has a fundamental solution $\Gamma(x_0;x,y)$ satisfying the following bound: for any positive integer k there exists a constant c_k (independent of x_0) such that*

$$\Gamma(x_0;x,y)\le\frac{c_k}{(1+\frac{|x-y|}{\rho(x)})^k}\cdot\frac{1}{|x-y|^{n-2}},\quad \text{for any}\ \ x,y\in\mathbb{R}^n,\ x\ne y,$$

where the auxiliary function $\rho(x)$ is defined as

$$\rho(x)=\sup\Big\{r>0:\frac{1}{r^{d-2}}\int_{B_r(x)}V(y)\mathrm{d}y\le 1\Big\},\quad x\in\mathbb{R}^n.$$

Lemma 3.2([3]) *Under the assumptions (2) and (3), $p\in(1,\infty)$, $\lambda\in(0,n)$, there exist positive constants C, such that for any $\Omega_r\subset\subset\Omega, r<\gamma_0$, and any weak solution $u\in W_0^{2,p}(\Omega_r)$ of $\mathcal{A}u=\sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j}=f$ such that $u_{x_ix_j}\in L^{p,\lambda}(\Omega_r)$, we have*

$$\|u_{x_ix_j}\|_{L^{p,\lambda}(\Omega_r)}\le C\|\mathcal{A}u\|_{L^{p,\lambda}(\Omega_r)},\quad i,j=1,\dots,n$$

where the constants C depend on n,p,λ,μ and the VMO module of the leading coefficients.

By the way, let us note that if $u\in W^2L^{p,\lambda}(\Omega)$ solves $\mathcal{L}u=f$ with $f\in L^{p,\lambda}(\Omega)$, then automatically $Vu\in L^{p,\lambda}(\Omega)$ (since $\mathcal{A}u\in L^{p,\lambda}(\Omega)$ by the boundedness of the coefficients), however, $u\in W^2L^{p,\lambda}(\Omega)$ does not imply in general that $\mathcal{L}u\in L^{p,\lambda}(\Omega)$.

Theorem 3.3 *Under the assumptions (2)–(4), for any $p\in(1,\frac{n}{2})$, $\lambda\in(0,n-2p)$, there exists a constant $C>0$ such that for any $\Omega_r\subset\subset\Omega, r<\gamma_0$, $\mathcal{L}u\in L^{p,\lambda}(\Omega_r)$, $u\in W^2L^{p,\lambda}(\Omega_r)\cap W_0^{1,p}(\Omega_r)$, then*

$$\|Vu\|_{L^{p,\lambda}(\Omega_r)}\le C\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega_r)},$$

where the constants C depend on n,p,λ,μ, the VMO module of the leading coefficients and the RH_q the constant of V.

Proof By a density argument, we might as well assume $u\in C_0^\infty(\Omega_r)$. For $x\in\Omega_r\equiv B_r(x_0)\cap\Omega$, and $\mathcal{L}_0u=(\mathcal{L}_0-\mathcal{L})u+\mathcal{L}u$ we can write

$$u(x)=\int_{\Omega_r}\Gamma(x_0;x,y)\mathcal{L}u(y)\mathrm{d}y+\sum_{i,j=1}^n\int_{\Omega_r}\Gamma(x_0;x,y)[a_{ij}(y)-a_{ij}(x_0)]u_{x_ix_j}(y)\mathrm{d}y.$$

Letting $x_0=x$, we get the representation formula

$$u(x)=\int_{\Omega_r}\Gamma(x;x,y)\mathcal{L}u(y)\mathrm{d}y+\sum_{i,j=1}^n\int_{\Omega_r}\Gamma(x;x,y)[a_{ij}(y)-a_{ij}(x)]u_{x_ix_j}(y)\mathrm{d}y,$$

which allows us to write the following pointwise bound, for every positive integer k,

$$\begin{aligned}|V(x)u(x)|\le&c_kV(x)\int_{\Omega_r}\frac{1}{\big(1+\frac{|x-y|}{\rho(x)}\big)^k}\cdot\frac{1}{|x-y|^{n-2}}\Big\{|\mathcal{L}u(y)|+\\&\sum_{i,j=1}^n|a_{ij}(y)-a_{ij}(x)||u_{x_ix_j}(y)|\Big\}\mathrm{d}y.\end{aligned}$$

Setting

$$V\cdot I_2f(x)=V(x)\int_{\Omega_r}\frac{1}{\big(1+\frac{|x-y|}{\rho(x)}\big)^k}\cdot\frac{1}{|x-y|^{n-2}}f(y)\mathrm{d}y,$$

$$V\cdot[a,I_2]f(x)=V(x)\int_{\Omega_r}\frac{1}{\big(1+\frac{|x-y|}{\rho(x)}\big)^k}\cdot\frac{1}{|x-y|^{n-2}}|a(y)-a(x)|f(y)\mathrm{d}y,$$

by Corollary 2.5 ($\alpha=2$) and Lemma 3.2, we have

$$\begin{aligned}\|Vu\|_{L^{p,\lambda}(\Omega_r)}&\le C\|V\cdot I_2(|\mathcal{L}u|)\|_{L^{p,\lambda}(\Omega_r)}+C\|V\cdot[a,I_2](|u_{x_ix_j}|)\|_{L^{p,\lambda}(\Omega_r)}\\&\le C\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega_r)}+C\varepsilon\|u_{x_ix_j}\|_{L^{p,\lambda}(\Omega_r)}\\&\le C\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega_r)}+C\varepsilon\|\mathcal{L}u-Vu\|_{L^{p,\lambda}(\Omega_r)}\\&\le (C+C\varepsilon)\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega_r)}+C\varepsilon\|Vu\|_{L^{p,\lambda}(\Omega_r)}.\end{aligned}$$

Now, using the interpolation methods of [4,5], we are in a position to state our main results.

Theorem 3.4 *Under the assumptions (2)–(4), for $p\in(1,\frac{n}{2})$, $\lambda\in(0,n-2p)$. Then if $u\in W^2_{\rm loc}L^{p,\lambda}(\Omega)\cap W^{1,p}_0(\Omega)$, $\mathcal{L}u\in L^{p,\lambda}_{\rm loc}(\Omega)$, and for given $\Omega'\subset\subset\Omega''\subset\subset\Omega$ we have $Vu\in L^{p,\lambda}(\Omega')$ and there exists a positive constant C such that*

$$\|D^2u\|_{L^{p,\lambda}(\Omega')}+\|Vu\|_{L^{p,\lambda}(\Omega')}\le C\big(\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega'')}+\|u\|_{L^{p,\lambda}(\Omega'')}\big),\tag{11}$$

where the constants C depend on n,p,λ,μ, the VMO module of the leading coefficients and the RH_q the constant of V.

Proof For any $x_0\in\Omega$, choose $\gamma_0>0$ such that $\Omega_{\gamma_0}=B_{\gamma_0}(x_0)\cap\Omega\subset\subset\Omega$. Let $0<r\le\gamma_0$ and $\eta(x)$ be a cut-off function on $B_{\tau'r}(x_0)$ relative to $B_{\tau r}(x_0)$, for $\tau\in(0,1)$ and $\tau'=\tau(3-\tau)/2>\tau$, we have $\eta(x)\in C_0^\infty(B_r(x_0))$ satisfying $\eta(x)\equiv1$ in $B_{\tau r}(x_0),\eta(x)\equiv0$ in $B^c_{\tau'r}(x_0)$ and

$$|D^s\eta(x)|\le C[\tau(1-\tau)r]^{-s},\ \text{for } s=0,1,2,\quad x\in B_r(x_0).$$

Set $v=\eta u$, then $v\in W^2L^{p,\lambda}(B_r(x_0))\cap W^{1,p}_0(B_r(x_0))$. By Theorem 3.3, we have

$$\begin{aligned}\|Vv\|_{L^{p,\lambda}(\Omega_{\tau'r})}&\le C\|\mathcal{L}(v)\|_{L^{p,\lambda}(\Omega_{\tau'r})}\\&\le C\Big(\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega_{\tau'r})}+\frac{\|Du\|_{L^{p,\lambda}(\Omega_{\tau'r})}}{\tau(1-\tau)r}+\frac{\|u\|_{L^{p,\lambda}(\Omega_{\tau'r})}}{[\tau(1-\tau)r]^2}\Big).\end{aligned}\tag{12}$$

Analogously, Lemma 3.2 implies

$$\|D^2v\|_{L^{p,\lambda}(\Omega_{\tau'r})}\le C\Big(\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega_{\tau'r})}+\|Vv\|_{L^{p,\lambda}(\Omega_{\tau'r})}+\frac{\|Du\|_{L^{p,\lambda}(\Omega_{\tau'r})}}{\tau(1-\tau)r}+\frac{\|u\|_{L^{p,\lambda}(\Omega_{\tau'r})}}{[\tau(1-\tau)r]^2}\Big).\tag{13}$$

Therefore, from the definition of v and the estimate of $\|Vv\|_{L^{p,\lambda}(\Omega_{\tau'r})}$, $\|D^2v\|_{L^{p,\lambda}(\Omega_{\tau'r})}$, we get that for any $0<r\le\gamma_0$,

$$\|D^2u\|_{L^{p,\lambda}(\Omega_{\tau r})}+\|Vu\|_{L^{p,\lambda}(\Omega_{\tau r})}\le C\Big(\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega_{\tau'r})}+\frac{\|Du\|_{L^{p,\lambda}(\Omega_{\tau'r})}}{\tau(1-\tau)r}+\frac{\|u\|_{L^{p,\lambda}(\Omega_{\tau'r})}}{[\tau(1-\tau)r]^2}\Big).\tag{14}$$

Denote

$$\Theta_s=\sup_{0<\tau<1}[\tau(1-\tau)r]^s\|D^su\|_{L^{p,\lambda}(\Omega_{\tau r})},\quad s=0,1,2.$$

Because of the choice of τ' we have $\tau(1-\tau)\le2\tau'(1-\tau')$. Thus, after standard transformations and taking the supremum with respect to $\tau\in(0,1)$, we can rewrite the inequality (14) as

$$\Theta_2+r^2\|Vu\|_{L^{p,\lambda}(\Omega_{\tau r})}\le C(r^2\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega_r)}+\Theta_1+\Theta_0).\tag{15}$$

By simple scaling arguments we get an interpolation inequality analogous to [12, Theorem 7.28]

$$\|Du\|_{L^{p,\lambda}(\Omega_r)} \le \delta \|D^2u\|_{L^{p,\lambda}(\Omega_r)} + \frac{C}{\delta}\|u\|_{L^{p,\lambda}(\Omega_r)}, \quad \delta \in (0,r). \tag{16}$$

There exists $\tau_0 \in (0,1)$ such that

$$\begin{aligned}\Theta_1 &\le 2[\tau_0(1-\tau_0)r]\|Du\|_{L^{p,\lambda}(\Omega_{\tau_0 r})}\\ &\le 2[\tau_0(1-\tau_0)r]\big(\delta\|D^2u\|_{L^{p,\lambda}(\Omega_{\tau_0 r})} + \frac{C}{\delta}\|u\|_{L^{p,\lambda}(\Omega_{\tau_0 r})}\big).\end{aligned}$$

For any $\varepsilon \in (0,2)$, we choose $\delta = \frac{\varepsilon}{2}[\tau_0(1-\tau_0)r] < \tau_0 r$. Thus $\Theta_1 \le \varepsilon\Theta_2 + \frac{C}{\varepsilon}\Theta_0$, for any $\varepsilon \in (0,2)$.

Interpolating Θ_1 in (15) and fixing $\tau = 1/2$, we obtain the following Caccioppoli-type inequality

$$\|D^2u\|_{L^{p,\lambda}(\Omega_{r/2})} + \|Vu\|_{L^{p,\lambda}(\Omega_{r/2})} \le C\big(\|\mathcal{L}u\|_{L^{p,\lambda}(\Omega_r)} + \frac{1}{r^2}\|u\|_{L^{p,\lambda}(\Omega_r)}\big). \tag{17}$$

From (17), by a finite covering of Ω' with balls $\Omega_{r/2}$, $r < \mathrm{dist}(\Omega', \Omega'')$, the estimate (11) follows.

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