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2-Local Superderivations on Basic Classical Lie Superalgebras

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Abstract Let $\mathbb{F}$ be an algebraically closed field of characteristic zero, and L be a basic classical Lie superalgebra except $A(n,n)$ over $\mathbb{F}$. In this paper, we prove that every 2-local superderivation on L is a superderivation. Furthermore, we give an example to show that a subalgebra of $\mathrm{spl}(2,2)$ admits a 2-local superderivation which is not a superderivation.

Keywords basic classical Lie superalgebras; 2-local superderivation; superderivation

MR(2010) Subject Classification 16W25; 17B20; 17B40

1. Introduction

The definition of 2-local derivation on the algebra was introduced by Šemerl [1]. In the reference [1], the author showed that each 2-local derivation on $\mathcal{B}(H)$ is a derivation, where $\mathcal{B}(H)$ is the algebra of all linear bounded operators on H. Similarly, some authors started to describe 2-local derivations on the different associative algebras such as semi-finite von Neumann algebras, matrix algebras over commutative regular algebras [2–6]. In 2015, the authors of the reference [7] investigated the 2-local derivations on a finite-dimensional semi-simple Lie algebra L over an algebraically closed field of characteristic zero. They proved that every 2-local derivation on L is a derivation and that a finite-dimensional nilpotent Lie algebra L with $\dim L > 1$ admits a 2-local derivation which is not a derivation. The reference [8] gave the definition of a 2-local superderivation on the associative superalgebra and proved that every 2-local superderivation on the associative superalgebra $M_n(\mathbb{C})$ is a superderivation. In this paper, we introduce the notion of 2-local superderivation on Lie superalgebra and prove that all 2-local superderivations on the basic classical Lie superalgebras are superderivation. Furthermore, we give an example to show that a subalgebra of $\mathrm{spl}(2,2)$ admits a 2-local superderivation which is not a superderivation.

In this paper, the algebras and vector spaces are finite-dimensional over an algebraically closed field $\mathbb{F}$ of characteristic zero. Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra. If H is a Cartan subalgebra of the Lie algebra $L_{\bar{0}}$, then we have the root space decomposition of L with respect to H. Let λ be any linear form on H and $L^\lambda = \{x \in L | \mathrm{ad}_L h(x) = \lambda(h)x, \forall h \in H\}$. Then $L = \bigoplus_{\lambda \in H^*} L^\lambda$. Let $\Delta_{\bar{0}} = \{\lambda \in H^* | \lambda \neq 0, L_{\bar{0}}^\lambda \neq \{0\}\}$, $\Delta_{\bar{1}} = \{\lambda \in H^* | L_{\bar{1}}^\lambda \neq \{0\}\}$. Then $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ is the set of roots of L with respect to H.

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Let $\mathbb{Z}$ be the integers set and $\mathbb{Z}_2$ the residue class modulo 2. The two elements of $\mathbb{Z}_2$ will be denoted by $\bar{0}$ and $\bar{1}$. Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a Lie superalgebra. A map $T : L \to L$ is called homogeneous of degree $\alpha, \alpha \in \mathbb{Z}_2$ if $T(L_\beta) \subseteq L_{\alpha+\beta}$ for all $\beta \in \mathbb{Z}_2$. Let $\mathrm{D}_\alpha(L), \alpha \in \mathbb{Z}_2$, be the subspace of all the homogeneous linear mapping δ of degree α of L such that

$$\delta([x, y]) = [\delta(x), y] + (-1)^{\alpha\beta}[x, \delta(y)], \quad \text{for all} \quad x \in L_\beta, y \in L, \beta \in \mathbb{Z}_2.$$

Define $\mathrm{D}(L) = \mathrm{D}_{\bar{0}}(L) \oplus \mathrm{D}_{\bar{1}}(L)$. The elements of $\mathrm{D}(L)$ are called superderivations of L. $\mathrm{D}(L)$ is called the Lie superalgebra of superderivations of L. For $a \in L$, the linear mapping $\mathrm{ad}_a : L \to L$ such that $\mathrm{ad}_a(b) = [a, b]$ for all $b \in L$ is a superderivation which is called inner.

Definition 1.1 *A homogeneous map $T : L \to L$ of degree α is called a 2-local homogeneous superderivation of degree α if for any two elements $x, y \in L$ there exists a superderivation $\delta_{x,y} : L \to L$(depending on x, y) such that $T(x) = \delta_{x,y}(x)$ and $T(y) = \delta_{x,y}(y)$.*

Let TD_α be the set of all 2-local homogeneous superderivations of degree α. The elements of $\mathrm{TD} = \mathrm{TD}_{\bar{0}} \oplus \mathrm{TD}_{\bar{1}}$ are called 2-local superderivations on L.

Obviously, ad_a for all $a \in L$ is a 2-local superderivations on L and the sum of two 2-local superderivations is also a 2-local superderivations on L.

In this paper, we will prove that 2-local superderivations on basic classical Lie superalgebras are superderivation.

2. Some results on basic classical Lie superalgebras

In 1977, Kac gave the classification of simple Lie superalgebras over an algebraically closed field of characteristic zero.

Definition 2.1 ([9]) *The simple Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is called classical if the representation of $L_{\bar{0}}$ on $L_{\bar{1}}$ is completely reducible; otherwise, Cartan type.*

Theorem 2.2 ([9]) *The classical Lie superalgebras consist of basic classical Lie superalgebras and two series $P(n)$ and $Q(n)$.*

The basic classical Lie superalgebras include:

(a) simple Lie algebras;

(b) simple Lie superalgebras of type

$A(m, n)$	$n, m \geq 0; m + n \geq 1$
$B(m, n)$	$m \geq 0, n \geq 1$
$C(n)$	$n \geq 3$
$D(m, , n)$	$m \geq 2, n \geq 1$
$D(2, 1, a)$	$a \neq 0, -1$
$G(3)$	
$F(4)$	

Table 1 The basic classical simple Lie superalgebras

Definition 2.3 ([10]) *For a bilinear form $f : L \times L \to \mathbb{F}$ we say that*

(1) f *is even if* $(L_\alpha, L_\beta) = 0$ *for* $\alpha \neq \beta$,

(2) f *is supersymmetric if* $(x, y) = (-1)^{\alpha\beta}(y, x)$,

(3) f *is invariant if* $([x, y], z) = (x, [y, z])$.

If L is a basic classical Lie superalgebra, then there exists a non-degenerate even supersymmetric invariant bilinear form on L.

Proposition 2.4 ([11]) *Let L be one of the basic classical Lie superalgebras listed above. Suppose that L is not equal to the algebras* $\mathrm{spl}(2, 2)/\mathbb{F} \cdot \mathrm{I}_4$. *We consider the roots and the root space decomposition of L with respect to some Cartan subalgebra H of $L_{\bar{0}}$.*

(1) $\dim L^\lambda = 1$ *for every* $\lambda \in \Delta$.

(2) $0 \notin \Delta_{\bar{1}}$ *and* $\Delta_{\bar{0}} \cap \Delta_{\bar{1}} = \emptyset$.

(3) Let $\alpha, \beta \in \mathbb{Z}_2$. *If* $\lambda \in \Delta_\alpha$, $\mu \in \Delta_\beta$, *and* $\lambda + \mu \in \Delta_{\alpha+\beta}$, *then*

$$[L^\lambda_\alpha, L^\mu_\beta] = L^{\lambda+\mu}_{\alpha+\beta}.$$

(4) $-\Delta_\alpha = \Delta_\alpha$ *for* $\alpha \in \mathbb{Z}_2$.

(5) We consider two roots λ and μ of L which are proportional:

$$\mu = r\lambda \text{ with some } r \in \mathbb{F}.$$

If λ, μ are both even or both odd, then $r = \pm 1$; if λ is odd and μ is even, then $r = \pm 2$.

(6) There exists a simple root system **B** *such that any root is a linear combination of simple roots with integer coefficients.*

Proposition 2.5 ([10]) *The basic classical Lie superalgebras except for $A(n, n)$ do not have any outer superderivations.*

In next section, we use the notation $\Re$ to represent the Lie superalgebras in Proposition 2.5.

3. 2-Local superderivations on Lie superalgebra $\Re$

The main result of this section is given as follows.

Theorem 3.1 *All 2-local superderivations on $\Re$ are superderivations.*

Since any superderivation of $\Re$ is inner, it follows that for such algebras the definition of 2-local homogeneous superderivation is reformulated as follows. A homogeneous map $T : \Re \to \Re$ of degree α is called a 2-local homogeneous superderivation of degree α if for any two elements $x, y \in \Re$ there exists an element $a_{x,y} \in \Re$ (depending on x, y) such that $T(x) = [a_{x,y}, x]$ and $T(y) = [a_{x,y}, y]$. If x and y are all homogeneous elements of $\Re$, then we can choose a homogeneous element $a_{x,y}$ of degree α. In the case, $a_{x,y}$ will be denoted by $t_{x,y}$.

Let H be a Cartan subalgebra of $\Re$. Then the root space decomposition of $\Re$ with respect to H is $\Re = H \bigoplus \bigoplus_{\lambda\in\Delta} \Re^\lambda$. By Proposition 2.4 for each $\lambda \in \Delta, \dim\Re^\lambda = 1$. Thus we can take a non zero element $E^\lambda \in \Re^\lambda$. Note that every element $x \in \Re$ has a unique decomposition of the form:

$$x = h + \sum_{\lambda \in \Delta} k^{\lambda} E^{\lambda}, \tag{3.1}$$

where $h \in H, k^{\lambda} \in \mathbb{F}$.

By the definition of the root subspaces it follows that

$$[h, E^{\lambda}] = \lambda(h) E^{\lambda} \text{ for all } h \in H, \lambda \in \Delta,$$

and $\Delta_{\bar{0}} \bigcap \Delta_{\bar{1}} = \emptyset$ implies that $E^{\lambda} \in L_{\alpha}^{\lambda}, \alpha \in \mathbb{Z}_2$.

Proposition 3.2 *If T is a 2-local superderivation on $\Re$, then T is linear.*

Proof We proceed in steps. Let $T \in \mathrm{TD}_{\alpha}, x \in \Re_{\beta}, y \in \Re_{\gamma}, z \in \Re_{\mu}, \alpha, \beta, \gamma, \mu \in \mathbb{Z}_2$. Suppose that $\mathrm{b}(\ ,\ )$ is a non-degenerate even supersymmetric invariant bilinear form on $\Re$.

(i) $\mathrm{b}(T(x), y) = -(-1)^{\alpha\beta} \mathrm{b}(x, T(y))$.

$$\begin{aligned} \mathrm{b}(T(x), y) =& \mathrm{b}([t_{x,y}, x], y) = -(-1)^{\alpha\beta} \mathrm{b}([x, t_{x,y}], y) \\ =& -(-1)^{\alpha\beta} \mathrm{b}(x, [t_{x,y}, y]) = -(-1)^{\alpha\beta} \mathrm{b}(x, T(y)) \end{aligned}$$

(ii) If $\beta = \gamma$, then

$$\begin{aligned} \mathrm{b}(T(x+y), z) =& -(-1)^{\alpha\beta} \mathrm{b}(x+y, T(z)) = -(-1)^{\alpha\beta} [\mathrm{b}(x, T(z)) + \mathrm{b}(y, T(z))] \\ =& \mathrm{b}(T(x), z) + \mathrm{b}(T(y), z) = \mathrm{b}(T(x) + T(y), z) \end{aligned}$$

(iii) If $\beta = \mu = \bar{0}$, $\gamma = \bar{1}$, then

$$\begin{aligned} \mathrm{b}(T(x+y), z) =& \mathrm{b}([a_0 + a_1, x+y], z) = \mathrm{b}([a_0, x], z) + \mathrm{b}([a_1, y], z) \\ =& -\mathrm{b}([x, a_0], z) + \mathrm{b}([y, a_1], z) = -\mathrm{b}(x, [a_0, z]) + \mathrm{b}(y, [a_1, z]) \\ =& -\mathrm{b}(x, [a_0 + a_1, z]) + \mathrm{b}(y, [a_1 + a_0, z]) = -\mathrm{b}(x, T(z)) + \mathrm{b}(y, T(z)) \\ =& \mathrm{b}(T(x), z) - (-1)^{\alpha} \mathrm{b}(T(y), z) = \mathrm{b}(T(x) + T(y), z) \end{aligned}$$

where $a_{x+y,z} = a_0 + a_1, a_0 \in R_{\bar{0}}, a_1 \in \Re_{\bar{1}}$.

(iv) If $\beta = \bar{0}$, $\gamma = \mu = \bar{1}$, then

$$\begin{aligned} \mathrm{b}(T(x+y), z) =& \mathrm{b}([a_0 + a_1, x+y], z) = \mathrm{b}([a_1, x], z) + \mathrm{b}([a_0, y], z) \\ =& -\mathrm{b}([x, a_1], z) - \mathrm{b}([y, a_0], z) = -\mathrm{b}(x, [a_1, z]) - \mathrm{b}(y, [a_0, z]) \\ =& -\mathrm{b}(x, [a_0 + a_1, z]) - \mathrm{b}(y, [a_0 + a_1, z]) = -\mathrm{b}(x, T(z)) - \mathrm{b}(y, T(z)) \\ =& \mathrm{b}(T(x), z) + (-1)^{\alpha} \mathrm{b}(T(y), z) = \mathrm{b}(T(x) + T(y), z) \end{aligned}$$

where $a_{x+y,z} = a_0 + a_1, a_0 \in \Re_{\bar{0}}, a_1 \in \Re_{\bar{1}}$.

(v) By (ii), (iii) and (iv), we get $\mathrm{b}(T(x+y), z) = \mathrm{b}(T(x) + T(y), z)$ for all $z \in \Re$. Because the form $\mathrm{b}(\ ,\ )$ is non-degenerate, we have

$$T(x+y) = T(x) + T(y) \quad \text{for all } x, y \in \Re.$$

(vi) Finally,

$$T(\lambda x) = [a_{\lambda x, x}, \lambda x] = \lambda [a_{\lambda x, x}, x] = \lambda T(x).$$

Hence T is linear. □

Lemma 3.3 *There exists an element $h_0 \in H$ such that $\lambda(h_0) \neq 0$ for all $\lambda \in \Delta$.*

Proof Let $\mathbf{B} = \{\lambda_1, \lambda_2, \dots, \lambda_l\}$ be the simple roots system of $\Re$, $\{h_1, h_2, \dots, h_l\}$ a basis in H which is dual to $\{\lambda_1, \lambda_2, \dots, \lambda_l\}$, i.e., $\lambda_i(h_j) = \delta_{ij}$ for all $i, j = 1, 2, \dots, l$. Set $h_0 = \sum_{k=1}^{l} t_0^k h_k$, where t_0 is a fixed algebraic number from $\mathbb{F}$ of degree bigger than $l = \dim H$. Let us take an arbitrary $\lambda \in \Delta$. There exist integers $r_1, r_2, \dots, r_l$ such that $\lambda = \sum_{k=1}^{l} r_k \lambda_k$. Then

$$\begin{aligned}\lambda(h_0) &= \sum_{k=1}^{l} r_k \lambda_k(h_0) = \sum_{k=1}^{l} r_k \lambda_k\Big(\sum_{s=1}^{l} t_0^s h_s\Big)\\ &= \sum_{k=1}^{l}\sum_{s=1}^{l} r_k t_0^s \lambda_k(h_s) = \sum_{k=1}^{l} r_k t_0^k \neq 0.\end{aligned}$$

Hence the conclusion is right. □

Lemma 3.4 *Let $a \in \Re$ be an element such that $[h_0, a] = 0$. Then $a \in H$.*

Proof We represent the element a in the form of (3.1):

$$a = h + \sum_{\lambda \in \Delta} k^{\lambda} E^{\lambda}.$$

Then

$$\begin{aligned}0 = [h_0, a] &= [h_0, h + \sum_{\lambda\in\Delta} k^{\lambda} E^{\lambda}]\\ &= [h_0, h] + \sum_{\lambda\in\Delta} k^{\lambda}[h_0, E^{\lambda}] = \sum_{\lambda\in\Delta} k^{\lambda}\lambda(h_0) E^{\lambda}.\end{aligned}$$

Thus $k^{\lambda}\lambda(h_0) = 0$ for all $\lambda \in \Delta$. Since $\lambda(h_0) \neq 0$, we get $k^{\lambda} = 0$ for all $\lambda \in \Delta$. Therefore $a = h \in H$. □

Remark 3.5 *Since h_0 is a homogeneous element of degree $\bar{0}$, $[h_0, a] = 0$ implies that $[a, h_0] = 0$.*

Lemma 3.6 *Let T be a 2-local superderivation on $\Re$ such that $T(h_0) = 0$. Then T annihilates the Cartan subalgebra H of $\Re_{\bar{0}}$, i.e., $T|_H \equiv 0$.*

Proof Let h be an arbitrary element of H. Since $\Re$ has no any outer superderivations, by definition of T there exists an element $a_{h,h_0} \in \Re$ such that

$$T(h) = [a_{h,h_0}, h], T(h_0) = [a_{h,h_0}, h_0].$$

Since $0 = T(h_0) = [a_{h,h_0}, h_0]$, Lemma 3.4 implies that $a_{h,h_0} \in H$. Therefore, $T(h) = [a_{h,h_0}, h] = 0$. □

Lemma 3.7 *Let T be a 2-local superderivation on $\Re$ such that $T(h_0) = 0$. Then*

(1) There exists $k^{\lambda} \in \mathbb{F}$ such that $T(E^{\lambda}) = k^{\lambda} E^{\lambda}$ for all $\lambda \in \Delta$;

(2) There exists $\mathfrak{h} \in H$ such that $T(E^{\lambda}) = \lambda(\mathfrak{h}) E^{\lambda}$ for all $\lambda \in \Delta$;

(3) $T = \mathrm{ad}\mathfrak{h}$.

Proof (1) Let h be an arbitrary element of H. Take an element $a_{h,E^\lambda} \in \Re$ such that

$$T(h) = [a_{h,E^\lambda}, h], \quad T(E^\lambda) = [a_{h,E^\lambda}, E^\lambda].$$

According to Lemma 3.6 we get $T(h) = 0$, i.e., $[a_{h,E^\lambda}, h] = 0$. Then

$$\begin{aligned}[h, T(E^\lambda)] &= [h, [a_{h,E^\lambda}, E^\lambda]] = [[h, a_{h,E^\lambda}], E^\lambda] + [a_{h,E^\lambda}, [h, E^\lambda]] \\ &= \lambda(h)[a_{h,E^\lambda}, E^\lambda] = \lambda(h)T(E^\lambda),\end{aligned}$$

i.e.,

$$[h, T(E^\lambda)] = \lambda(h)T(E^\lambda) \text{ for all } h \in H, \lambda \in \Delta.$$

This means that $T(E^\lambda) \in \Re^\lambda$. Since $\dim\Re^\lambda = 1$, there exists $k^\lambda \in \mathbb{F}$, such that $T(E^\lambda) = k^\lambda E^\lambda$ for all $\lambda \in \Delta$.

(2) Now put $x = \sum_{\lambda\in\Delta} E^\lambda$. Take an element $a_{h_0,x} \in \Re$ such that

$$T(h_0) = [a_{h_0,x}, h_0], T(x) = [a_{h_0,x}, x].$$

Since $0 = T(h_0) = [a_{h_0,x}, h_0]$, Lemma 3.4 implies that $a_{h_0,x} \in H$. Write $a_{h_0,x} = \mathfrak{h}$. Then

$$T(x) = [a_{h_0,x}, x] = [\mathfrak{h}, x] = [\mathfrak{h}, \sum_{\lambda\in\Delta} E^\lambda] = \sum_{\lambda\in\Delta} [\mathfrak{h}, E^\lambda] = \sum_{\lambda\in\Delta} \lambda(\mathfrak{h})E^\lambda,$$

i.e.,

$$T(x) = \sum_{\lambda\in\Delta} \lambda(\mathfrak{h})E^\lambda. \tag{3.2}$$

On the other hand, taking into account the linearity of T we obtain

$$T(x) = T(\sum_{\lambda\in\Delta} E^\lambda) = \sum_{\lambda\in\Delta} T(E^\lambda) = \sum_{\lambda\in\Delta} k^\lambda E^\lambda,$$

i.e.,

$$T(x) = \sum_{\lambda\in\Delta} k^\lambda E^\lambda. \tag{3.3}$$

Combining (3.2) and (3.3) we have

$$\sum_{\lambda\in\Delta} \lambda(\mathfrak{h})E^\lambda = \sum_{\lambda\in\Delta} k^\lambda E^\lambda.$$

Thus $k^\lambda = \lambda(\mathfrak{h})$ for all $\lambda \in \Delta$, i.e., $T(E^\lambda) = \lambda(\mathfrak{h})E^\lambda$ for all $\lambda \in \Delta$.

(3) Finally, let x be an arbitrary element of $\Re$. We represent x in the form of (3.1):

$$x = h + \sum_{\lambda\in\Delta} k^\lambda E^\lambda.$$

Then $T(x) = T(h) + \sum_{\lambda\in\Delta} k^\lambda \lambda(\mathfrak{h})E^\lambda$. Due to Lemma 3.6 we get $T(h) = 0$, i.e., $T(x) = \sum_{\lambda\in\Delta} k^\lambda \lambda(\mathfrak{h})E^\lambda$. On the other hand,

$$\mathrm{ad}\mathfrak{h}(x) = \sum_{\lambda\in\Delta} k^\lambda [\mathfrak{h}, E^\lambda] = \sum_{\lambda\in\Delta} k^\lambda \lambda(\mathfrak{h})E^\lambda.$$

Thus $T = \mathrm{ad}\mathfrak{h}$. □

The Proof of Theorem 3.1 Let T be a 2-local superderivation. Take an element $a \in \Re$ such that $T(h_0) = [a, h_0]$. Set $T_0 = T - \mathrm{ad}a$. Then $T_0(h_0) = 0$. By Lemma 3.7 there exists $\mathfrak{h} \in H$ such that $T_0 = \mathrm{ad}\mathfrak{h}$. Therefore, $T = \mathrm{ad}\mathfrak{h} + \mathrm{ad}a$ is a superderivation. □

4. 2-Local superderivation on a subalgebra of Lie superalgebra spl(2, 2)

In this section, we give an example of 2-local superderivation on a Lie superalgebra which is not superderivation.

Suppose that L is a Lie superalgebra over $\mathbb{F}$. $\mathrm{Z}(L)$ and $[L, L]$ denote the center and derived algebra of L, respectively. Let $\delta : L \to L$ be a linear map which is homogeneous of degree α such that $\delta|_{[L,L]} \equiv 0$ and $\delta(L) \subseteq \mathrm{Z}(L)$. Then δ is a superderivation. Indeed, for every $x \in L_\beta, y \in L, \alpha, \beta \in \mathbb{Z}_2$ we have

$$\delta([x, y]) = 0 = [\delta(x), y] + (-1)^{\alpha\beta}[x, \delta(y)].$$

Let S be a subalgebra of Lie superalgebra $\mathrm{spl}(2, 2)$. S consists of the elements as follows:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_1 & 0 & d_1 & d_2 \\ c_1 & 0 & 0 & b_2 \\ c_2 & 0 & 0 & 0 \end{pmatrix}$$

where $b_i, c_i, d_i \in \mathbb{F}$, $i = 1, 2$. If

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_1 & 0 & d_1 & d_2 \\ c_1 & 0 & 0 & b_2 \\ c_2 & 0 & 0 & 0 \end{pmatrix}, \quad X' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b'_1 & 0 & d'_1 & d'_2 \\ c'_1 & 0 & 0 & b'_2 \\ c'_2 & 0 & 0 & 0 \end{pmatrix},$$

then

$$[X, X'] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ d_1c'_1 + d_2c'_2 + d'_1c_1 + d'_2c_2 & 0 & 0 & d_1b'_2 - d'_1b_2 \\ b_2c'_2 - b'_2c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to prove that $\mathrm{Z}(S) = \mathbb{F}E_{21}$, $[S, S] = \mathbb{F}E_{21} \oplus \mathbb{F}E_{31} \oplus \mathbb{F}E_{24}$, where E_{ij} is a 4×4 matrix with 1 in the (i, j) position and 0 elsewhere. We can give the decomposition of S in the following form

$$S = [S, S] \oplus \mathbb{F}E_{41} \oplus \mathbb{F}E_{23} \oplus \mathbb{F}E_{34}.$$

Let us define a function f on $\mathbb{F}^2$ as follows

$$f(k_1, k_2) = \begin{cases} \dfrac{k_1^2}{k_2}, & \text{if } k_2 \neq 0, \\ 0, & \text{if } k_2 = 0, \end{cases}$$

where $k_1, k_2 \in \mathbb{F}$. Define a map T on S by

$$T(x) = f(k_1, k_2)E_{21}, \text{ for } x = x_1 + k_1E_{41} + k_2E_{23} + k_3E_{34} \in S,$$

where $x_1 \in [S, S]$, $k_1, k_2, k_3 \in \mathbb{F}$. The map is not a superderivation since it is not linear.

In the following, we will show that T is a 2-local superderivation of degree $\bar{1}$ on S. Obviously,

$$S_{\bar{0}} = \mathbb{F}E_{21} \oplus \mathbb{F}E_{34}, \quad S_{\bar{1}} = \mathbb{F}E_{31} \oplus \mathbb{F}E_{24} \oplus \mathbb{F}E_{41} \oplus \mathbb{F}E_{23}.$$

Thus $T(S_{\bar{0}}) = 0 \in S_{\bar{1}}$, $T(S_{\bar{1}}) \subseteq S_{\bar{0}}$, i.e., T is homogeneous map of degree $\bar{1}$.

Define a linear map δ on S by

$$\delta(x) = (ak_1 + bk_2)E_{21}, \text{ for } x = x_1 + k_1E_{41} + k_2E_{23} + k_3E_{34} \in S,$$

where $a, b \in \mathbb{F}$. Since $\delta|_{[L,L]} \equiv 0$ and $\delta(L) \subseteq \mathrm{Z}(L)$, δ is a superderivation. $\delta(S_{\bar{0}}) = 0 \in S_{\bar{1}}$ and $\delta(S_{\bar{1}}) \subseteq S_{\bar{0}}$ imply that δ is a superderivation of degree $\bar{1}$.

Let $x = x_1 + k_1E_{41} + k_2E_{23} + k_3E_{34}$ and $y = y_1 + l_1E_{41} + l_2E_{23} + l_3E_{34}$ be elements of S. We are going to choose the elements a and b such that

$$T(x) = \delta(x), \quad T(y) = \delta(y).$$

Let us rewrite the above equalities as a system equations with respect to unknowns a, b as follows

$$\begin{cases} k_1a + k_2b = f(k_1, k_2), \\ l_1a + l_2b = f(l_1, l_2). \end{cases}$$

According to the definition of f, we know that the rank of matrix of coefficients equals to the rank of augmented matrix. Therefore the system of equations has a solution. As a result, T is a 2-local superderivation of degree $\bar{1}$. Thus we have the following conclusion:

Proposition 4.1 *There exists a 2-local superderivation on S which is not a superderivation.*

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