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# On Some Properties of $c$-Frames

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**Abstract** In this paper we discuss about $c$-frames, namely continuous frames. Since, $c$-frames are generalizations of discrete frames, we generalize some results of discrete frames to continuous version. We explain some results about relations of projections in Hilbert spaces and $c$-frames to characterize these frames. Also, we will specify (precisely) the synthesis and frame operators of Bochner integrable $c$-frames. Finally, we classify Hilbert-Schmidt operators by $c$-frames and express some new identities for Parseval $c$-frames.

**Keywords** Banach space; Hilbert space; frame; $c$-frame; Parseval $c$-frame; Bochner measurable

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## 1. Introduction

Duffin and Schaeffer introduced the concept of discrete frames in Hilbert spaces in 1952 to study some deep problems in nonharmonic Fourier series [1]. After the fundamental paper [2] by Daubechies, Grossmann and Meyer, frames usage began to be raised. In signal processing, image and data compression and sampling theory, the concept of frame has a fundamental impact. Frames provide an alternative to orthonormal bases in Hilbert spaces. Indeed, a discrete frame is a countable family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements. For more details about discrete frames we refer to [3]. Various kind of frames have been introduced till now, which are generalization of discrete frames. For more studies about some types of frames, the interested reader can refer to [4–11].

In this paper we generalize some concepts of discrete frames and some results in [12] to $c$-frames. The paper is organized as follows. In Section 2, we verify relations between projections and $c$-frames. Our aim in Section 3 is study of effects of Bochner integrability on $c$-frames. Section 4 is devoted to classifying Hilbert-Schmidt operators by $c$-frames. Finally, in the last section we show some new identities for Parseval $c$-frames.

Throughout this paper $H$ and $K$ stand for Hilbert spaces, and $X$ and $Y$ stand for Banach spaces.

Suppose $(\Omega, \Sigma, \mu)$ is a measure space, where $\mu$ is a positive measure.

At first we give some definitions to introduce Bochner measurable and Bochner integrable mappings.

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**Definition 1.1** *A function $f : \Omega \longrightarrow X$ is called simple if there exist $x_1, \ldots, x_n \in X$ and $E_1, \ldots, E_n \in \Sigma$ such that $f = \sum_{i=1}^{n} x_i \chi_{E_i}$, where $\chi_{E_i}(\omega) = 1$ if $\omega \in E_i$ and $\chi_{E_i}(\omega) = 0$ if $\omega \in E_i^c$. If $\mu(E_i)$ is finite, whenever $x_i \neq 0$, then the simple function $f$ is integrable, and the integral is then defined by*

$$\int_\Omega f(\omega) \mathrm{d}\mu(\omega) = \sum_{i=1}^{n} \mu(E_i) x_i.$$

**Definition 1.2** *A function $f : \Omega \longrightarrow X$ is called Bochner measurable if there exists a sequence of simple functions $\{f_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \|f_n - f\| = 0$, $\mu$-almost everywhere.*

**Definition 1.3** *A Bochner measurable function $f : \Omega \longrightarrow X$ is called Bochner integrable if there exists a sequence of integrable simple functions $\{f_n\}_{n=1}^{\infty}$ such that*

$$\lim_{n\to\infty} \int_\Omega \|f_n(\omega) - f(\omega)\| \mathrm{d}\mu(\omega) = 0.$$

*In this case, $\int_E f(\omega) d\mu(\omega)$ is defined by*

$$\int_E f(\omega) \mathrm{d}\mu(\omega) = \lim_{n\to\infty} \int_E f_n(\omega) \mathrm{d}\mu(\omega), \quad E \in \Sigma.$$

Now, we review the definition of continuous frames.

**Definition 1.4** *A mapping $f : \Omega \longrightarrow H$ is called a continuous frame or c-frame for $H$ if:*

*(i) For each $h \in H$, $\omega \longmapsto \langle h, f(\omega)\rangle$ is a measurable function;*

*(ii) There exist positive constants $A$ and $B$ such that*

$$A\|h\|^2 \leq \int_\Omega |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \leq B\|h\|^2, \quad h \in H. \tag{1.1}$$

The constants $A$ and $B$ are called $c$-frame bounds. $f$ is called a tight $c$-frames if $A = B$ and it is called a Parseval $c$-frame if $A = B = 1$. The mapping $f$ is called $c$-Bessel mapping if the second inequality in (1.1) holds. In this case, $B$ is called the Bessel constant.

For a $c$-Bessel mapping, there are two important associated operators as below.

**Proposition 1.5** ([11]) *Let $f$ be a c-Bessel mapping for $H$. Then the operator*

$$T : L^2(\Omega, \mu) \longrightarrow H$$

*weakly defined by*

$$\langle T\varphi, h\rangle = \int_\Omega \varphi(\omega) \langle f(\omega), h\rangle \mathrm{d}\mu(\omega), \quad h \in H, \tag{1.2}$$

*is well defined, linear, bounded and its adjoint is given by*

$$T^* : H \longrightarrow L^2(\Omega, \mu), \quad T^* h(\omega) = \langle h, f(\omega)\rangle, \ \omega \in \Omega. \tag{1.3}$$

*The operator $T$ is called the pre-frame operator or the synthesis operator and $T^*$ is called the analysis operator of $f$.*

If $f$ is a $c$-Bessel mapping for $H$, then the operator $S : H \longrightarrow H$ defined by $S = TT^*$, is called the frame operator of $f$. Thus

$$\langle Sh, k\rangle = \int_\Omega \langle h, f(\omega)\rangle \langle f(\omega), k\rangle \mathrm{d}\mu(\omega), \quad h, k \in H.$$

It can be easily shown that if $f$ is a $c$-frame for $H$, then $S$ is invertible.

The following Lemma provides a right inverse for a closed range operator.

**Lemma 1.6** ([13]) *Let $H, K$ be Hilbert spaces, and suppose that $U : K \longrightarrow H$ is a bounded operator with closed range $R_U$. Then there exists a bounded operator $U^\dagger : K \longrightarrow H$ for which*

$$N_{U^\dagger} = R_U^\perp, \quad R_{U^\dagger} = N_U^\perp, \quad UU^\dagger x = x, \ x \in R_U.$$

The operator $U^\dagger$ is called the pseudo-inverse of $U$.

Now, we state the definition of a Hilbert-Schmidt operator.

**Definition 1.7** *A linear operator $V \in B(H)$ is Hilbert-Schmidt if, for any orthonormal basis $\{e_i\}_{i=1}^\infty$, we have*

$$\|V\|_{HS}^2 = \sum_{i=1}^{\infty} \|Ve_i\|^2 < \infty.$$

## 2. Projections and $c$-frames

We start by a result that shows the alternative conditions of being $c$-frame.

**Theorem 2.1** *Let $(\Omega, \mu)$ be a measure space where $\mu$ is $\sigma$-finite. The mapping $f : \Omega \longrightarrow H$ is a $c$-frame for $H$ with bounds $A$ and $B$ if and only if the following conditions hold.*

*(i)* $\{h \in H : \langle h, f(\omega)\rangle = 0, \text{ a.e. } [\mu]\} = \{0\}$.

*(ii) The operator $T$ defined by (1.2) is well defined and*

$$A\|\varphi\|_2^2 \leq \|T\varphi\|^2 \leq B\|\varphi\|_2^2, \quad \varphi \in N_T^\perp. \tag{2.1}$$

**Proof** Let $f : \Omega \longrightarrow H$ be a $c$-frame for $H$. It is clear that

$$\|T\varphi\|^2 \leq B\|\varphi\|_2^2, \quad \varphi \in L^2(\Omega, \mu).$$

If $h \in H$ such that $\langle h, f(\omega)\rangle = 0$, a.e. $[\mu]$, then

$$\int_\Omega |\langle f(\omega), h\rangle|^2 \mathrm{d}\mu(\omega) = 0.$$

Hence $h = 0$. By [11, Theorem 2.9], $R_T = H$, so $R_{T^*}$ is closed and

$$N_T^\perp = \overline{R_{T^*}} = R_{T^*},$$

i.e., $N_T^\perp$ consists of all families of the form $\{\langle h, f(\omega)\rangle\}_{\omega\in\Omega}$, $h \in H$. Now, for given $h \in H$,

$$\begin{aligned}\Big(\int_\Omega |\langle f(\omega), h\rangle|^2 \mathrm{d}\mu(\omega)\Big)^2 &= |\langle Sh, h\rangle|^2 \leq \|Sh\|^2\|h\|^2 \\ &\leq \|Sh\|^2 \frac{1}{A} \int_\Omega |\langle f(\omega), h\rangle|^2 \mathrm{d}\mu(\omega),\end{aligned}$$

where $S$ is the frame operator of $f$. Therefore

$$A\Big(\int_\Omega |\langle f(\omega), h\rangle|^2 \mathrm{d}\mu(\omega)\Big)^2 \leq \|Sh\|^2 = \|T\{\langle h, f(\omega)\rangle\}_{\omega\in\Omega}\|^2, \quad h \in H.$$

Now, we prove the other implication. Since $T$ is bounded below, $R_T$ is closed. By [11, Theorem 2.7], $f : \Omega \longrightarrow H$ is a $c$-Bessel mapping. We have

$$\begin{aligned}\{0\} &= \{h \in H : \langle h, f(\omega)\rangle = 0, \text{ a.e. } [\mu]\} = \{h \in H : (T^*h)(\omega) = 0, \text{ a.e. } [\mu]\}\\ &= \{h \in H : T^*h = 0\}.\end{aligned}$$

So $N_{T^*} = \{0\}$. Hence $H = \{0\}^\perp = N_{T^*}^\perp = \overline{R_T} = R_T$. Let $T^\dagger$ denote the pseudo inverse of $T$. By Lemma 1.6, $T^\dagger T$ is the orthogonal projection onto $N_T^\perp$, and $TT^\dagger$ is the orthogonal projection onto $R_T = H$. Thus for each $\varphi \in L^2(\Omega, \mu)$, the inequality (2.1) implies that

$$A\|T^\dagger T\varphi\|^2 \leq \|TT^\dagger T\varphi\|^2 = \|T\varphi\|^2. \tag{2.2}$$

Since $N_{T^\dagger} = R_T^\perp$, (2.2) gives that $\|T^\dagger\|^2 \leq \frac{1}{A}$. Thus $\|(T^*)^\dagger\|^2 \leq \frac{1}{A}$. But $(T^*)^\dagger T^*$ is the orthogonal projection onto

$$R_{(T^*)^\dagger} = R_{(T^\dagger)^*} = N_{T^\dagger}^\perp = R_T = H,$$

so for all $h \in H$,

$$\|h\|^2 = \|(T^*)^\dagger T^* h\|^2 \leq \frac{1}{A}\|T^*h\|^2 = \frac{1}{A}\int_\Omega |\langle f(\omega), h\rangle|^2 \mathrm{d}\mu(\omega). \quad \square$$

Let $f : \Omega \longrightarrow H$ be a $c$-frame for $H$ and $P : H \longrightarrow K$ be an orthogonal projection. Then $Pf : \Omega \longrightarrow K$ is a $c$-frame for $K = PH$ and $PS^{-1}f$ is a dual of $Pf$, since for each $h, k \in H$

$$\begin{aligned}\langle Ph, Pk\rangle &= \int_\Omega \langle Ph, f(\omega)\rangle\langle S^{-1}f(\omega), Pk\rangle \mathrm{d}\mu(\omega)\\ &= \int_\Omega \langle Ph, Pf(\omega)\rangle\langle PS^{-1}f(\omega), Pk\rangle \mathrm{d}\mu(\omega).\end{aligned}$$

**Theorem 2.2** *Let $f : \Omega \longrightarrow H$ be a c-frame for $H$ and $P : H \longrightarrow K$ be an orthogonal projection and $S$ and $\tilde{S}$ be the frame operators of $f$ and $Pf$, respectively. Then $SP = PS$ if and only if $PS^{-1}f = \tilde{S}^{-1}Pf$.*

**Proof** It is obvious that $SP = PS$ if and only if $S^{-1}P = PS^{-1}$. Let $PS^{-1}f = \tilde{S}^{-1}Pf$. Considering $\tilde{S}^{-1}P$ as an operator in $B(H)$, for each $h, k \in H$, we have

$$\begin{aligned}\langle P\tilde{S}^{-1}h, k\rangle &= \int_\Omega \langle P\tilde{S}^{-1}h, f(\omega)\rangle\langle S^{-1}f(\omega), k\rangle \mathrm{d}\mu(\omega)\\ &= \int_\Omega \langle h, \tilde{S}^{-1}Pf(\omega)\rangle\langle S^{-1}f(\omega), k\rangle \mathrm{d}\mu(\omega)\\ &= \int_\Omega \langle h, PS^{-1}f(\omega)\rangle\langle S^{-1}f(\omega), k\rangle \mathrm{d}\mu(\omega)\\ &= \int_\Omega \langle S^{-1}Ph, f(\omega)\rangle\langle S^{-1}f(\omega), k\rangle \mathrm{d}\mu(\omega)\\ &= \langle S^{-1}Ph, k\rangle,\end{aligned}$$

thus $P\tilde{S}^{-1} = S^{-1}P$. We have $P\tilde{S}^{-1} = PS^{-1}P$ so $S^{-1}P = PS^{-1}P$. By taking adjoint on both sides, we get $PS^{-1} = PS^{-1}P$. Therefore, $S^{-1}P = PS^{-1}$. Conversely, suppose $S^{-1}P = PS^{-1}$.

For each $h, k \in H$, we have

$$\langle Ph, k\rangle = \langle Ph, Pk\rangle = \int_\Omega \langle Ph, \tilde{S}^{-1}Pf(\omega)\rangle\langle Pf(\omega), Pk\rangle \mathrm{d}\mu(\omega),$$

so for each $\nu \in \Omega$ and $k \in H$,

$$\begin{aligned}\langle \tilde{S}^{-1}Pf(\nu), k\rangle &= \langle P\tilde{S}^{-1}Pf(\nu), k\rangle = \langle Pf(\nu), \tilde{S}^{-1}Pk\rangle \\ &= \int_\Omega \langle Pf(\nu), \tilde{S}^{-1}Pf(\omega)\rangle\langle Pf(\omega), \tilde{S}^{-1}Pk\rangle \mathrm{d}\mu(\omega) \\ &= \int_\Omega \langle \tilde{S}^{-1}Pf(\nu), Pf(x)\rangle\langle \tilde{S}^{-1}Pf(\omega), Pk\rangle \mathrm{d}\mu(\omega) \\ &= \int_\Omega \langle P\tilde{S}^{-1}Pf(\nu), f(\omega)\rangle\langle f(\omega), P\tilde{S}^{-1}Pk\rangle \mathrm{d}\mu(\omega) \\ &= \langle P\tilde{S}^{-1}Pf(\nu), SP\tilde{S}^{-1}Pk\rangle.\end{aligned}$$

Therefore for each $\nu \in \Omega$ and $k \in H$,

$$\langle \tilde{S}^{-1}Pf(\nu), k\rangle = \langle \tilde{S}^{-1}PSP\tilde{S}^{-1}Pf(\nu), Pk\rangle = \langle \tilde{S}^{-1}PSP\tilde{S}^{-1}Pf(\nu), k\rangle,$$

$$\tilde{S}^{-1}Pf(\nu) = \tilde{S}^{-1}PSP\tilde{S}^{-1}Pf(\nu).$$

Consequently, for each $\nu \in \Omega$,

$$Pf(\nu) = PSP\tilde{S}^{-1}Pf(\nu) = SP\tilde{S}^{-1}Pf(\nu),$$

this implies that $PS^{-1}f = \tilde{S}^{-1}Pf$. □

**Corollary 2.3** *Let $f : \Omega \longrightarrow H$ be a c-frame for $H$. Then $f$ is a tight c-frame for $H$ if and only if for every orthogonal projection $P \in B(H)$,*

$$PS^{-1}f = \tilde{S}^{-1}Pf,$$

*where $S$ and $\tilde{S}$ are the frame operators of $f$ and $Pf$, respectively.*

**Proof** Let $f$ be a tight $c$-frame for $H$ with bound $A$ $P \in B(H)$ being an orthogonal projection. Therefore $S = AI_H$ and $Pf$ is a tight $c$-frame for $PH$ with bound $A$ and $\tilde{S} = AI_{PH}$. So $S^{-1} = A^{-1}I_H$ and $\tilde{S}^{-1} = A^{-1}I_{PH}$ and we have

$$\tilde{S}^{-1}Pf = A^{-1}I_{PH}Pf = A^{-1}Pf = PA^{-1}f = PA^{-1}I_Hf = PS^{-1}f.$$

Conversely, suppose for every orthogonal projection $P \in B(H)$, $PS^{-1}f = \tilde{S}^{-1}Pf$. By Theorem 2.2, for every orthogonal projection $P \in B(H)$, $PS = SP$, so $SPH = PSH = PH$. Thus for each closed subspace $K \subseteq H$, $SK = K$. Let $\{e_\alpha\}_{\alpha\in I}$ be an orthonormal basis of $H$. For each $h \in H$, consider

$$K_h = \{\lambda h : \lambda \in \mathbb{C}\}.$$

Each $K_h$ is a closed subspace of $H$, so by injectivity of $S$, there exists a unique $\lambda_h$ such that $Sh = \lambda_h h$.

By a simple calculation, for every $\alpha, \beta \in I$, we have

$$\lambda_{e_\alpha} = \lambda_{e_\beta}.$$

Let $\lambda$ be the common value of $\lambda_{e_\alpha}$'s. For each $h \in H$ we have

$$Sh = S\Big(\sum_{\alpha\in\mathfrak{A}} \langle h, e_\alpha\rangle e_\alpha\Big) = \sum_{\alpha\in\mathfrak{A}} \langle h, e_\alpha\rangle \lambda e_\alpha = \lambda h.$$

Therefore $S = \lambda I_H$ and $f$ is a tight $c$-frame for $H$ with bound $\lambda$. □

**Theorem 2.4** *Let $(\Omega, \mu)$ be a measure space and $H$ be a Hilbert space such that $\dim H = \operatorname{card}\Omega$. Fix an orthonormal basis $\{e_\omega\}_{\omega\in\Omega}$ for $H$. Suppose that $P$ and $Q$ are projections in $B(H)$ and let $M = PH$ and $N = QH$. Let $f : \Omega \longrightarrow M$ and $g : \Omega \longrightarrow N$ defined by*

$$f(\omega) = Pe_\omega, \quad g(\omega) = Qe_\omega$$

*be Parseval c-frames for $M$ and $N$, respectively. Then $f$ and $g$ are unitarily equivalent if and only if $P = Q$.*

**Proof** Suppose $f$ and $g$ are unitarily equivalent. Then there is a unitary $U \in B(M, N)$ such that $Uf = g$. This determines a partial isometry $\tilde{U} \in B(H)$ with initial and final spaces $M$ and $N$, respectively, such that $\tilde{U}f = g$. So $\tilde{U}^*\tilde{U} = P$, $\tilde{U}\tilde{U}^* = Q$ and $\tilde{U} = Q\tilde{U}P = Q\tilde{U} = \tilde{U}P$. Note that $\tilde{U}Pe_\omega = Qe_\omega$, $\omega \in \Omega$. Therefore via $\tilde{U}P = \tilde{U}$ we obtain

$$\tilde{U}e_\omega = Qe_\omega, \quad \omega \in \Omega.$$

So $\tilde{U} = Q$ and hence $P = Q$. □

## 3. Bochner integrability and $c$-frames

**Lemma 3.1** *Let $f : \Omega \longrightarrow H$ be a Bochner integrable function and $V \in B(H, K)$. Then*

$$\int_\Omega Vf(\omega)\mathrm{d}\mu(x) = V\int_\Omega f(\omega)\mathrm{d}\mu(\omega).$$

**Proof** Since $f$ is Bochner integrable, there exist a sequence of integrable simple functions $\{f_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty}\int_\Omega \|f_n(\omega) - f(\omega)\|\mathrm{d}\mu(\omega) = 0,$$

and

$$\int_\Omega f(\omega)\mathrm{d}\mu(\omega) = \lim_{n\to\infty}\int_\Omega f_n(\omega)\mathrm{d}\mu(\omega).$$

So

$$V\int_\Omega f(\omega)\mathrm{d}\mu(\omega) = \lim_{n\to\infty} V\int_\Omega f_n(\omega)\mathrm{d}\mu(\omega). \tag{3.1}$$

Now, for each $h \in H$, we have

$$\begin{aligned}\Big|\int_\Omega Vf_n(\omega)\mathrm{d}\mu(\omega) - \int_\Omega Vf(\omega)\mathrm{d}\mu(\omega)\Big| &= \Big|\int_\Omega V(f_n(\omega) - f(\omega))\mathrm{d}\mu(\omega)\Big| \\ \le \int_\Omega \|V(f_n(\omega) - f(\omega))\|\mathrm{d}\mu(\omega) &\le \|V\|\int_\Omega \|f_n(\omega) - f(\omega)\|\mathrm{d}\mu(\omega),\end{aligned}$$

then

$$\lim_{n\to\infty}\int_\Omega Vf_n(\omega)\mathrm{d}\mu(\omega) = \int_\Omega Vf(\omega)\mathrm{d}\mu(\omega). \tag{3.2}$$

Considering $f_n = \sum_{i=1}^{k^{(n)}} x_i^{(n)} \chi_{E_i^{(n)}}$, we have

$$\int_\Omega f_n(\omega)\mathrm{d}\mu(\omega) = \sum_{i=1}^{k^{(n)}} x_i^{(n)} \mu({E_i}^{(n)}),$$

so

$$V\int_\Omega f_n(\omega)\mathrm{d}\mu(\omega) = \sum_{i=1}^{k^{(n)}} \mu({E_i}^{(n)}) V(x_i^{(n)}).$$

Also,

$$Vf_n = \sum_{i=1}^{k^{(n)}} \chi_{{E_i}^{(n)}} V(x_i^{(n)}),$$

therefore

$$\int_\Omega Vf_n(\omega)\mathrm{d}\mu(\omega) = \sum_{i=1}^{k^{(n)}} \mu({E_i}^{(n)}) V(x_i^{(n)}).$$

Thus

$$\lim_{n\to\infty} \int_\Omega Vf_n(\omega)\mathrm{d}\mu(\omega) = \lim_{n\to\infty} V\int_\Omega f_n(\omega)\mathrm{d}\mu(\nu),$$

consequently by (3.1) and (3.2)

$$\int_\Omega Vf(\omega)\mathrm{d}\mu(x) = V\int_\Omega f(\omega)\mathrm{d}\mu(\omega). \quad \square$$

**Corollary 3.2** *Let $f : \Omega \longrightarrow H$ be a Bochner integrable function. Then for each $h \in H$ we have*

$$\int_\Omega \langle f(\omega), h\rangle \mathrm{d}\mu(\omega) = \Big\langle \int_\Omega f(\omega)\mathrm{d}\mu(\omega), h\Big\rangle.$$

**Theorem 3.3** *Let $f : \Omega \longrightarrow H$ be a c-frame for $H$ and $f$ be Bochner integrable. If $T$ and $S$ are synthesis and frame operators of $f$, respectively, then*

$$T\varphi = \int_\Omega \varphi(\omega) f(\omega)\mathrm{d}\mu(\omega), \quad \varphi \in L^2(\Omega,\mu),$$

$$Sh = \int_\Omega \langle h, f(\omega)\rangle f(\omega)\mathrm{d}\mu(\omega), \quad h \in H.$$

**Proof** By Corollary 3.2, for each $\varphi \in L^2(\Omega,\mu)$,

$$\langle T\varphi, h\rangle = \int_\Omega \varphi(\omega)\langle f(\omega), h\rangle \mathrm{d}\mu(\omega) = \Big\langle \int_\Omega \varphi(\omega) f(\omega)\mathrm{d}\mu(\omega), h\Big\rangle,$$

thus

$$T\varphi = \int_\Omega \varphi(\omega) f(\omega)\mathrm{d}\mu(\omega).$$

In a similar manner it can be shown that

$$Sh = \int_\Omega \langle h, f(\omega)\rangle f(\omega)\mathrm{d}\mu(\omega), \quad h \in H. \quad \square$$

**Theorem 3.4** *Let $f, g : \Omega \longrightarrow H$ be c-Bessel mappings and $f$ and $g$ be Bochner integrable. Then the following statements are equivalent.*

*(i) For each $h \in H$, $h = \int_\Omega \langle h, g(\omega)\rangle f(\omega)\mathrm{d}\mu(\omega)$;*

*(ii)* For each $h \in H$, $h = \int_\Omega \langle h, f(\omega)\rangle g(\omega)\mathrm{d}\mu(\omega)$;
*(iii)* For each $h, k \in H$, $\langle h, k\rangle = \int_\Omega \langle h, f(\omega)\rangle\langle g(\omega), k\rangle \mathrm{d}\mu(\omega)$.

**Proof** The proof is similar to discrete case [3, Lemma 5.6.2]. □

## 4. Classifying Hilbert-Schmidt operators by $c$-frames

**Lemma 4.1** *Let $f : \Omega \longrightarrow H$ be c-frame for $H$ with bounds $A, B$ and $\{e_\alpha\}_{\alpha\in I}$ be an orthonormal basis of $H$. Let $V \in B(H)$. Then*

$$A\sum_{\alpha\in I}\|V^*e_\alpha\|^2 \le \int_\Omega \|Vf(\omega)\|^2\mathrm{d}\mu(\omega) \le B\sum_{\alpha\in I}\|V^*e_\alpha\|^2.$$

**Proof** By [14, Theorem 1.27], we have

$$A\sum_{\alpha\in\mathfrak{A}}\|V^*e_\alpha\|^2 \le \sum_{\alpha\in\mathfrak{A}}\int_\Omega |\langle f(\omega), V^*e_\alpha\rangle|^2\mathrm{d}\mu(\omega) = \int_\Omega\sum_{\alpha\in\mathfrak{A}}|\langle Vf(\omega), e_\alpha\rangle|^2\mathrm{d}\mu(\omega)$$
$$= \int_\Omega \|Vf(\omega)\|^2\mathrm{d}\mu(\omega) \le B\sum_{\alpha\in\mathfrak{A}}\|V^*e_\alpha\|^2. \quad \Box$$

**Corollary 4.2** *Let $f : \Omega \longrightarrow H$ be a c-Bessel mapping with Bessel constant $B$ and $\{e_\alpha\}_{\alpha\in I}$ be an orthonormal basis of $H$. If $V \in B(H)$, then*

$$\int_\Omega \|Vf(\omega)\|^2\mathrm{d}\mu(\omega) \le B\sum_{\alpha\in I}\|V^*e_\alpha\|^2.$$

**Theorem 4.3** *An operator $V \in B(H)$ is Hilbert Schmidt if and only if*

$$\int_\Omega \|Vf(\omega)\|^2\mathrm{d}\mu(\omega) < \infty$$

*for one (and therefore for all) c-frame(s) for $H$. Moreover*

$$\sqrt{A}\|V\|_{HS} \le \sqrt{\int_\Omega \|Vf(\omega)\|^2\mathrm{d}\mu(\omega)} \le \sqrt{B}\|V\|_{HS},$$

*in which $A$ and $B$ are c-frame bounds. In particular for tight c-frames (with bound $A$) we have*

$$\|V\|_{HS} = \frac{1}{A}\sqrt{\int_\Omega \|Vf(\omega)\|^2\mathrm{d}\mu(\omega)}.$$

## 5. Some points about parseval $c$-frames

**Lemma 5.1** *Let $f : \Omega \longrightarrow H$ be a c-frame for $H$ with frame operator $S$ and $V \in B(H)$ be an invertible operator such that $V^*Vf = S^{-1}f$. Then $Vf : \Omega \longrightarrow H$ is a Parseval c-frame for $H$.*

**Proof** For each $h, k \in H$, we have

$$\langle h, k\rangle = \int_\Omega \langle h, S^{-1}f(\omega)\rangle\langle f(\omega), k\rangle\mathrm{d}\mu(\omega) = \int_\Omega \langle h, V^*Vf(\omega)\rangle\langle f(\omega), k\rangle\mathrm{d}\mu(\omega),$$

so

$$\langle V^{-1}h, k\rangle = \int_\Omega \langle V^{-1}h, V^*Vf(\omega)\rangle\langle f(\omega), k\rangle\mathrm{d}\mu(\omega) = \int_\Omega \langle h, Vf(\omega)\rangle\langle f(\omega), k\rangle\mathrm{d}\mu(\omega).$$

Then

$$\|h\|^2 = \langle h, h\rangle = \langle V^{-1}h, V^*h\rangle = \int_\Omega \langle h, Vf(\omega)\rangle\langle f(\omega), V^*h\rangle \mathrm{d}\mu(\omega)$$
$$= \int_\Omega \langle h, Vf(\omega)\rangle\langle Vf(\omega), h\rangle \mathrm{d}\mu(\omega) = \int_\Omega |\langle h, Vf(\omega)\rangle|^2 \mathrm{d}\mu(\omega).$$

Thus $Vf$ is a Parseval $c$-frame for $H$. □

**Remark 5.2** Let $f_1, f_2, \dots, f_k$ be $c$-frames for Hilbert spaces $H_1, H_2, \dots, H_k$, respectively. Let $H_1 \oplus H_2 \oplus \cdots \oplus H_k$ be the direct sum of $H_1, H_2, \dots, H_k$. We define

$$f_1 \oplus f_2 \oplus \cdots \oplus f_k : \Omega \longrightarrow H_1 \oplus H_2 \oplus \cdots \oplus H_k$$
$$f_1 \oplus f_2 \oplus \cdots \oplus f_k(\omega) = (f_1(\omega), f_2(\omega), \dots, f_k(\omega)).$$

It is obvious that $f_1 \oplus f_2 \oplus \cdots \oplus f_k$ is weakly measurable.

For each $(h_1, h_2, \dots, h_k) \in H_1 \oplus H_2 \oplus \cdots \oplus H_k$, we have

$$\begin{aligned}
&\int_\Omega \big|\langle (h_1, h_2, \dots, h_k), f_1 \oplus f_2 \oplus \cdots \oplus f_k(\omega)\rangle\big|^2 \mathrm{d}\mu(\omega)\\
&= \int_\Omega \Big|\sum_{i=1}^k \langle h_i, f_i(\omega)\rangle\Big|^2 \mathrm{d}\mu(\omega) \le \int_\Omega \Big(\sum_{i=1}^k |\langle h_i, f_i(\omega)\rangle|\Big)^2 \mathrm{d}\mu(\omega)\\
&\le \int_\Omega \big[2^{k-1}(|\langle h_1, f_1(\omega)\rangle|^2 + |\langle h_2, f_2(\omega)\rangle|^2) +\\
&\quad 2^{k-2}|\langle h_3, f_3(\omega)\rangle|^2 \cdots + 2|\langle h_k, f_k(\omega)\rangle|^2\big] \mathrm{d}\mu(\omega)\\
&\le 2^{k-1}(B_1\|h_1\|^2 + B_2\|h_2\|^2) + 2^{k-2}B_3\|h_3\|^2 + \cdots + 2B_k\|h_k\|^2\\
&\le \max\{2^{k-1}B_1, 2^{k-1}B_2, 2^{k-2}B_3, \dots, 2B_k\}\|(h_1, h_2, \dots, h_k)\|^2.
\end{aligned}$$

So $f_1 \oplus f_2 \oplus \cdots \oplus f_k$ is a $c$-Bessel mapping for $H_1 \oplus H_2 \oplus \cdots \oplus H_k$.

**Theorem 5.3** *(i) If $f$ is a $c$-frame for $H$ and $V \in B(H)$ is a co-isometry, then $Vf$ is a $c$-frame for $H$. Moreover if $f$ is a Parseval $c$-frame for $H$, then $Vf$ is a Parseval $c$-frame for $H$.*

*(ii) Let $f, g$ be Parseval $c$-frames for $H$ and $K$, respectively, and $V \in B(H, K)$ be an operator such that $Vf = g$. Then $V$ is a co-isometry. Moreover if $V$ is invertible, then it is unitary.*

*(iii) If $f : \Omega \longrightarrow H$, $g : \Omega \longrightarrow K$ are Parseval $c$-frames for $H$ such that $f \oplus g$ is a Parseval $c$-frame and if $r : \Omega \longrightarrow M$ is a Parseval $c$-frame which is unitarily equivalent to $g$, then $f \oplus g$ is also a Parseval $c$-frame.*

**Proof** (i) For each $h \in H$,

$$A\|h\|^2 = A\|V^*h\|^2 \le \int_\Omega |\langle V^*h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) \le B\|V^*h\|^2 = B\|h\|^2.$$

(ii) For each $k \in K$,

$$\|V^*k\|^2 = \int_\Omega |\langle V^*k, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) = \int_\Omega |\langle k, Vf(\omega)\rangle|^2 \mathrm{d}\mu(\omega) = \|k\|^2,$$

so $V^*$ is an isometry. It is clear that if $V$ is invertible, then it is unitary.

(iii) Let $U \in B(H, M)$ be a unitary such that $Ug = r$. Then $V = I_H \oplus U$ is a unitary such that $V(f \oplus g) = f \oplus r$. So $f \oplus r$ is a Parseval $c$-frame. □

Let $f : \Omega \longrightarrow H$ be a $c$-Bessel mapping for $H$ and $E \subseteq \Omega$ be measurable. Define the operator $S_E : H \longrightarrow H$ weakly by

$$\langle S_E h, k\rangle = \int_E \langle h, f(\omega)\rangle\langle f(\omega), k\rangle \mathrm{d}\mu(\omega).$$

It is obvious that $S_E$ is well defined. If $f : \Omega \longrightarrow H$ is a $c$-frame for $H$ with frame operator $S$, then $S = S_E + S_{E^c}$.

**Lemma 5.4** *If $T$ and $S$ are two operators on $H$ such that $S + T = I_H$, then $S - T = S^2 - T^2$.*

**Proof** It is an easy calculation. □

**Theorem 5.5** *Let $f : \Omega \longrightarrow H$ be a c-frame for $H$ with canonical dual frame $\tilde{f} = S^{-1}f$. Then for each measurable set $E \subseteq \Omega$ we have*

$$\int_E |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) - \int_\Omega |\langle S_E h, \tilde{f}(\omega)\rangle|^2 \mathrm{d}\mu(\omega)$$
$$= \int_{E^c} |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) - \int_\Omega |\langle S_E h, \tilde{f}(\omega)\rangle|^2 \mathrm{d}\mu(\omega).$$

**Proof** Since $S = S_E + S_{E^c}$, so $I_H = S^{-1}S_E + S^{-1}S_{E^c}$. By using Lemma 5.4 for $S^{-1}S_E$ and $S^{-1}S_{E^c}$, we have

$$S^{-1}S_E - S^{-1}S_E S^{-1}S_E = S^{-1}S_{E^c} - S^{-1}S_{E^c}S^{-1}S_{E^c}. \tag{5.1}$$

Also for each $h, k \in H$,

$$\langle S^{-1}S_E h, k\rangle - \langle S^{-1}S_E S^{-1}S_E h, k\rangle = \langle S_E h, S^{-1}k\rangle - \langle S^{-1}S_E h, S^{-1}S_E k\rangle. \tag{5.2}$$

Now let $k = Sh$. Then the equality (5.2) can be continued as:

$$= \langle S_E h, h\rangle - \langle S^{-1}S_E h, S_E h\rangle = \int_E |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) - \int_\Omega |\langle S_E h, \tilde{f}(\omega)\rangle|^2 \mathrm{d}\mu(\omega).$$

Similarly, we can write the equation (5.2) for $E^c$. Therefore

$$\int_E |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) - \int_\Omega |\langle S_E h, \tilde{f}(\omega)\rangle|^2 \mathrm{d}\mu(\omega)$$
$$= \int_{E^c} |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) - \int_\Omega |\langle S_{E^c} h, \tilde{f}(\omega)\rangle|^2 \mathrm{d}\mu(\omega). \quad \square$$

**Theorem 5.6** *Let $f : \Omega \longrightarrow H$ be a Parseval c-frame for $H$. Then for each measurable set $E \subseteq \Omega$ we have*

$$\int_E |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) - \|S_E h\|^2 = \int_{E^c} |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) - \|S_{E^c} h\|^2.$$

**Proof** By using Theorem 5.5, it is obvious. □

**Proposition 5.7** *Let $f : \Omega \longrightarrow H$ be a Parseval c-frame for $H$. Then for each measurable set*

*$E \subseteq \Omega$, measurable set $F \subseteq E^c$ and each $h \in H$ we have*

$$\|S_{E\cup F}h\|^2 - \|S_{E^c\setminus F}h\|^2 = \|S_E h\|^2 - \|S_{E^c}h\|^2 + 2\int_F |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega).$$

**Proof** Using Theorem 5.6 twice implies the result. □

**Corollary 5.8** *Let $f : \Omega \longrightarrow H$ be a $\lambda$-tight $c$-frame for $H$. Then for each measurable set $E \subseteq \Omega$ we have*

$$\lambda \int_E |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) - \|S_E h\|^2 = \lambda \int_{E^c} |\langle h, f(\omega)\rangle|^2 \mathrm{d}\mu(\omega) - \|S_{E^c}h\|^2.$$

**Proof** Since $\frac{1}{\sqrt{\lambda}}f$ is a Parseval $c$-frame, using Theorem 5.6 yields the result. □

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