A Family of Convexity-Preserving Subdivision Schemes

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Abstract A new five-point binary subdivision scheme with high continuity and convexity preservation is proposed in this paper. It is shown that the limit curves are \( C^k \) \((k = 0, 1, \ldots, 7)\) continuous for the certain range of the parameter. The range of the parameter for the property of convexity preservation of the limit curves is also provided. Experimental results demonstrate the efficiency and flexibility of the scheme.

Keywords Binary subdivision; convergence and smoothness; convexity preservation

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1. Introduction

Subdivision method has become a powerful tool to generate curves and surfaces in many fields such as in CAGD, CG and geometric modeling because of its efficiency and simplicity. Many classical curve and surface subdivision schemes have been proposed. Dyn et al. [1] gave a 4-point interpolatory subdivision scheme with a tension parameter, which can generate only \( C^1 \) continuous curves. But this scheme had the convexity preserving property and approximation order two, which were examined by Dyn et al. [2]. Weissman [3] proposed a 6-point binary interpolatory subdivision scheme in his master thesis. Siddiqi and Ahmad [4] proved the scheme to be \( C^2 \) by using the Laurent polynomial. Hassan et al. [5] provided a 4-point ternary interpolatory subdivision scheme with a tension parameter and proved the resulting curves to be \( C^2 \) for a certain range of tension parameter. Cai [6] discussed the convexity preserving property of this subdivision scheme. All the above are the subdivision schemes with even number of control points. Hassan and Dodgson [7] investigated schemes with an odd number of control points, specifically 3-point schemes, which included a binary 3-point approximating scheme, a ternary 3-point approximating scheme and a ternary 3-point interpolatory scheme. Siddiqi and Ahmad [8] presented a binary 3-point approximating scheme using quadratic B-spline basis function. Siddiqi and Rehan [9] introduced a ternary 3-point approximating scheme which generated \( C^2 \) limit curve and its limit function had a support on [3, 2]. In [10] they gave modified form of binary and ternary 3-point subdivision schemes presented in [8,9]. For 5-point scheme, Siddiqi and Ahmad [11] introduced a new scheme which can generate \( C^4 \) limit curves. Mustafa [12]...

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gave a new five-point scheme which can generate $C^6$ limit curves. Cao and Tan [13] presented a new five-point scheme with up to $C^5$ continuity based on the classical four-point interpolating subdivision scheme. Tan et al. [14] rewrote the scheme in [13] and discussed the conditions for the scheme to be convexity preserving. Zheng et al. [15] proposed a general formula which can obtain a five-point scheme for generating $C^7$ limit curves. Mustafa et al. got a new 5-point symmetric binary approximating scheme with free parameter in [16]. And in [17], Mustafa et al. presented a general algorithm to generate a new class of binary approximating subdivision schemes.

Generally, $C^2$ continuity is enough in industrial design, but the design of precision instruments in manufacturing industry requires higher continuous curves and surfaces. Based on these, we present a new five-point binary subdivision scheme with high continuity and convexity preservation in this paper. The rest of the paper is organized as follows. In Section 2, we introduce the preliminaries. In Section 3, a new five-point binary subdivision scheme is introduced. The conditions of the convergence and smoothness of the limit curve are discussed in Section 4. In Section 5, the conditions of convexity preservation of the limit curve are presented. In Section 6, some numerical examples are given to illustrate the effectiveness of our scheme. Finally, we conclude the paper in Section 7.

2. Preliminaries

Given a set of initial control points, let $P^k = \{p_i^k\}_{i \in \mathbb{Z}}$ be the set of control points at level $k$ $(k \geq 0, k \in \mathbb{Z})$. Define $\{p_i^{k+1}\}_{i \in \mathbb{Z}}$ recursively by the following binary subdivision rules:

$$p_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{2j-i} p_j^k, \quad i \in \mathbb{Z}, \quad \text{(1)}$$

where the finite set $a = \{a_i\}_{i \in \mathbb{Z}}$ is called the mask. The symbol of the scheme is defined by $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i$.

**Theorem 2.1** ([18]) Let a binary subdivision scheme $S$ be convergent. Then the mask $a(z) = \sum_{i \in \mathbb{Z}} a_i z^i$ satisfies

$$\sum_{i \in \mathbb{Z}} a_{2i} = \sum_{i \in \mathbb{Z}} a_{2i+1} = 1. \quad \text{(2)}$$

**Theorem 2.2** ([18]) Let a subdivision scheme $S$ with a mask $a = \{a_i\}_{i \in \mathbb{Z}}$ satisfy (2). Then there exists a subdivision scheme $S_1$ (first order difference scheme of $S$) which satisfies the property

$$dP^k = S_1 dP^{k-1},$$

where $P^k = S_k P^0$, $dP^k = \{(dp^k)_i = 2^k(p_{i+1}^k - p_i^k)|i \in \mathbb{Z}\}$. The symbol of $S_1$ is $a^{(1)}(z) = \frac{2z}{1+z} a(z)$. Generally, if $S_n$ exists and $S_n$ is the nth order difference scheme of $S$ with a mask $a^{(n)} = \{a_i^{(n)}\}_{i \in \mathbb{Z}}$, then the symbol of $S_n$ is $a^{(n)}(z) = \sum_{i \in \mathbb{Z}} a_i^{(n)} z^i = (\frac{2z}{1+z})^n a(z)$.

**Theorem 2.3** ([18]) Let a subdivision scheme $S$ have a mask $a^{(0)} = \{a_i^{(0)}\}_{i \in \mathbb{Z}}$, and its jth order difference scheme $S_j$ $(j = 1, 2, \ldots, n+1)$ exist and have the mask $a^{(j)} = \{a_i^{(j)}\}_{i \in \mathbb{Z}}$ satisfying

$$\sum_{i \in \mathbb{Z}} a_{2i}^{(j)} = \sum_{i \in \mathbb{Z}} a_{2i+1}^{(j+1)} = 1, \quad j = 0, 1, \ldots, n. \quad \text{(3)}$$
If there exists an integer \( L \geq 1 \), such that \( \| (\frac{1}{2} S_{n+1})^L \|_\infty < 1 \), then the subdivision scheme \( S \) is \( C^n \) continuous, where
\[
\| (\frac{1}{2} S_{n+1})^L \|_\infty = \max_{i \in \mathbb{Z}} \{ \sum_{p} \| b_i(z) \|_{2^L} : 0 \leq i < 2^L \},
\]
\[
b^{[L]}(z) = b(z)b(z^2) \cdots b(z^{2^{L-1}}), \quad b(z) = \frac{1}{2} a^{(n+1)}(z).
\]
Especially, when \( L = 1 \), \( \| \frac{1}{2} S_{n+1} \|_\infty = \frac{1}{2} \max \{ \sum_{i \in \mathbb{Z}} |a_i^{(n+1)}|, \sum_{i \in \mathbb{Z}} |a_{i+1}^{(n+1)}| \}. \)

### 3. A new five-point binary subdivision scheme

The new five-point binary subdivision scheme is defined as
\[
\begin{align*}
p^{k+1}_{2i} &= \frac{1}{256} \left( (9 - 7\mu)p_{i-2}^k + (84 - 28\mu)p_{i-1}^k + (126 + 14\mu)p_i^k + (36 + 20\mu)p_{i+1}^k + (1 + \mu)p_{i+2}^k, \right. \\
p^{k+1}_{2i+1} &= \frac{1}{256} \left( (1 - \mu)p_{i-2}^k + (36 - 20\mu)p_{i-1}^k + (126 - 14\mu)p_i^k + (84 + 28\mu)p_{i+1}^k + (9 + 7\mu)p_{i+2}^k \right)
\end{align*}
\]
where \( \mu \in \mathbb{R} \) is a tension parameter. For \( \mu = 0 \), we can get the scheme
\[
\begin{align*}
p^{k+1}_{2i} &= \frac{9}{256} p_{i-2}^k + \frac{21}{64} p_{i-1}^k + \frac{63}{128} p_i^k + \frac{9}{64} p_{i+1}^k + \frac{1}{256} p_{i+2}^k, \\
p^{k+1}_{2i+1} &= \frac{1}{256} p_{i-2}^k + \frac{1}{64} p_{i-1}^k + \frac{63}{128} p_i^k + \frac{21}{64} p_{i+1}^k + \frac{9}{256} p_{i+2}^k.
\end{align*}
\]
It is actually the eighth-power uniform \( B \)-spline curves. When \( \mu = 1 \), the five-point binary approximating scheme in [14] can be obtained.
\[
\begin{align*}
p^{k+1}_{2i} &= \frac{1}{128} p_{i-2}^k + \frac{7}{32} p_{i-1}^k + \frac{35}{64} p_i^k + \frac{7}{32} p_{i+1}^k + \frac{1}{128} p_{i+2}^k, \\
p^{k+1}_{2i+1} &= \frac{7}{16} p_{i-1}^k + \frac{7}{16} p_{i+1}^k + \frac{1}{16} p_{i+2}^k.
\end{align*}
\]
Actually, the scheme (4) can be seen as the combination of the scheme (5) and (6).

### 4. Convergence and smoothness

In this section, the conditions of the convergence and smoothness of the limit curve generated by the subdivision scheme (4) are given and proved according to Theorem 2.3.

**Theorem 4.1** The limit curves generated by the subdivision scheme (4) are \( C^0 \) continuous in the range \(-6.2 < \mu < 6.2\), \( C^1 \) continuous in the range \(-5.4 < \mu < 5.4\), \( C^2 \) continuous in the range \(-5.3 < \mu < 5.3\), and \( C^3 \) continuous in the range \(-\frac{13}{4} < \mu < \frac{13}{4}\), \( C^4 \) continuous in the range \(-4 < \mu < 4\), \( C^5 \) continuous in the range \(-3 < \mu < 3\), \( C^6 \) continuous in the range \(-2 < \mu < 2\), and \( C^7 \) continuous when \( \mu = 0 \).

**Proof** The generating polynomial for the mask of the subdivision scheme (4) can be written as:
\[
a(z) = \frac{1}{256} \left( 1 - \mu \right) z^{-5} + (9 - 7\mu) z^{-4} + (36 - 20\mu) z^{-3} + (84 - 28\mu) z^{-2} + (126 - 14\mu) z^{-1} + (126 + 14\mu) + (84 + 28\mu) z + (36 + 20\mu) z^2 + (9 + 7\mu) z^3 + (1 + \mu) z^4 \right).
\]
It is easy to verify that \( a(z) \) satisfies Eq. (2). According to Theorem 2.2, we have the generating polynomials for \( S_j \) \((j = 1, 2, \ldots, 8)\) as follows:

\[
\begin{align*}
  a^{(1)}(z) &= \frac{1}{128}(1-\mu)z^{-4} + (8 - 6\mu)z^{-3} + (28 - 14\mu)z^{-2} + (56 - 14\mu)z^{-1} + 70 + (56 + 14\mu)z + (28 + 14\mu)z^2 + (8 + 6\mu)z^3 + (1 + \mu)z^4, \\
  a^{(2)}(z) &= \frac{1}{64}[(1-\mu)z^{-3} + (7 - 5\mu)z^{-2} + (21 - 9\mu)z^{-1} + (35 - 5\mu) + (35 + 5\mu)z + (21 + 9\mu)z^2 + (7 + 5\mu)z^3 + (1 + \mu)z^4], \\
  a^{(3)}(z) &= \frac{1}{32}[(1-\mu)z^{-2} + (6 - 4\mu)z^{-1} + (15 - 5\mu) + 20z + (15 + 5\mu)z^2 + (6 + 4\mu)z^3 + (1 + \mu)z^4], \\
  a^{(4)}(z) &= \frac{1}{16}[(1-\mu)z^{-1} + (5 - 3\mu) + (10 - 2\mu)z + (10 + 2\mu)z^2 + (5 + 3\mu)z^3 + (1 + \mu)z^4], \\
  a^{(5)}(z) &= \frac{1}{8}[(1-\mu) + (4 - 2\mu)z + 6z^2 + (4 + 2\mu)z^3 + (1 + \mu)z^4], \\
  a^{(6)}(z) &= \frac{1}{4}[(1-\mu)z + (3 - \mu)z^2 + (3 + \mu)z^3 + (1 + \mu)z^4], \\
  a^{(7)}(z) &= \frac{1}{2}[(1-\mu)z^2 + 2z^3 + (1 + \mu)z^4], \\
  a^{(8)}(z) &= z^3 + z^4.
\end{align*}
\]

It is easy to verify that \( a^j(z) \) \((j = 1, 2, \ldots, 8)\) satisfy Eq. (2). And when \(-6.2 < \mu < 6.2\), we have

\[
\| \frac{1}{2} S_1 \|_\infty = \frac{1}{256} \max \{ |1-\mu| + |28 - 14\mu| + 70 + |28 + 14\mu| + |1 + \mu|, |8 - 6\mu| + |56 - 14\mu| + |56 + 14\mu| + |8 + 6\mu| \} < 1,
\]

when \(-5.4 < \mu < 5.4\), we have

\[
\| \frac{1}{2} S_2 \|_\infty = \frac{1}{128} \max \{ |1-\mu| + |21 - 9\mu| + |35 + 5\mu| + |7 + 5\mu|, |7 - 5\mu| + |35 - 5\mu| + |21 + 9\mu| + |1 + \mu| \} < 1,
\]

when \(-5.3 < \mu < 5.3\), we have

\[
\| \frac{1}{2} S_3 \|_\infty = \frac{1}{64} \max \{ |1-\mu| + |15 - 5\mu| + |15 + 5\mu| + |1 + \mu|, |6 - 4\mu| + 20 + |6 + 4\mu| \} < 1,
\]

when \(-\frac{13}{3} < \mu < \frac{13}{3}\), we have

\[
\| \frac{1}{2} S_4 \|_\infty = \frac{1}{32} \max \{ |1-\mu| + |10 - 2\mu| + |5 + 3\mu|, |5 - 3\mu| + |10 + 2\mu| + |1 + \mu| \} < 1,
\]

when \(-4 < \mu < 4\), we have

\[
\| \frac{1}{2} S_5 \|_\infty = \frac{1}{16} \max \{ |1-\mu| + 6 + |1 + \mu|, |4 - 2\mu| + |4 + 2\mu| \} < 1,
\]

when \(-3 < \mu < 3\), we have

\[
\| \frac{1}{2} S_6 \|_\infty = \frac{1}{8} \max \{ |1-\mu| + |3 + \mu|, |3 - \mu| + |1 + \mu| \} < 1,
\]

when \(-2 < \mu < 2\), we have

\[
\| \frac{1}{2} S_7 \|_\infty = \frac{1}{4} \max \{ |1-\mu| + |1 + \mu|, 2 \} < 1,
\]

when \(\mu > 2\), we have

\[
\| \frac{1}{2} S_8 \|_\infty = \frac{1}{2} \max \{ |1-\mu|, 4 \} < 1.
\]
Suppose $d_0$ and $d_1$ are strictly convex, i.e., $d_i > 0$, for all $i$. When $d_i = 0$, we have

$$\|\frac{1}{2}S_n\|_{\infty} = \frac{1}{2}\max\{1, 1\} < 1.$$ 

Consequently, it follows from Theorem 2.3 that the proof of Theorem 4.1 is completed. □

5. Convexity preservation

Shape preservation of subdivision curve is an important property in geometric design. In this section, we will prove the convexity preservation of the subdivision scheme (4).

Given a set of initial control points $\{p_i^0\}_{i \in \mathbb{Z}}$, $p_i^0 = (x_i^0, f_i^0)$, which are strictly convex, where $\{x_i^0\}_{i \in \mathbb{Z}}$ are equidistant points. For simplicity, let $\Delta x_i^0 = x_i^0 - x_i^0 = 1$. We have $\Delta x_i^{k+1} = x_i^{k+1} - x_i^{k+1} = \frac{1}{2}\Delta x_i^k = \frac{\mu}{2^k}$ and let

$$d_i^k = f[x_i^{k-1}, x_i^{k}, x_i^{k+1}] = 2^{2k-1}(f_i^{k-1} - 2f_i^k + f_i^{k+1})$$

denote the second order divided differences.

**Theorem 5.1** Suppose the initial control points $\{p_i^0\}_{i \in \mathbb{Z}}$, $p_i^0 = (x_i^0, f_i^0)$ are strictly convex, i.e., $d_i^k > 0$, for all $i$. Then the subdivision scheme (4) is directly convex when $-1 \leq \mu \leq 1$.

**Proof** Using the subdivision scheme (4), we can get

$$d_i^{k+1} = 2^{2k+1}(f_i^{k+1} - 2f_i^k + f_i^{k+1})$$

$$= \frac{1}{64}[(1 - \mu)d_{i-2}^k + (21 - 9\mu)d_{i-1}^k + (35 + 5\mu)d_i^k + (7 + 5\mu)d_{i+1}^k].$$

(7)

$$d_i^{k+1} = 2^{2k+1}(f_i^{k+1} - 2f_i^k + f_i^{k+1})$$

$$= \frac{1}{64}[(7 - 5\mu)d_{i-2}^k + (35 - 5\mu)d_{i-1}^k + (21 + 9\mu)d_i^k + (1 + \mu)d_{i+1}^k].$$

(8)

Suppose $d_i^k > 0$ for some $k \geq 0$, $i \in \mathbb{Z}$. Then when $-1 \leq \mu \leq 1$, from (7) and (8) it follows $d_i^{k+1} > 0$ and $d_i^{k+1} > 0$. Therefore $d_i^k > 0$, for all $i \geq 0$, $i \in \mathbb{Z}$.

**Remark 5.2** When $-1 \leq \mu \leq 1$, the scheme (4) generates a convexity preserving $C^6$ limit curve; when $\mu = 0$, the scheme (4) generates a convexity preserving $C^7$ limit curve.

6. Numerical examples

In this section, four examples are depicted in Figures 1–4 to illustrate our scheme.

**Example 6.1** Chinese character in a fancy style generated by scheme (4) is shown in Figure 1 under the condition of $\mu = 1$. In the figure, the blue dashed lines indicate initial control polygons, and the red solid lines indicate continuous curves generated by subdivision schemes after three subdivision steps.

**Example 6.2** Figure 2 depicts the limit curves after five times refinement based on the scheme (4) under the condition of $\mu = 0$, 1, 2, 3, 4, 5, which can achieve the $C^k$ ($k = 7, 6, 5, 4, 3, 2$) continuities respectively. It indicates that the subdivision scheme (4) has great flexibility for curves’ generation. Figure 2 corresponds to the given initial control points $((0, 0), (0, 2), (2, 2), (2, 0))$ marked by “o”. The initial control polygon is square in the figure.
Example 6.3  As shown in Figure 3 and Table 1, we compare our scheme with the existing 5-point subdivision schemes presented in [11–15]. In Figure 3 the red curve is generated by our proposed scheme by taking $\mu = 2.5$ whose limit curve is $C^5$, the curves generated by the scheme in [11–15] are marked in green, mauve, cyan and black, respectively, which can get the $C^k$ ($k = 4, 6, 4, 6$) continuities. In the figure, the initial control points are marked by “⋄”, blue dashed lines indicate initial control polygons, and solid lines indicate continuous curves generated by subdivision schemes after six subdivision steps. We list the differences between our scheme with those in [11–15] in Table 1.

Example 6.4  The convexity preservation of the subdivision scheme (4) is well demonstrated in Figure 4. In the figure, the initial control points are marked by “∗”, the initial control polygons are shown by the blue dashed lines, and the limit curves after five iterations are shown by the red solid curves.

7. Conclusions

In this paper, a new five-point binary subdivision scheme is presented and analyzed. We discussed the uniform convergence and continuity of this subdivision scheme by using generating polynomial method. The examples illustrate that our proposed scheme gives great flexibility to geometric designers by choosing appropriate parameters. Our other work is aimed at the study of
the conditions of the scheme’s convexity preservation. Extending the curves subdivision scheme to surfaces is our further work.

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Table 1. Comparison of proposed existing 5-point schemes

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References