Coefficient Estimates for the Subclasses of Analytic Functions and Bi-Univalent Functions Associated with the Strip Domain

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Abstract The Sălăgean operator is used here to introduce a new subclass of analytic functions associated with the strip domain. We obtain the bounds of coefficients and Fekete-szegő inequality for functions in this class and coefficient estimates of bi-univalent functions for certain subclasses of this class. The results presented here extend some of the earlier results.

Keywords analytic functions; strip domain; Sălăgean operator; subordination

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1. Introduction

Let \( A \) denote the class of functions \( f(z) \) normalized by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disc \( U = \{ z \in \mathbb{C} : |z| < 1 \} \). Also, let \( S \) denote the subclass of \( A \) consisting of all functions which are univalent in \( U \) (see [1]).

It is well known that every function \( f \in S \) of the form (1.1) has an inverse \( f^{-1} \), defined by

\[
f^{-1}(f(z)) = z(z \in \mathbb{C}) \text{ and } f^{-1}(f(\omega)) = \omega \text{ (|\omega| < r; r \geq \frac{1}{2})},
\]

where

\[
f^{-1}(\omega) = \omega - a_2 \omega^2 + (2a_2 - a_3) \omega^3 - (5a_2^2 - 5a_2a_3 + a_4) \omega^4 + \cdots \quad (1.2)
\]

A function \( f \in A \) is bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \). Let \( \Sigma \) denote the class of bi-univalent functions defined in the open unit disk \( U \). Recently, the bounds of coefficients of analytic and bi-univalent functions have been studied by many authors [2–7].

Let \( u(z) \) and \( v(z) \) be analytic in \( A \). We say that the function \( u(z) \) is subordinate to \( v(z) \) in \( U \), and write \( u(z) \prec v(z) \), if there exists a Schwarz function \( \omega(z) \), which is analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) such that \( u(z) = v(\omega(z)) \) \( (z \in U) \).

Furthermore, if the function \( v \) is univalent in \( U \), then we have the following equivalence:

\[
u(z) \prec v(z) \quad (z \in U) \iff u(0) = v(0) \quad \text{and} \quad u(U) \subset v(U).
\]

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Let $\mathcal{P}$ denote the class of functions $p(z)$ of the form:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

which are analytic in $U$. If $\Re(p(z)) > 0$ ($z \in U$), we say that $p(z)$ is the Caratheodory function [1].

Let $S^*(\alpha)$ and $K(\alpha)$ ($0 \leq \alpha < 1$) denote the subclass consisting of all functions, which are defined, respectively, by

$$\Re\{zf'(z)/f(z)\} > \alpha$$

and

$$\Re\{1 + zf''(z)/f'(z)\} > \alpha, \quad f(z) \in \mathcal{A}.$$

The classes $S^*(\alpha)$ and $K(\alpha)$ were introduced by Robertson [8]. Obviously, for $\alpha = 0$, we have the well-known classes $S^*$ and $K$, respectively.

Also, let $M(\beta)$ and $N(\beta)$ ($\beta > 1$) denote the subclasses consisting of all functions, which are defined, respectively, by

$$\Re\{zf'(z)/f(z)\} < \beta$$

and

$$\Re\{1 + zf''(z)/f'(z)\} < \beta, \quad f(z) \in \mathcal{A}.$$  

The classes $M(\beta)$ and $N(\beta)$ were investigated by Uralegaddi, Ganigi and Sarangi [9] (see also [10]).

In [11], Kuroki and Owa defined an analytic function $S_{\alpha,\beta}(z) : U \to \mathbb{C}$ as follows.

\textbf{Definition 1.1} ([11]) Let $\alpha$ and $\beta$ be real numbers with $\alpha < 1$ and $\beta > 1$. Then the function $S_{\alpha,\beta}(z)$ defined by

$$S_{\alpha,\beta}(z) = 1 + \frac{\beta - \alpha}{\pi} \log \left( \frac{1 - e^{\frac{2\pi i(1-\alpha)}{\beta - \alpha}}}{1 - z} \right), \quad z \in U$$

(1.4)
is analytic and univalent in $U$ with $S_{\alpha,\beta}(0) = 1$. In addition, $S_{\alpha,\beta}(z)$ maps $U$ onto the strip domain $\omega$ with $\alpha < \Re\{\omega\} < \beta$.

We note that the function $S_{\alpha,\beta}(z)$ defined by (1.4) has the form [11]

$$S_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} B_n z^n,$$

(1.5)

where

$$B_n = \frac{\beta - \alpha}{n\pi} i (1 - e^{-\frac{2\pi i(1-\alpha)}{\beta - \alpha}}), \quad n \in \mathbb{N}.$$  

(1.6)

\textbf{Definition 1.2} ([12]) Let $-1 \leq B < A \leq 1$, $C \neq D$ and $-1 \leq D \leq 1$. Then the analytic function $p(z) \in P(A; B; C; D)$ if and only if $p(z)$ satisfies each of the following two subordination relationships:

$$p(z) \prec h_1(z) = \frac{1 + Az}{1 + Bz}$$

(1.7)
earlier works: the function $f$ let 

\[ p(z) < h_2(z) = \frac{1 + Cz}{1 + Dz}. \]  

(1.8)

For $A = 1 - 2\alpha$ ($0 \leq \alpha < 1$), $B = -1$, $C = 1 - 2\beta$ ($\beta > 1$) and $D = -1$ in $P(A, B; C, D)$, we obtain the following relationship:

\[ p(z) \in P(\alpha, \beta) = P(1 - 2\alpha, -1, 1 - 2\beta, -1) \iff \alpha < \Re\{p(z)\} < \beta. \]  

(1.9)

From (1.4) and (1.9), we have

\[ p(z) \in P(\alpha, \beta) \iff p(z) < S_{\alpha, \beta}(z). \]  

(1.10)

Also, from Definition 1.2, we introduce the following subclass of $p(z) \in P(A, B; C, D)$.

**Definition 1.3** Let

\[ \hat{P}(\rho_1) = \{ p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n : \Re\{p(z)\} > \rho_1 \}, \]

\[ \hat{P}(\rho_2) = \{ p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n : \Re\{p(z)\} < \rho_2 \}, \]

\[ \hat{P}(\rho_1, \rho_2) = \{ p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n : \rho_1 < \Re\{p(z)\} < \rho_2 \} \]

and

\[ \hat{P}(\rho_3, \rho_4) = \{ p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n : \rho_3 < \Re\{p(z)\}, \Re\{2 - p(z)\} < 1 + \rho_4 \}, \]

where

\[ \begin{cases} 
\rho_1 = \max\{\frac{1 - A}{1 - B}, \frac{1 + C}{1 + D}\}, & -1 < B < A \leq 1, \ -1 < C < D < 1, \\
\rho_2 = \min\{\frac{1 + A}{1 + B}, \frac{1 - C}{1 - D}\}, & -1 < B < A \leq 1, \ -1 < C < D < 1, \\
\rho_3 = \frac{1 - A}{2}, & B = -1, \\
\rho_4 = \frac{1 + C}{2}, & D = 1. 
\end{cases} \]  

(1.11)

In [13], Sălăgean defined the operator $D^m f(z) : A \to A$ as follows:

\[ D^0 f(z) = f(z), \ D^1 f(z) = D f(z) = z f'(z), \]

in general,

\[ D^m f(z) = D(D^{m-1} f(z)) = z + \sum_{n=2}^{\infty} n^m a_n z^n, \ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \]  

(1.12)

By using the operator $D^m$, we introduce the following two new subclasses of $A$.

**Definition 1.4** Let $m \in \mathbb{N}_0$, $0 \leq \lambda$, $-1 \leq B < A \leq 1$, $-1 < C < D \leq 1$, and $f(z) \in A$. Then the function $f(z) \in S_{m, \lambda}(A, B; C, D)$ if and only if $f(z)$ satisfies the following condition:

\[ \psi(f; m, \lambda) = \frac{D^{m+1} f(z)}{D^m f(z)} + \lambda^2 \frac{(D^m f(z))^n}{D^m f(z)} \in P(A, B; C, D). \]  

(1.13)

From the class $S_{m, \lambda}(A, B; C, D)$, we obtain the following subclasses which were studied in many earlier works:
2. Preliminary results

To prove the main results in the paper, we need the following lemmas.

Lemma 2.1 ([12]) \textit{The function }$p(z) \in P(A, B; C, D)$ \textit{if and only if }$p(z)$ \textit{satisfies each of the following two conditions:

\[
\left\{
\begin{array}{ll}
|p(z) - \sigma_i| < r_i, & i = 1, 2; \ -1 < B < A \leq 1; \ -1 < C < D < 1, \\
\rho_3 < \Re\{p(z)\}, & B = -1, \ \Re\{2 - p(z)\} < 1 + \rho_4, \ D = 1,
\end{array}
\right.
\]

(2.1)

where

\[
\left\{
\begin{array}{ll}
\sigma_1 = \frac{1 - AB}{1 - B^2} \text{ and } r_1 = \frac{A - B}{1 - B^2}, \\
\sigma_2 = \frac{1 - CD}{1 - D^2} \text{ and } r_2 = \frac{D - C}{1 - D^2},
\end{array}
\right.
\]

(2.2)

and $\rho_3, \rho_4$ are given by (1.11).

Lemma 2.2 ([12]) \textit{Let }$j = 1, 2, 3, 4; \ -1 < B < A \leq 1 $ \textit{and }$-1 < C < D < 1 $; $S_{\alpha, \beta}(z)$ \textit{is defined by (1.4). If }$p(z) \in P(A, B; C, D)$, \textit{then}

\[
p(z) \prec p_j(z) = \left\{
\begin{array}{ll}
p_1(z) = S_{\frac{A}{1+B}, \frac{-C}{1+B}}(z), & BC - AD \geq |A - B + C - D|, \ j = 1, \\
p_2(z) = S_{\frac{A}{1+B}, \frac{C}{1+B}}(z), & AD - BC \geq |A - B + C - D|, \ j = 2, \\
p_3(z) = S_{\frac{A}{1+B}, \frac{C}{1+B}}(z), & |AD - BC| \leq B - A + D - C, \ j = 3, \\
p_4(z) = S_{\frac{A}{1+B}, \frac{-C}{1+B}}(z), & |AD - BC| \leq A - B + C - D, \ j = 4,
\end{array}
\right.
\]

(2.3)

where $p_j(0) = 1$ and

\[
p_j(z) = \left\{
\begin{array}{ll}
p_1(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, & j = 1, \\
p_2(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, & j = 2, \\
p_3(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, & j = 3, \\
p_4(z) = 1 + \sum_{n=1}^{\infty} B_n z^n, & j = 4,
\end{array}
\right.
\]

(2.4)
Also, similarly as the proof in (i), it is easy to prove that

$$B_{n,j} = \begin{cases} 
B_{n,1} = \frac{1+4j}{n\pi} \frac{i(1-e^{2n\pi i(1-\frac{1}{1-a})/(1-\frac{1}{1-b})})}{j=1,} \\
B_{n,2} = \frac{1+4j}{1+n\pi} \frac{i(1-e^{2n\pi i(1-\frac{1}{1+a})/(1+\frac{1}{1+b})})}{j=2,} \\
B_{n,3} = \frac{1+4j}{1+n\pi} \frac{i(1-e^{2n\pi i(1-\frac{1}{1-a})/(1+\frac{1}{1+b})})}{j=3,} \\
B_{n,4} = \frac{1+4j}{1+n\pi} \frac{i(1-e^{2n\pi i(1-\frac{1}{1-a})/(1+\frac{1}{1+b})})}{j=4,}
\end{cases} (2.5)$$

**Proof** (i) Let \( p(z) \in P(A, B; C, D) \) with \( BC - AD \geq |A - B + C - D| \). Let \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \in P(A, B; C, D) \). Then, from Definition 1.2 and the definition of subordination, we get

$$\begin{cases} 
p(0) = h_1(0), & p(U) \subset h_1(U), \\
p(0) = h_2(0), & p(U) \subset h_2(U),
\end{cases} (2.6)$$

where \( h_1(z) \) and \( h_2(z) \) are given by (1.7) and (1.8), respectively. Therefore, we have

$$\begin{cases} 
p(z) = h_1(\omega_1(z)), & \omega_1(0) = 0, |\omega_1(z)| < 1, \\
p(z) = h_2(\omega_2(z)), & \omega_2(0) = 0, |\omega_2(z)| < 1.
\end{cases}$$

We also deduce that

$$\begin{cases} 
|\omega_1(z)| = \frac{|p(z) - 1|}{A - Bp(z)} < 1, & p(z) = u + iv, \\
|\omega_2(z)| = \frac{|p(z) - 1|}{C - Dp(z)} < 1, & p(z) = u + iv.
\end{cases} (2.7)$$

From (2.7), we find that

$$\begin{cases} 
2u(1 - AB) > 1 - A^2 + (1 - B^2)(u^2 + v^2), \\
2u(1 - CD) > 1 - C^2 + (1 - D^2)(u^2 + v^2).
\end{cases} (2.8)$$

Since

$$|p(z)|^2 \geq |\Re(p(z))|^2, (2.9)$$

from (2.8) and (2.9) we have

$$\begin{cases} 
\frac{1}{1-A} < u = \Re(p(z)) < \frac{1+4}{1+B}, \\
\frac{1}{1-C} < u = \Re(p(z)) < \frac{1+4}{1+D}.
\end{cases} (2.10)$$

Then, from (2.10) we obtain

$$\frac{1-A}{1-B} < \Re(p(z)) < \frac{1-C}{1-D}.$$ 

By using (1.9), we get

$$p(z) < p_1(z) = S_{\frac{1}{1-A} \frac{1+4}{1+B}}(z), \quad BC - AD \geq |A - B + C - D|.$$ 

Also, similarly as the proof in (i), it is easy to prove that

(ii) \( p(z) < p_2(z) = S_{\frac{1+4}{1+B} \frac{1}{1-C}}(z), \quad AD - BC \geq |A - B + C - D|,$$

(iii) \( p(z) < p_3(z) = S_{\frac{1}{1-B} \frac{1+4}{1+C}}(z), \quad |AD - BC| \leq B - A + D - C$$

and

(iv) \( p(z) < p_4(z) = S_{\frac{1+4}{1+C} \frac{1}{1-D}}(z), \quad |AD - BC| \leq A - B + C - D.$$ 

Therefore, we complete the proof of Lemma 2.2. \( \Box \)

The functions \( p_j \ (j = 1, 2, 3, 4) \) maps \( U \) onto the strip domain (see Figures 1-1, 1-2, 1-3 and 1-4).
Lemma 2.3 ([20]) Let \( p(z) = 1 + c_1z + c_2z^2 + \cdots \) be analytic and univalent in \( U \), and suppose that \( p(z) \) maps \( U \) onto a convex domain. If \( q(z) = 1 + q_1z + q_2z^2 + \cdots \) is analytic in \( U \) and satisfies the following subordination:

\[
q(z) \prec p(z), \quad z \in U
\]

then

\[
|q_n| \leq |c_1|, \quad n = 1, 2, \ldots.
\]

Using Definition 1.1, Lemma 2.3 and the definition of subordination, we can obtain the following lemma.

Lemma 2.4 ([12]) Let \(-1 \leq B < A \leq 1, -1 < C < D \leq 1, i = 1, 2; j = 1, 2, 3, 4 \) and
\( \tilde{P}(p_1), \tilde{P}(p_2), \tilde{P}(p_1, p_2) \) and \( \tilde{P}(p_3, p_4) \) are given by Definition 1.3. If \( p(z) = 1 + c_1z + c_2z^2 + \cdots \in P(A, B; C, D) \), then

\[
|c_n| \leq \chi(\delta; \rho_j) = \begin{cases} 
2\delta_1, & p \in \tilde{P}(p_1), \\
2\delta_2, & p \in \tilde{P}(p_2), \\
2\min\{\delta_1, \delta_2\}, & p \in \tilde{P}(p_1, p_2), \\
2\min\{\frac{1+\lambda_1}{2}, \frac{1-C}{2}\}, & p \in \tilde{P}(p_3, p_4),
\end{cases}
\tag{2.11}
\]

where

\[
\begin{aligned}
\delta_1 &= \min\{\frac{A-B}{1-B}, \frac{D-C}{1+C}\}, \\
\delta_2 &= \min\{\frac{A-B}{1+B}, \frac{D-C}{1-B}\},
\end{aligned}
\tag{2.12}
\]

and \( \rho_j \) are given by (1.11).

**Lemma 2.5** ([21]) Let the function \( p(z) \) be given by (1.3). If \( p(z) \in \mathcal{P} \), then for any complex number \( \gamma \),

\[
|c_2 - \gamma c_1^2| \leq 2\max\{1, |2\gamma - 1| \},
\]

and the result is sharp for the functions given by \( p(z) = \frac{1+z^2}{1-z^2}, p(z) = \frac{1+z}{1-z} \).

### 3. Main results

Using Lemma 2.1 and Definition 1.4, we easily get

**Theorem 3.1** Let \( \psi(f; m, \lambda) \) be defined by (1.13). The function \( f(z) \in S_{m, \lambda}(A, B; C, D) \) if and only if \( f(z) \) satisfies each of the following two conditions:

\[
\begin{cases}
|\psi(f; m, \lambda) - \sigma_i| < r_i, & i = 1, 2; -1 < B < A \leq 1; -1 < C < D < 1, \\
\rho_3 < \Re\{\psi(f; m, \lambda)\}, & B = -1, \ \Re\{2 - \psi(f; m, \lambda)\} < 1 + \rho_4, \ D = 1,
\end{cases}
\]

where \( \sigma_i \) and \( r_i \) (\( i = 1, 2 \)) are given by (2.2) and \( \rho_k \) (\( k = 3, 4 \)) are given by (1.11).

**Theorem 3.2** Let \( m \in \mathbb{N}_0, \lambda \geq 0, |a_1| = 1 \) and the function \( f(z) \) be given by (1.1). If \( f(z) \in S_{m, \lambda}(A, B; C, D) \), then

\[
|a_n| \leq M_{n,j}(m, \lambda) = \begin{cases} 
\frac{|B_{1,j}|}{2^{m(2\lambda+1)}}, & n = 2, \\
\frac{|B_{1,j}|}{(n-1)(n\lambda+1)} \prod_{k=2}^{n-1} (1 + \frac{|B_{1,j}|}{(k-1)(k\lambda+1)}), & n \geq 3,
\end{cases}
\tag{3.1}
\]

where \( |B_{1,j}| \) (\( j = 1, 2, 3, 4 \)) are defined by (2.5).

**Proof** According to Definition 1.2 and the subordination relationship, we have

\[
\frac{D^{m+1}f(z)}{D^mf(z)} + \lambda z^2(D^mf(z))'' \in h_1(\mathbb{U})
\tag{3.2}
\]

and

\[
\frac{D^{m+1}f(z)}{D^mf(z)} + \lambda z^2(D^mf(z))'' \in h_2(\mathbb{U}),
\tag{3.3}
\]

where the functions \( h_1(z) \) and \( h_2(z) \) are given by (1.7) and (1.8), respectively.
Applying (3.2) and (3.3), we get

\[
\frac{D^{m+1}f(z)}{D^mf(z)} + \lambda \frac{z^2(D^mf(z))'}{D^mf(z)} = p(z), \quad \exists p(z) = 1 + c_1z + c_2z^2 + \cdots \in P(A, B; C, D),
\]

or, equivalently,

\[
D^{m+1}f(z) + \lambda^2(D^mf(z))' = p(z)D^mf(z), \quad \exists p(z) = 1 + c_1z + c_2z^2 + \cdots \in P(A, B; C, D). \tag{3.4}
\]

Then, comparing the coefficients of \(z^n\) in the both sides of (3.4), we have

\[
(n-1)(n\lambda + 1)n^m a_n = (c_{n-1} + c_{n-2}2^m a_2 + \cdots + c_1(n-1)^m a_{n-1}). \tag{3.5}
\]

Using Lemma 2.2, Lemma 2.3 and (3.5), we obtain

\[
|a_n| \leq \frac{1}{(n-1)(n\lambda + 1)n^m} \left( |c_{n-1}| + |c_{n-2}| 2^m |a_2| + \cdots + |c_1| (n-1)^m |a_{n-1}| \right)
\]

\[
\leq \frac{|B_{1,j}|}{(n-1)(n\lambda + 1)n^m} \sum_{k=1}^{n-1} k^m |a_k|.
\]

Hence, we have \(|a_2| \leq M_{2,j}(m, \lambda)\). To prove the remaining part of the theorem, we need to show that

\[
\sum_{k=1}^{n-1} k^m |a_k| \leq \prod_{k=2}^{n-1} \left( 1 + \frac{|B_{1,j}|}{(k-1)(k\lambda + 1)} \right), \tag{3.6}
\]

for \(n = 3, 4, 5, \ldots\). We use induction to prove (3.6). The case \(n = 3\) is clear. Next, assume that the inequality (3.6) holds for \(n = p\). Then, a straightforward calculation gives

\[
\sum_{k=1}^{p} k^m |a_k| = \sum_{k=1}^{p-1} k^m |a_k| + p^m |a_p|
\]

\[
\leq \left( 1 + \frac{|B_{1,j}|}{(p-1)(p\lambda + 1)} \right) \sum_{k=1}^{p-1} k^m |a_k|
\]

\[
\leq \left( 1 + \frac{|B_{1,j}|}{(p-1)(p\lambda + 1)} \right) \prod_{k=2}^{p-1} \left( 1 + \frac{|B_{1,j}|}{(k-1)(k\lambda + 1)} \right)
\]

\[
= \prod_{k=2}^{p} \left( 1 + \frac{|B_{1,j}|}{(k-1)(k\lambda + 1)} \right)
\]

which implies that the inequality (3.6) holds for \(n = p + 1\). Hence, the desired estimate for \(|a_n|\) \((n \geq 3)\) follows, as asserted in (3.1). This completes the proof of Theorem 3.2. \(\Box\)

**Remark 3.3** Taking \(m = 0, A = 1 - 2\alpha (0 \leq \alpha \leq 1), B = -1, C = 1 - 2\beta (1 < \alpha), D = -1,\) we obtain the improved result of Theorem 3.1 in the paper [16]. Also, setting \(m = 0, \lambda = 0,\) we obtain the improved result of Theorem 3.2 in the paper [12].

Also, using Lemma 2.4 and Definition 1.4, we get

**Theorem 3.4** Let \(m \in \mathbb{N}_0, \lambda \geq 0, |a_1| = 1\) and the function \(f(z)\) be given by (1.1). If
Proof If \( f(z) \in S_m, \lambda (A, B; C, D), \) then

\[
|a_n| \leq \Psi_{n,j}(m, \lambda) = \begin{cases} \frac{\chi(\delta_i, \rho_j)}{2^n(2\lambda+1)^n}, & n = 2, \\ \left(\frac{\chi(\delta_i, \rho_j)}{(n-1)(n\lambda+1)n}\right)^{n-1} \prod_{k=2}^{n-1} (1 + \frac{\chi(\delta_i, \rho_j)}{(k-1)(k\lambda+1)}) & n \geq 3, \end{cases} \tag{3.7}
\]

where \( \chi(\delta_i, \rho_j) \) (i = 1, 2; j = 1, 2, 3, 4) are defined by (2.11).

Remark 3.5 Setting \( m = 0, \lambda = 0, \) we obtain the improved result of Theorem 3.1 in [12].

Theorem 3.6 Let \( m \in \mathbb{N}_0, \lambda \geq 0, -1 < B < A \leq 1, -1 < C < D < 1, 0 \leq \mu \leq 1 \) and \( p_j(z) = 1 + \sum_{n=1}^{\infty} B_{n,j} z^n (j = 1, 2, 3, 4). \) If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_m(A, B; C, D), \) then

\[
|a_3 - \mu a_2^2| \leq \frac{|B_{1,j}|}{2 \cdot 3^m(3\lambda + 1)^2} \max\{1, |\frac{B_{2,j}}{B_{1,j}}| = \frac{2 (3\lambda + 1) (\frac{3}{4})^m \mu - (2\lambda + 1)}{(2\lambda + 1)^2} B_{1,j}, \}
\tag{3.8}
\]

where \( |B_{1,i}| (i = 1, 2, j = 1, 2, 3, 4) \) are defined by (2.5).

Proof If \( f(z) \in S_m(A, B; C, D), \) then there exists a Schwarz function \( \omega(z) \) in \( U \) such that

\[
\frac{D^{m+1} f(z)}{D^m f(z)} + \lambda \frac{z^2 (D^m f(z))''}{D^m f(z)} = p_j(\omega(z)), \quad z \in U, \tag{3.9}
\]

where \( p_j(z) (j = 1, 2, 3, 4) \) are defined by (2.3).

Let the function \( p(z) \) be given by

\[
p(z) = \frac{D^{m+1} f(z)}{D^m f(z)} + \lambda \frac{z^2 (D^m f(z))''}{D^m f(z)} \tag{3.10}
\]

Then, from (3.9) and (3.10) we have \( p(z) \prec p_j(z). \) Let

\[
q(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + q_1 z + q_2 z^2 + \cdots. \tag{3.11}
\]

Then \( q(z) \) is analytic and has positive real part in \( U. \) From (3.11), we get

\[
\omega(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} q_1 z + (q_2 - \frac{q_1^2}{2}) z^2 + \cdots. \tag{3.12}
\]

We see from (3.12) that

\[
p(z) = p_j(\frac{q(z) - 1}{q(z) + 1}) = 1 + \frac{1}{2} B_{1,j} q_1 z + \left(\frac{1}{2} B_{1,j} (q_2 - \frac{q_1^2}{2}) + \frac{B_{2,j} q_1^2}{4}\right) z^2 + \cdots. \tag{3.13}
\]

Using (3.10) and (3.13), we obtain

\[
(2\lambda + 1)^2 a_2 = \frac{B_{1,j} q_1}{2},
\]

\[
2(3\lambda + 1)^3 a_3 - (2\lambda + 1)^4 a_2^2 = \frac{B_{1,j} q_2}{2} - \frac{q_1^2}{4} (B_{1,j} - B_{2,j}),
\]

which imply that

\[
a_3 - \mu a_2^2 = \frac{B_{1,j}}{4 \cdot 3^m (3\lambda + 1)} \left[q_2 - \gamma_j q_1^2\right], \tag{3.14}
\]

where, for convenience,

\[
\gamma_j = \frac{1}{2} [1 - \frac{B_{2,j}}{B_{1,j}} + \frac{2 (3\lambda + 1) (\frac{3}{4})^m \mu - (2\lambda + 1)}{(2\lambda + 1)^2} B_{1,j}],
\]
Then, applying Lemma 2.5, we have

\[
|a_3 - \mu a_2^2| \leq \frac{|B_{1,j}|}{4 \cdot 3^m(3\lambda + 1)} |q_2 - \gamma q_1^2| \leq \frac{|B_{1,j}|}{2 \cdot 3^m(3\lambda + 1)} \max\{1, |1 - 2\gamma_j|\}
\]

\[
\leq \frac{|B_{1,j}|}{2 \cdot 3^m(3\lambda + 1)} \max\{1, \frac{B_{2,j}}{B_{1,j}} - \frac{2(3\lambda + 1)(\frac{3}{2})^m\mu - (2\lambda + 1)}{(2\lambda + 1)^2} B_{1,j}\}.
\]

The estimate is sharp for the function \(f_j(z) (j = 1, 2, 3, 4)\) defined by

\[
f_j(z) = D^{-m} \left[ \int_0^z \left( \exp \left( \int_0^\eta \frac{p_j(\xi)}{\xi} d\xi \right) \right) d\eta \right], \tag{3.15}
\]

where the function \(p_j(z) (j = 1, 2, 3, 4)\) are given by (2.3) (see Figures 2-1, 2-2, 2-3 and 2-4).

Hence we complete the proof of Theorem 3.6. \(\square\)

**Figure 2-1** The image of \(U\) under \(f_1(z)\) for \(A = 0.1, B = -0.5, C = -0.5, D = 0.2, m = 0\)

**Figure 2-2** The image of \(U\) under \(f_2(z)\) for \(A = 0.7, B = 0.4, C = 0.1, D = 0.8, m = 0\)

**Figure 2-3** The image of \(U\) under \(f_3(z)\) for \(A = 0.7, B = 0.4, C = -0.1, D = 0.8, m = 0\)

**Figure 2-4** The image of \(U\) under \(f_4(z)\) for \(A = 0.9, B = 0.1, C = 0.1, D = 0.4, m = 0\)

**Remark 3.7** Setting \(m = 0, A = 1 - 2\alpha (0 \leq \alpha \leq 1), B = -1; C = 1 - 2\beta (1 < \alpha), D = -1,\)
we obtain the improved result of Theorem 2 in the paper [16]. Also, taking \( m = 0, \lambda = 0 \), we have the improved result of Theorem 3.3 in the paper [12].

Using Theorem 3.6, we can easily get the following result.

**Corollary 3.8** Let \( m \in \mathbb{N}_0, \lambda \geq 0, -1 < B < A \leq 1, -1 < C < D < 1 \), and \( f^{-1} \) be the inverse function of \( f \). If \( f(z) \in S_m(A, B; C, D) \), then

\[
|b_2| \leq \frac{|B_{1,j}|}{2^m(2\lambda + 1)} \quad \text{and} \quad |b_3| \leq \frac{|B_{1,j}|}{2 \cdot 3^m(3\lambda + 1)} \max\{1, |B_{2,j}|(\frac{2(2\lambda + 1)B_{1,j}}{B_{1,j} - B_{2,j}}) - \frac{4(3\lambda + 1)(\frac{3}{2})^m - (2\lambda + 1)}{(2\lambda + 1)^2}B_{1,j}m\},
\]

where \( B_{1,j} \) \((i = 1, 2; j = 1, 2, 3, 4)\) are defined by (2.5).

**Proof** The relations (1.2) and \( f^{-1}(\omega) = \omega + b_2\omega^2 + \cdots \) yield \( b_2 = -a_2 \) and \( b_3 = 2a_2^2 - a_3 \). Thus, in view of (3.1) and the identity \( |b_2| = |a_2| \), the estimate for \( |b_2| \) follows immediately. Furthermore, applying Theorem 3.6 with \( \mu = 2 \) gives the estimate for \( |b_1| \). \( \square \)

Finally, we will estimate some initial coefficients for the bi-univalent functions \( f \).

**Theorem 3.9** Let \( m \in \mathbb{N}_0, \lambda \geq 0, -1 < B < A \leq 1, -1 < C < D < 1 \). If \( f \in \Sigma_{m,\lambda}(A, B; C, D) \), then

\[
|a_2| \leq \frac{|B_{1,j}|}{\sqrt{|B_{1,j}|^2[2(3\lambda + 1)^3m - (2\lambda + 1)^4m] + 4^m(2\lambda + 1)B_{1,j} - B_{2,j}]}}
\]

and

\[
|a_3| \leq \frac{|B_{1,j}|\{2[4(3\lambda + 1)^3m - (2\lambda + 1)^4m] + 2(2\lambda + 1)^4m\} + 8(3\lambda + 1)3^m|B_{1,j} - B_{2,j}|}{4(3\lambda + 1)^3m[4(3\lambda + 1)^3m - 2(2\lambda + 1)^4m]} \quad (3.16)
\]

where \( B_{i,j} \) \((i = 1, 2; j = 1, 2, 3, 4)\) are defined by (2.5).

**Proof** If \( f(z) \in \Sigma_{m}(A, B; C, D) \), then \( f(z) \in S_{m,\lambda}(A, B; C, D) \) and \( g = f^{-1} \in S_{m,\lambda}(A, B; C, D) \).

Hence

\[
G(z) = \frac{D^{m+1}f(z)}{D^m f(z)} + \lambda \frac{z^2(D^m f(z))^n}{D^m f(z)} \prec p_j(z), \quad z \in \mathbb{U}; \quad j = 1, 2, 3, 4
\]

and

\[
H(z) = \frac{D^{m+1}g(z)}{D^m g(z)} + \lambda \frac{z^2(D^m g(z))^n}{D^m g(z)} \prec p_j(z), \quad z \in \mathbb{U}; \quad j = 1, 2, 3, 4,
\]

where the function \( p_j(z) \) is given by (2.3). Let

\[
\varsigma(z) = \frac{1 + p_j^{-1}(G(z))}{1 - p_j^{-1}(G(z))} = 1 + \varsigma_1 z + \varsigma_2 z^2 + \cdots, \quad z \in \mathbb{U}; \quad j = 1, 2, 3, 4
\]

and

\[
\tau(z) = \frac{1 + p_j^{-1}(H(z))}{1 - p_j^{-1}(H(z))} = 1 + \tau_1 z + \tau_2 z^2 + \cdots, \quad z \in \mathbb{U}; \quad j = 1, 2, 3, 4.
\]
Then $\zeta$ and $\tau$ are analytic and have positive real part in $U$, and satisfy the estimates

$$|\kappa_n| \leq 2 \text{ and } |\tau_n| \leq 2, \quad n \in \mathbb{N}. \quad (3.17)$$

Therefore, we have

$$G(z) = p_j \left( \frac{\zeta(z) - 1}{\zeta(z) + 1} \right) \text{ and } H(z) = p_j \left( \frac{\tau(z) - 1}{\tau(z) + 1} \right), \quad z \in U; \quad j = 1, 2, 3, 4.$$ 

By comparing the coefficients, we get

$$(2\lambda + 1)2^m a_2 = \frac{B_{1,j} \zeta_1}{2}, \quad (3.18)$$

$$2(3\lambda + 1)3^m a_3 - (2\lambda + 1)2^m a_2^2 = \frac{B_{1,j} \tau_2}{2} + \frac{\zeta_1^2}{4} (B_{1,j} - B_{2,j}), \quad (3.19)$$

and

$$-(2\lambda + 1)2^m a_2 = \frac{B_{1,j} \tau_1}{2} \quad (3.20)$$

Also, from (3.19)–(3.22), we see that

$$a_2^2 = \frac{B_{1,j}^2 (\zeta_2 + \tau_2)}{4B_{1,j}^2 [2(3\lambda + 1)3^m - (2\lambda + 1)4^m]} + \frac{4^m + 1}{4^m} (2\lambda + 1)^2 (B_{1,j} - B_{2,j})$$

and

$$a_3 = \frac{B_{1,j} ([4(3\lambda + 1)3^m - (2\lambda + 1)4^m] \zeta_2 + (2\lambda + 1)4^m \tau_2) - 2(3\lambda + 1)3^m (B_{1,j} - B_{2,j}) \xi_1^2}{4(3\lambda + 1)3^m [4(3\lambda + 1)3^m - 2(2\lambda + 1)4^m]}.$$ 

These equations, together with (3.17), give the bounds on $|a_2|$ and $|a_3|$ as asserted in (3.16). This completes the proof of Theorem 3.9. \(\square\)

**Remark 3.10** Letting $m = 0, A = 1 - 2\alpha$ ($0 \leq \alpha \leq 1$), $B = -1; \ C = 1 - 2\beta$ ($1 < \alpha$), $D = -1$, we get the improved result of Theorem 3.6 in the paper [16].

**References**


