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Uncertainty Principles for the Segal-Bargmann Transform

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Abstract We recall some properties of the Segal-Bargmann transform; and we establish for this transform qualitative uncertainty principles: local uncertainty principle, Heisenberg uncertainty principle, Donoho-Stark's uncertainty principle and Matolcsi-Szücs uncertainty principle.

Keywords Segal-Bargmann space; Segal-Bargmann transform; local uncertainty principle; Heisenberg uncertainty principle; Donoho-Stark's uncertainty principle

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1. Introduction

Many uncertainty principles have already been proved for the Fourier transform: Heisenberg-Pauli-Weyl inequality [1–3], Cowling-Price's inequality [2], local uncertainty inequality [4–6], Donoho-Stark's inequality [7], Benedicks inequality [8] and Matolcsi-Szücs inequality [9]. Laeng-Morpurgo [10] and Morpurgo [11] obtained Heisenberg inequality involving a combination of L^1-norms and L^2-norms. Folland-Sitaram [12], next Nemri-Soltani [13–15] proved a general forms of the Heisenberg-Pauli-Weyl inequality and the Donoho-Stark's inequality. And one expects that the results in this work will be useful when discussing the uncertainty principles for the Segal-Bargmann transform.

Let $H(\mathbb{C})$ denote the space of entire functions on $\mathbb{C}$, and $L^2(\Omega)$, the space of measurable functions f on $\mathbb{C}$ satisfying

$$\|f\|^2_{L^2(\Omega)} := \int_{\mathbb{C}} |f(z)|^2 \mathrm{d}\Omega(z) < \infty,$$

where $z = x + iy$ and $\mathrm{d}\Omega(z) := \frac{e^{-|z|^2}}{\pi}\mathrm{d}x\mathrm{d}y$.

The Fock space $F(\mathbb{C})$ (called also Segal-Bargmann space [16]) is the space of functions in $H(\mathbb{C}) \cap L^2(\Omega)$, equipped with the norm $\|f\|_{F(\mathbb{C})} := \|f\|_{L^2(\Omega)}$. This space was introduced by Bargmann in [17] and it was the tool of many works [16,18,19]. Some uncertainty principles have already been proved on the Segal-Bargmann space $F(\mathbb{C})$ by Kehe Zhu in his paper [20].

Let $L^2(\mu)$ be the space of measurable functions on $\mathbb{R}$, for which

$$\|f\|^2_{L^2(\mu)} := \int_{\mathbb{R}} |f(z)|^2 \mathrm{d}\mu(z) < \infty,$$

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where $\mathrm{d}\mu(z) := \frac{1}{\sqrt{2\pi}}\mathrm{d}z$. The Segal-Bargmann transform B is defined on $L^2(\mu)$ by

$$B(f)(w) := 2^{1/4}\int_{\mathbb{R}} \exp\big(-\frac{w^2+z^2}{2}+\sqrt{2}wz\big)f(z)\mathrm{d}\mu(z), \quad w\in\mathbb{C}.$$

This transform (see [17]) is an isometric isomorphism of $L^2(\mu)$ onto $F(\mathbb{C})$ and has a pointwise inversion formula. Next, it is the background of some applications in this work; especially, we give Heisenberg-Pauli-Weyl uncertainty principle, Donoho-Stark uncertainty principle and Matolcsi-Szücs uncertainty principle for the Segal-Bargmann transform B.

Building on the ideas of Faris [4] and Price [5,6] for the Fourier transform, we show a local uncertainty principle for the Segal-Bargmann transform B. More precisely, we will show the following result. Let W be a measurable subset of $\mathbb{C}$ such that $0 < m(W) < \infty$, and $a > 0$. If $f \in L^2(\mu)$, then

$$\|\chi_W B(f)\|_{L^2(\Omega)} \le \begin{cases} K_1(a)(m(W))^a\|\,|x|^a f\|_{L^2(\mu)}, & a\in(0,\frac{1}{2}),\\ K_2(a)(m(W))^{1/2}\|f\|_{L^2(\mu)}^{1-\frac{1}{2a}}\|\,|x|^a f\|_{L^2(\mu)}^{\frac{1}{2a}}, & a>1/2,\\ 2K_1(\frac{1}{4})(m(W))^{1/4}\|f\|_{L^2(\mu)}^{1/2}\|\,|x|^{1/2} f\|_{L^2(\mu)}^{1/2}, & a=1/2,\end{cases}$$

where χ_W is the characteristic function of the set W, $\mathrm{d}m(z) := \frac{1}{\pi}\mathrm{d}x\mathrm{d}y$ and $K_1(a)$, $K_2(a)$ are positive constants given explicitly by Theorem 3.1. We shall use the local uncertainty principle for the Segal-Bargmann transform B to show for it the following Heisenberg-Pauli-Weyl uncertainty principle

$$\|f\|_{L^2(\mu)} \le K(a,b)\|\,|x|^a f\|_{L^2(\mu)}^{\frac{b}{2a+b}}\|\,|w|^b B(f)\|_{L^2(\Omega)}^{\frac{2a}{2a+b}}, \quad a,b>0,$$

where $K(a,b)$ is a positive constant given explicitly by Theorem 3.3. Next, by using Clarkson-type inequality and Nash-type inequality for the Segal-Bargmann transform B we establish for it Heisenberg uncertainty principle involving $L^1(\mu)$ and $L^2(\mu)$-norms.

Finally, based on the techniques of Donoho-Stark [7], we will show uncertainty principles of concentration-type on $L^2(\mu)$ and on $L^1\cap L^2(\mu)$ for the Segal-Bargmann transform B. And based on the ideas of Ghobber-Jaming [21] we deduce uncertainty principle of Matolcsi-Szücs-type involving $L^1(\mu)$ and $L^2(\mu)$-norms for the Segal-Bargmann transform B.

The contents of the paper are as follows. In Section 2, we recall some properties of the Fock space $F(\mathbb{C})$ and the Segal-Bargmann transform B. In Section 3, we establish local uncertainty principle and Heisenberg-Pauli-Weyl uncertainty principle for the Segal-Bargmann transform B. In Section 4, we show Heisenberg uncertainty principle involving $L^1(\mu)$ and $L^2(\mu)$-norms for the Segal-Bargmann transform B. The last section is devoted to present Donoho-Stark uncertainty principle and Matolcsi-Szücs uncertainty principle for the Segal-Bargmann transform B.

2. Segal-Bargmann transform

We denote by

- $H(\mathbb{C})$ the space of entire functions on $\mathbb{C}$.

• $\mathrm{d}\Omega(z)$, the measure defined on $\mathbb{C}$, by

$$\mathrm{d}\Omega(z) := \frac{e^{-|z|^2}}{\pi}\mathrm{d}x\mathrm{d}y, \quad z = x + iy.$$

• $L^q(\Omega)$, $1 \le q < \infty$, the space of measurable functions f on $\mathbb{C}$ satisfying

$$\|f\|_{L^q(\Omega)} := \Big[\int_{\mathbb{C}} |f(z)|^q \mathrm{d}\Omega(z)\Big]^{1/q} < \infty.$$

We define the pre Hilbert space $F(\mathbb{C})$, to be the space of functions in $H(\mathbb{C}) \cap L^2(\Omega)$, equipped with the inner product

$$\langle f, g\rangle_{F(\mathbb{C})} := \int_{\mathbb{C}} f(z)\overline{g(z)}\mathrm{d}\Omega(z),$$

and the squared norm

$$\|f\|^2_{F(\mathbb{C})} := \int_{\mathbb{C}} |f(z)|^2 \mathrm{d}\Omega(z).$$

The following properties are proved in [17].

(a) If $f, g \in F(\mathbb{C})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle f, g\rangle_{F(\mathbb{C})} = \sum_{n=0}^{\infty} a_n \overline{b_n}\, n!.$$

(b) The function K given for $w, z \in \mathbb{C}$, by

$$K(w, z) = e^{\overline{w}z},$$

is the reproducing kernel for the Fock space $F(\mathbb{C})$, that is,

(i) for every $w \in \mathbb{C}$, the function $z \to K(w, z)$ belongs to $F(\mathbb{C})$;

(ii) for all $w \in \mathbb{C}$ and $f \in F(\mathbb{C})$, we have

$$\langle f, K(w, .)\rangle_{F(\mathbb{C})} = f(w).$$

(c) If $f \in F(\mathbb{C})$, then

$$|f(w)| \le e^{|w|^2/2}\|f\|_{F(\mathbb{C})}, \quad w \in \mathbb{C}.$$

(d) The space $F(\mathbb{C})$ equipped with the inner product $\langle ., .\rangle_{F(\mathbb{C})}$ is a Hilbert space; and the set $\{\frac{w^n}{\sqrt{n!}}\}_{n\in\mathbb{N}}$ forms a Hilbertian basis for the space $F(\mathbb{C})$.

We denote by

• $L^p(\mu)$, $1 \le p < \infty$, the space of measurable functions on $\mathbb{R}$, for which

$$\|f\|_{L^p(\mu)} := \Big[\int_{\mathbb{R}} |f(z)|^p \mathrm{d}\mu(z)\Big]^{1/p} < \infty.$$

Here, μ is the measure defined on $\mathbb{R}$ by

$$\mathrm{d}\mu(z) := \frac{1}{\sqrt{2\pi}}\mathrm{d}z.$$

• U the kernel given for $w \in \mathbb{C}$ and $z \in \mathbb{R}$, by

$$U(w, z) := 2^{1/4}\exp(-\frac{w^2 + z^2}{2} + \sqrt{2}wz).$$

The kernel U satisfies the following properties [17].

(a) For $w \in \mathbb{C}$ and $z \in \mathbb{R}$, we have

$$|U(w,z)| \leq 2^{1/4} e^{|w|^2/2}. \tag{2.1}$$

(b) For $w \in \mathbb{C}$ and $z \in \mathbb{R}$, we have

$$U(w,z) = \sum_{n=0}^{\infty} \frac{h_n(z)}{\sqrt{n!}} w^n,$$

where h_n are the Hermite functions given by

$$h_n(z) := 2^{1/4}(2^{-n} n!)^{1/2} e^{-z^2/2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{k!\,(n-2k)!}(2z)^{n-2k},$$

with $\lfloor n/2 \rfloor$ being the integer part of $n/2$.

(c) For all $v, w \in \mathbb{C}$, we have

$$e^{vw} = \int_{\mathbb{R}} U(v,z)U(w,z)\mathrm{d}\mu(z).$$

(d) For all $w \in \mathbb{C}$, the function $z \to U(w,z) \in L^2(\mu)$, and

$$\|U(w,.)\|^2_{L^2(\mu)} = e^{|w|^2}. \tag{2.2}$$

The kernel U gives rise to an integral transform B, which is called the Segal-Bargmann transform on $\mathbb{C}$, and defined for f in $L^2(\mu)$, by

$$B(f)(w) := \int_{\mathbb{R}} U(w,z)f(z)\mathrm{d}\mu(z), \quad w \in \mathbb{C}.$$

From relations (2.1) and (2.2) the Segal-Bargmann transform satisfies the following properties.

Theorem 2.1 *(i) For $f \in L^1(\mu)$ and $w \in \mathbb{C}$ we have*

$$|B(f)(w)| \leq 2^{1/4} e^{|w|^2/2} \|f\|_{L^1(\mu)}.$$

(ii) For $f \in L^2(\mu)$ and $w \in \mathbb{C}$ we have

$$|B(f)(w)| \leq e^{|w|^2/2} \|f\|_{L^2(\mu)}.$$

The following theorem is proved in [17].

Theorem 2.2 *The Segal-Bargmann transform B is an isometric isomorphism of $L^2(\mu)$ onto $F(\mathbb{C})$. In particular, we have*

$$\|B(f)\|_{F(\mathbb{C})} = \|f\|_{L^2(\mu)}, \quad f \in L^2(\mu).$$

Let β denote the family of entire functions g such that

$$\sup_{w \in \mathbb{C}} \left\{ |g(w)| \exp(-\frac{|w|^2}{4} + \lambda |w|) \right\} < \infty,$$

for every $\lambda > 0$. Since $\{\frac{w^n}{\sqrt{n!}}\}_{n \in \mathbb{N}} \subset \beta$, the family β is dense in $F(\mathbb{C})$. The family β simplifies the presentation of the inverse operator B^{-1} (see [17]).

Theorem 2.3 *(i) If $g \in \beta$, we have*

$$B^{-1}(g)(z) = \int_{\mathbb{C}} g(w)\overline{U(w,z)}e^{-|w|^2}\mathrm{d}m(w), \quad z \in \mathbb{R}.$$

(ii) If $g \in F(\mathbb{C})$ and $z \in \mathbb{R}$, we have

$$B^{-1}(g)(z) = \lim_{n\to\infty}\int_{\mathbb{C}} g_n(w)\overline{U(w,z)}e^{-|w|^2}\mathrm{d}m(w), \quad \text{in } L^2(\mu)\text{-sense},$$

where the sequence $\{g_n\} \subset \beta$ and converges to g in $F(\mathbb{C})$.

3. Heisenberg uncertainty principle

In this section we begin by establishing local uncertainty principle for the Segal-Bargmann transform B, more precisely we will show the following theorem.

Theorem 3.1 *Let W be a measurable subset of $\mathbb{C}$ such that $0 < m(W) < \infty$, and $a > 0$. If $f \in L^2(\mu)$, then*

$$\|\chi_W B(f)\|_{L^2(\Omega)} \le \begin{cases} K_1(a)(m(W))^a \|\,|x|^a f\|_{L^2(\mu)}, & a \in (0,\frac{1}{2}), \\ K_2(a)(m(W))^{1/2}\|f\|_{L^2(\mu)}^{1-\frac{1}{2a}}\|\,|x|^a f\|_{L^2(\mu)}^{\frac{1}{2a}}, & a > 1/2, \\ 2K_1(\frac{1}{4})(m(W))^{1/4}\|f\|_{L^2(\mu)}^{1/2}\|\,|x|^{1/2}f\|_{L^2(\mu)}^{1/2}, & a = 1/2, \end{cases}$$

where

$$K_1(a) = \frac{(1-2a)^{a-1}}{2^a\pi^{a/2}a^{2a}}, \quad K_2(a) = \frac{(4\pi)^{1/4}(2a-1)^{\frac{1-2a}{4a}}}{\sqrt{\sin(\frac{\pi}{2a})}}.$$

Proof (i) Let $f \in L^2(\mu)$. The first inequality holds if $\|\,|x|^a f\|_{L^2(\mu)} = \infty$. Assume that $\|\,|x|^a f\|_{L^2(\mu)} < \infty$. By Minkowski's inequality, for all $r > 0$,

$$\begin{aligned}\|\chi_W B(f)\|_{L^2(\Omega)} &\le \|\chi_W B(f\chi_{|x|<r})\|_{L^2(\Omega)} + \|\chi_W B(f\chi_{|x|>r})\|_{L^2(\Omega)} \\ &\le \|\chi_W B(f\chi_{|x|<r})\|_{L^2(\Omega)} + \|B(f\chi_{|x|>r})\|_{F(\mathbb{C})}.\end{aligned}$$

But

$$\|\chi_W B(f\chi_{|x|<r})\|_{L^2(\Omega))} = \Big(\int_W |B(f\chi_{|x|<r})(w)|^2\mathrm{d}\Omega(w)\Big)^{1/2}.$$

By Theorem 2.1 (i), for $w \in \mathbb{C}$, we have

$$|B(f\chi_{|x|<r})(w)| \le 2^{1/4}e^{|w|^2/2}\|f\chi_{|x|<r}\|_{L^1(\mu)}.$$

Thus,

$$\|\chi_W B(f\chi_{|x|<r})\|_{L^2(\Omega)} \le 2^{1/4}(m(W))^{1/2}\|f\chi_{|x|<r}\|_{L^1(\mu)}.$$

Hence, it follows from Theorem 2.2 that

$$\|\chi_W B(f)\|_{L^2(\Omega)} \le 2^{1/4}(m(W))^{1/2}\|f\chi_{|x|<r}\|_{L^1(\mu)} + \|f\chi_{|x|>r}\|_{L^2(\mu)}.$$

On the other hand, by Hölder's inequality and hypothesis $a < 1/2$,

$$\|f\chi_{|x|<r}\|_{L^1(\mu)} \le \|\,|x|^{-a}\chi_{|x|<r}\|_{L^2(\mu)}\|\,|x|^a f\|_{L^2(\mu)}$$

$$\leq \frac{2^{1/4} r^{-a+(1/2)}}{\pi^{1/4}(1-2a)^{1/2}} \| \, |x|^a f \|_{L^2(\mu)}.$$

Moreover,

$$\| f \chi_{|x|>r} \|_{L^2(\mu)} \leq r^{-a} \| \, |x|^a f \|_{L^2(\mu)}.$$

Thus, we deduce that

$$\| \chi_W B(f) \|_{L^2(\Omega)} \leq \big[r^{-a} + \frac{2^{1/2}(m(W))^{1/2}}{\pi^{1/4}(1-2a)^{1/2}} r^{-a+(1/2)} \big] \| \, |x|^a f \|_{L^2(\mu)}.$$

We choose $r = \frac{2a^2\sqrt{\pi}}{1-2a}(m(W))^{-1}$, we obtain the first inequality.

(ii) The second inequality holds if $\|f\|_{L^2(\mu)} = \infty$ or $\| \, |x|^a f\|_{L^2(\mu)} = \infty$. Assume that $\|f\|_{L^2(\mu)} + \| \, |x|^a f\|_{L^2(\mu)} < \infty$. From [6] and hypothesis $a > 1/2$, we have

$$\|f\|_{L^1(\mu)} \leq \frac{(2\pi)^{1/4}(2a-1)^{\frac{1-2a}{4a}}}{\sqrt{\sin(\frac{\pi}{2a})}} \|f\|_{L^2(\mu)}^{1-\frac{1}{2a}} \| \, |x|^a f\|_{L^2(\mu)}^{\frac{1}{2a}}.$$

Then, according to this inequality we deduce that

$$\begin{aligned} \| \chi_W B(f) \|_{L^2(\Omega)} &\leq 2^{1/4}(m(W))^{1/2} \|f\|_{L^1(\mu)} \\ &\leq K_2(a)(m(W))^{1/2} \|f\|_{L^2(\mu)}^{1-\frac{1}{2a}} \| \, |x|^a f\|_{L^2(\mu)}^{\frac{1}{2a}}, \end{aligned}$$

which gives the second inequality.

(iii) Let $r > 0$. From the inequality $(\frac{|x|}{r})^{1/4} \leq 1 + (\frac{|x|}{r})^{1/2}$, it follows that

$$\| \, |x|^{1/4} f\|_{L^2(\mu)} \leq r^{1/4} \|f\|_{L^2(\mu)} + r^{-1/4} \| \, |x|^{1/2} f\|_{L^2(\mu)}.$$

Optimizing in r, we get

$$\| \, |x|^{1/4} f\|_{L^2(\mu)} \leq 2 \|f\|_{L^2(\mu)}^{1/2} \| \, |x|^{1/2} f\|_{L^2(\mu)}^{1/2}.$$

Thus, we deduce that

$$\begin{aligned} \| \chi_W B(f) \|_{L^2(\Omega)} &\leq K_1(\frac{1}{4})(m(W))^{1/4} \| \, |x|^{1/4} f\|_{L^2(\mu)} \\ &\leq 2K_1(\frac{1}{4})(m(W))^{1/4} \|f\|_{L^2(\mu)}^{1/2} \| \, |x|^{1/2} f\|_{L^2(\mu)}^{1/2}, \end{aligned}$$

which gives the result for $a = 1/2$. □

Remark 3.2 Let $a > 0$. If $f \in L^2(\mu)$, then

$$\|B(f)\|_{L^{\frac{2}{1-2a},2}(\Omega)} \leq K_1(a) \| \, |x|^a f\|_{L^2(\mu)}, \quad 0 < a < 1/2,$$

$$\|B(f)\|_{L^{\infty,2}(\Omega)} \leq K_2(a) \|f\|_{L^2(\mu)}^{1-\frac{1}{2a}} \| \, |x|^a f\|_{L^2(\mu)}^{\frac{1}{2a}}, \quad a > 1/2,$$

$$\|B(f)\|_{L^{4,2}(\Omega)} \leq 2K_1(\frac{1}{4}) \|f\|_{L^2(\mu)}^{1/2} \| \, |x|^{1/2} f\|_{L^2(\mu)}^{1/2}, \quad a = 1/2,$$

where $L^{s,q}(\Omega)$ is the Lorentz-space defined by the norm

$$\|f\|_{L^{s,q}(\Omega)} := \sup_{\substack{W \subset \mathbb{C} \\ 0 < m(W) < \infty}} \big((m(W))^{\frac{1}{s}-\frac{1}{q}} \| \chi_W f\|_{L^q(\Omega)} \big).$$

We shall use the local uncertainty principle (Theorem 3.1) to obtain the following Heisenberg-Pauli-Weyl uncertainty principle for the Segal-Bargmann transform B, more precisely we will show the following theorem.

Theorem 3.3 *Let $a, b > 0$. If $f \in L^2(\mu)$, then*

$$\|f\|_{L^2(\mu)} \le K(a,b) \| \, |x|^a f\|_{L^2(\mu)}^{\frac{b}{2a+b}} \| \, |w|^b B(f)\|_{L^2(\Omega)}^{\frac{2a}{2a+b}},$$

where

$$K(a,b) = \begin{cases} \left[(\frac{b}{2a})^{\frac{2a}{2a+b}} + (\frac{2a}{b})^{\frac{b}{2a+b}} \right]^{1/2} (K_1(a))^{\frac{b}{2a+b}}, & a \in (0, \frac{1}{2}), \ b > 0, \\ \left[b^{\frac{1}{b+1}} + b^{-\frac{b}{b+1}} \right]^{\frac{ab+a}{2a+b}} (K_2(a))^{\frac{2ab}{2a+b}}, & a > 1/2, \ b > 0, \\ \left[(2b)^{\frac{1}{2b+1}} + (2b)^{-\frac{2b}{2b+1}} \right]^{\frac{2b+1}{2b+2}} (2K_1(\frac{1}{4}))^{\frac{2b}{b+1}}, & a = 1/2, \ b > 0. \end{cases}$$

Here $K_1(a)$ and $K_2(a)$ are the constants given by Theorem 3.1.

Proof (i) Let $0 < a < 1/2$, $b > 0$ and $r > 0$. Then

$$\|B(f)\|_{F(\mathbb{C})}^2 = \int_{|w|<r} |B(f)(w)|^2 \mathrm{d}\Omega(w) + \int_{|w|>r} |B(f)(w)|^2 \mathrm{d}\Omega(w).$$

Firstly,

$$\int_{|w|>r} |B(f)(w)|^2 \mathrm{d}\Omega(w) \le r^{-2b} \| \, |w|^b B(f)\|_{L^2(\Omega)}^2.$$

From Theorem 3.1, we get

$$\int_{|w|<r} |B(f)(w)|^2 \mathrm{d}\Omega(w) \le (K_1(a))^2 r^{4a} \| \, |x|^a f\|_{L^2(\mu)}^2.$$

Combining the precedent relations, by Theorem 2.2 we obtain

$$\|f\|_{L^2(\mu)}^2 \le (K_1(a))^2 r^{4a} \| \, |x|^a f\|_{L^2(\mu)}^2 + r^{-2b} \| \, |w|^b B(f)\|_{L^2(\Omega)}^2.$$

Choosing $r = (\frac{b}{2a})^{\frac{1}{4a+2b}} (\frac{\| \, |w|^b B(f)\|_{L^2(\Omega)}}{K_1(a) \| \, |x|^a f\|_{L^2(\mu)}})^{\frac{1}{2a+b}}$, we get the first inequality.

(ii) Let $a > 1/2$, $b > 0$ and $r > 0$. From Theorem 3.1, we get

$$\int_{|w|<r} |B(f)(w)|^2 \mathrm{d}\Omega(w) \le (K_2(a))^2 r^2 \|f\|_{L^2(\mu)}^{2-(1/a)} \| \, |x|^a f\|_{L^2(\mu)}^{1/a}.$$

Thus,

$$\|B(f)\|_{F(\mathbb{C})}^2 \le (K_2(a))^2 r^2 \|f\|_{L^2(\mu)}^{2-(1/a)} \| \, |x|^a f\|_{L^2(\mu)}^{1/a} + r^{-2b} \| \, |w|^b B(f)\|_{L^2(\Omega)}^2.$$

Choosing $r = (\frac{b \| \, |w|^b B(f)\|_{L^2(\Omega)}^2}{(K_2(a))^2 \|f\|_{L^2(\mu)}^{2-(1/a)} \| \, |x|^a f\|_{L^2(\mu)}^{1/a}})^{\frac{1}{2b+2}}$, we get the second inequality.

(iii) Let $a = 1/2$, $b > 0$ and $r > 0$. From Theorem 3.1, we get

$$\int_{|w|<r} |B(f)(w)|^2 \mathrm{d}\Omega(w) \le (2K_1(\frac{1}{4}))^2 r \|f\|_{L^2(\mu)} \| \, |x|^{1/2} f\|_{L^2(\mu)}.$$

Therefore,

$$\|B(f)\|_{F(\mathbb{C})}^2 \le (2K_1(\frac{1}{4}))^2 r \|f\|_{L^2(\mu)} \| \, |x|^{1/2} f\|_{L^2(\mu)} + r^{-2b} \| \, |w|^b B(f)\|_{L^2(\Omega)}^2.$$

Choosing $r = (\frac{b \| \, |w|^b B(f)\|_{L^2(\Omega)}^2}{2(K_1(\frac{1}{4}))^2 \|f\|_{L^2(\mu)} \| \, |x|^{1/2} f\|_{L^2(\mu)}})^{\frac{1}{2b+1}}$, we get the third inequality. □

Remark 3.4 Let $f \in L^2(\mu)$. In particular case $(a = b = 1)$, we obtain the following Heisenberg's inequality for the Segal-Bargmann transform B,

$$\|f\|^3_{L^2(\mu)} \le 8\sqrt{\pi}\| \, |x|f\|_{L^2(\mu)}\| \, |w|B(f)\|^2_{L^2(\Omega)}. \tag{3.1}$$

Let Λ be all $f \in L^2(\mu)$ such that

$$\Delta_1(f) = \frac{\| \, |x|f\|_{L^2(\mu)}}{\|f\|_{L^2(\mu)}}, \quad \Delta_2(f) = \frac{\| \, |w|B(f)\|_{L^2(\Omega)}}{\|f\|_{L^2(\mu)}}.$$

We obtain a characterization of the region of Heisenberg's inequality (see Figure 1),

$$\{(\Delta_1(f), \Delta_2(f)), f \in \Lambda\} \subset \{(x, y), x, y > 0, xy^2 \ge \frac{1}{8\sqrt{\pi}}\}.$$

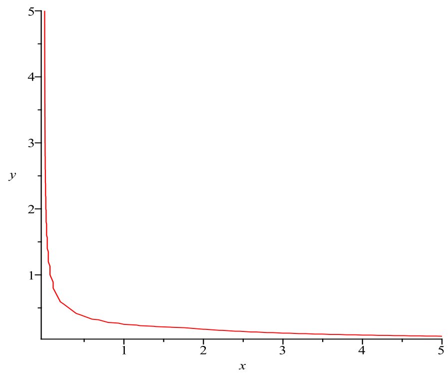

Figure 1 Region of the concentrated Heisenberg's inequality (3.1)

Theorem 3.5 (Nash-type inequality) *Let $b > 0$. If $f \in L^2(\mu)$, then*

$$\|f\|_{L^2(\mu)} \le D(b)\| \, |w|^b B(f)\|_{L^2(\Omega)},$$

where

$$D(b) = \left[b^{\frac{1}{b+1}} + b^{-\frac{b}{b+1}}\right]^{\frac{b+1}{2}}.$$

Proof Let $f \in L^2(\mu)$, $b > 0$ and $r > 0$. Then

$$\|B(f)\|^2_{F(\mathbb{C})} = \int_{|w|<r} |B(f)(w)|^2 \mathrm{d}\Omega(w) + \int_{|w|>r} |B(f)(w)|^2 \mathrm{d}\Omega(w).$$

Firstly,

$$\int_{|w|>r} |B(f)(w)|^2 \mathrm{d}\Omega(w) \le r^{-2b}\| \, |w|^b B(f)\|^2_{L^2(\Omega)}.$$

From Theorem 2.1 (ii), we get

$$\int_{|w|<r} |B(f)(w)|^2 \mathrm{d}\Omega(w) \le r^2\|f\|^2_{L^2(\mu)}.$$

Combining the precedent relations, by Theorem 2.2 we obtain

$$\|f\|^2_{L^2(\mu)} \le r^2\|f\|^2_{L^2(\mu)} + r^{-2b}\| \, |w|^b B(f)\|^2_{L^2(\Omega)}.$$

Choosing $r = b^{\frac{1}{2b+2}}\left(\frac{\| \, |w|^b B(f)\|_{L^2(\Omega)}}{\|f\|_{L^2(\mu)}}\right)^{\frac{1}{b+1}}$, we get the desired inequality. □

4. Heisenberg principle involving L^1 and L^2-norms

In this section we will establish uncertainty principles for the Segal-Bargmann transform B involving L^1 and L^2-norms. More precisely we will show the following theorems.

Theorem 4.1 (Clarkson-type inequality) *Let $a > 0$. If $f \in L^1 \cap L^2(\mu)$, then*

$$\|f\|_{L^1(\mu)} \le D_1(a)\|f\|_{L^2(\mu)}^{\frac{2a}{2a+1}} \| \, |x|^a f\|_{L^1(\mu)}^{\frac{1}{2a+1}},$$

where

$$D_1(a) = (\frac{2}{\pi})^{\frac{a}{2(2a+1)}} \big[(2a)^{\frac{1}{2a+1}} + (2a)^{-\frac{2a}{2a+1}}\big].$$

Proof Let $f \in L^1 \cap L^2(\mu)$ and $r > 0$. Then

$$\|f\|_{L^1(\mu)} = \|f\chi_{|x|<r}\|_{L^1(\mu)} + \|(1-\chi_{|x|<r})f\|_{L^1(\mu)}.$$

Firstly, $\|(1-\chi_{|x|<r})f\|_{L^1(\mu)} \le r^{-a}\| \, |x|^a f\|_{L^1(\mu)}$. By Hölder's inequality, we get

$$\|f\chi_{|x|<r}\|_{L^1(\mu)} \le (\mu(|x|<r))^{1/2}\|f\|_{L^2(\mu)} \le \big(r\sqrt{\frac{2}{\pi}}\big)^{1/2}\|f\|_{L^2(\mu)}.$$

Combining the precedent relations, we obtain

$$\|f\|_{L^1(\mu)} \le \big(r\sqrt{\frac{2}{\pi}}\big)^{1/2}\|f\|_{L^2(\mu)} + r^{-a}\| \, |x|^a f\|_{L^1(\mu)}.$$

Choosing $r = (\frac{2a\| \, |x|^a f\|_{L^1(\mu)}}{(\frac{2}{\pi})^{1/4}\|f\|_{L^2(\mu)}})^{\frac{2}{2a+1}}$, we get the desired inequality. □

Theorem 4.2 (Nash-type inequality) *Let $b > 0$. If $f \in L^1 \cap L^2(\mu)$, then*

$$\|f\|_{L^2(\mu)} \le D_2(b)\|f\|_{L^1(\mu)}^{\frac{b}{b+1}} \| \, |w|^b B(f)\|_{L^2(\Omega)}^{\frac{1}{b+1}},$$

where

$$D_2(b) = \big[\sqrt{2}(\frac{b}{\sqrt{2}})^{\frac{1}{b+1}} + (\frac{b}{\sqrt{2}})^{-\frac{b}{b+1}}\big]^{1/2}.$$

Proof Let $f \in L^1 \cap L^2(\mu)$, $b > 0$ and $r > 0$. Then

$$\|B(f)\|_{F(\mathbb{C})}^2 = \int_{|w|<r} |B(f)(w)|^2 \mathrm{d}\Omega(w) + \int_{|w|>r} |B(f)(w)|^2 \mathrm{d}\Omega(w).$$

Firstly,

$$\int_{|w|>r} |B(f)(w)|^2 \mathrm{d}\Omega(w) \le r^{-2b}\| \, |w|^b B(f)\|_{L^2(\Omega)}^2.$$

From Theorem 2.1 (i), we get

$$\int_{|w|<r} |B(f)(w)|^2 \mathrm{d}\Omega(w) \le \sqrt{2}r^2\|f\|_{L^1(\mu)}^2.$$

Combining the precedent relations, by Theorem 2.2 we obtain

$$\|f\|_{L^2(\mu)}^2 \le \sqrt{2}r^2\|f\|_{L^1(\mu)}^2 + r^{-2b}\| \, |w|^b B(f)\|_{L^2(\Omega)}^2.$$

Choosing $r = (\frac{b}{\sqrt{2}})^{\frac{1}{2b+2}}(\frac{\| \, |w|^b B(f)\|_{L^2(\Omega)}}{\|f\|_{L^1(\mu)}})^{\frac{1}{b+1}}$, we get the desired inequality. □

By combining the Clarkson-type inequality (Theorem 4.1) and the Nash-type inequality (Theorem 4.2) we obtain the following uncertainty inequality of Heisenberg-type.

Theorem 4.3 *Let $a, b > 0$. If $f \in L^1 \cap L^2(\mu)$, then*

(i) $\|f\|_{L^1(\mu)}^{\frac{1}{2b+1}} \|f\|_{L^2(\mu)}^{\frac{1}{2a+1}} \le C_1 \| \, |x|^a f\|_{L^1(\mu)}^{\frac{1}{2a+1}} \| \, |w|^{2b} B(f)\|_{L^2(\Omega)}^{\frac{1}{2b+1}}$, *where* $C_1 = D_1(a) D_2(2b)$.

(ii) $\|f\|_{L^2(\mu)} \le C_2 \| \, |x|^a f\|_{L^1(\mu)}^{\frac{2b}{2a+2b+1}} \| \, |w|^{2b} B(f)\|_{L^2(\Omega)}^{\frac{2a+1}{2a+2b+1}}$, *where*

$$C_2 = (D_1(a))^{\frac{2b(2a+1)}{2a+2b+1}} (D_2(2b))^{\frac{(2a+1)(2b+1)}{2a+2b+1}}.$$

(iii) $\|f\|_{L^1(\mu)} \le C_3 \| \, |x|^a f\|_{L^1(\mu)}^{\frac{2b+1}{2a+2b+1}} \| \, |w|^{2b} B(f)\|_{L^2(\Omega)}^{\frac{2a}{2a+2b+1}}$, *where*

$$C_3 = (D_1(a))^{\frac{(2a+1)(2b+1)}{2a+2b+1}} (D_2(2b))^{\frac{2a(2b+1)}{2a+2b+1}}.$$

Remark 4.4 The uncertainty principles given by Theorem 2.4, are the analogs of the results obtained by Laeng-Morpurgo [10] and Morpurgo [11] for the Fourier transform, and the results obtained by Ghobber [22] for the Dunkl transform. In particular case, if $a = b$, we obtain the following Heisenberg's inequalities for the Segal-Bargmann transform B.

(i) For $a > 0$ and $f \in L^1 \cap L^2(\mu)$:

$$\|f\|_{L^1(\mu)} \|f\|_{L^2(\mu)} \le C \| \, |x|^a f\|_{L^1(\mu)} \| \, |w|^{2a} B(f)\|_{L^2(\Omega)},$$

where $C = (D_1(a) D_2(2a))^{2a+1}$. If $a = b = 1$:

$$\|f\|_{L^1(\mu)} \|f\|_{L^2(\mu)} \le \frac{(\sqrt{3})^9}{4\sqrt{\pi}} \| \, |x| f\|_{L^1(\mu)} \| \, |w|^2 B(f)\|_{L^2(\Omega)}. \quad (4.1)$$

Let Λ be all $f \in L^1 \cap L^2(\mu)$ such that

$$\Delta_1(f) = \frac{\| \, |x| f\|_{L^1(\mu)}}{\|f\|_{L^1(\mu)}}, \quad \Delta_2(f) = \frac{\| \, |w|^2 B(f)\|_{L^2(\Omega)}}{\|f\|_{L^2(\mu)}}.$$

We obtain a characterization of the region of Heisenberg's inequality (see Figure 2),

$$\{(\Delta_1(f), \Delta_2(f)), f \in \Lambda\} \subset \{(x, y), x, y > 0, xy \ge \frac{4\sqrt{\pi}}{(\sqrt{3})^9}\}.$$

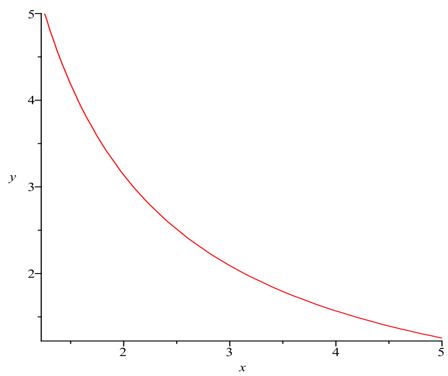

Figure 2 Region of the concentrated Heisenberg's inequality (4.1)

(ii) For $a = b = 1$ and $f \in L^1 \cap L^2(\mu)$:

$$\|f\|_{L^2(\mu)}^5 \le \frac{(\sqrt{2})^3 (\sqrt{3})^{21}}{\pi} \| \, |x| f\|_{L^1(\mu)}^2 \| \, |w|^2 B(f)\|_{L^2(\Omega)}^3. \quad (4.2)$$

Let Λ be all $f \in L^1 \cap L^2(\mu)$ such that

$$\Delta_1(f) = \frac{\| \, |x| f \|_{L^1(\mu)}}{\|f\|_{L^2(\mu)}}, \quad \Delta_2(f) = \frac{\| \, |w|^2 B(f) \|_{L^2(\Omega)}}{\|f\|_{L^2(\mu)}}.$$

We obtain a characterization of the region of Heisenberg's inequality (see Figure 3),

$$\{(\Delta_1(f), \Delta_2(f)), f \in \Lambda\} \subset \{(x, y), x, y > 0, x^2 y^3 \geq \frac{\pi}{(\sqrt{2})^3 (\sqrt{3})^{21}}\}.$$

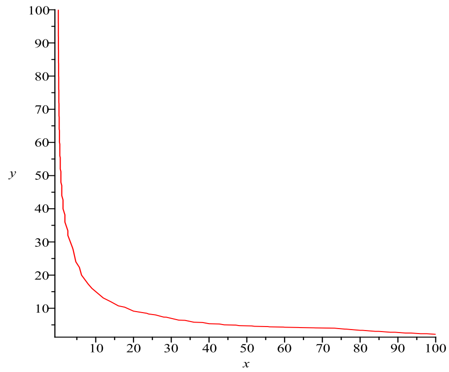

Figure 3 Region of the concentrated Heisenberg's inequality (4.2)

(iii) For $a = b = 1$ and $f \in L^1 \cap L^2(\mu)$:

$$\|f\|^5_{L^2(\mu)} \leq \frac{3^{12}}{\sqrt{\pi}(\sqrt{2})^{11}} \| \, |x| f \|^3_{L^1(\mu)} \| \, |w|^2 B(f) \|^2_{L^2(\Omega)}. \tag{4.3}$$

Let Λ be all $f \in L^1 \cap L^2(\mu)$ such that

$$\Delta_1(f) = \frac{\| \, |x| f \|_{L^1(\mu)}}{\|f\|_{L^2(\mu)}}, \quad \Delta_2(f) = \frac{\| \, |w|^2 B(f) \|_{L^2(\Omega)}}{\|f\|_{L^2(\mu)}}.$$

We obtain a characterization of the region of Heisenberg's inequality (see Figure 4),

$$\{(\Delta_1(f), \Delta_2(f)), f \in \Lambda\} \subset \{(x, y), x, y > 0, x^3 y^2 \geq \frac{\sqrt{\pi}(\sqrt{2})^{11}}{3^{12}}\}.$$

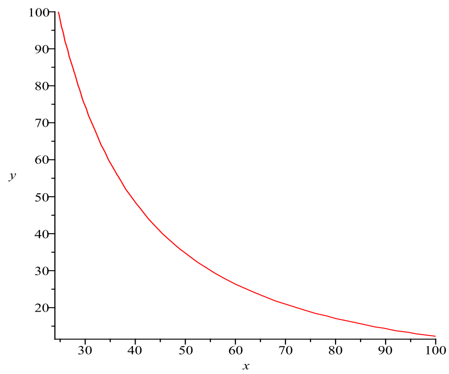

Figure 4 Region of the concentrated Heisenberg's inequality (4.3)

5. Donoho-Stark uncertainty principle

Let E be a measurable subset of $\mathbb{R}$. We say that a function $f \in L^p(\mu)$, $p = 1, 2$, is ε-concentrated on E in $L^p(\mu)$-norm, if

$$\|f - \chi_E f\|_{L^p(\mu)} \leq \varepsilon \|f\|_{L^p(\mu)}. \tag{5.1}$$

Let W be a measurable subset of $\mathbb{C}$ and let $f \in L^2(\mu)$. We say that $B(f)$ is η-concentrated on W in $L^2(\Omega)$-norm, if

$$\|B(f) - \chi_W B(f)\|_{L^2(\Omega)} \leq \eta \|B(f)\|_{F(\mathbb{C})}. \tag{5.2}$$

The Donoho-Stark uncertainty principle for the Segal-Bargmann transform B is given by the following theorem.

Theorem 5.1 (Donoho-Stark-type inequality) *Let E be a measurable subset of $\mathbb{R}$, W be a measurable subset of $\mathbb{C}$ and $f \in L^2(\mu)$. If f is ε-concentrated on E in $L^2(\mu)$-norm, $B(f)$ is η-concentrated on W in $L^2(\Omega)$-norm and $\varepsilon + \eta < 1$, then*

$$\mu(E)(m(W) \geq \frac{1}{\sqrt{2}}(1 - \eta - \varepsilon)^2. \tag{5.3}$$

Proof Let $f \in L^2(\mu)$. Assume that $\mu(E) < \infty$ and $m(W) < \infty$. From (5.1), (5.2) and Theorem 2.2 it follows that

$$\begin{aligned}\|B(f) - \chi_W B(\chi_E f)\|_{L^2(\Omega)} &\leq \|B(f) - \chi_W B(f)\|_{L^2(\Omega)} + \|\chi_W B(f - \chi_E f)\|_{L^2(\Omega)} \\ &\leq \eta \|B(f)\|_{F(\mathbb{C})} + \|B(f - \chi_E f)\|_{F(\mathbb{C})} \\ &\leq (\eta + \varepsilon)\|f\|_{L^2(\mu)}.\end{aligned}$$

Then the triangle inequality shows that

$$\begin{aligned}\|B(f)\|_{F(\mathbb{C})} &\leq \|\chi_W B(\chi_E f)\|_{L^2(\Omega)} + \|B(f) - \chi_W B(\chi_E f)\|_{L^2(\Omega)} \\ &\leq \|\chi_W B(\chi_E f)\|_{L^2(\Omega)} + (\eta + \varepsilon)\|f\|_{L^2(\mu)}.\end{aligned}$$

But

$$\|\chi_W B(\chi_E f)\|_{L^2(\Omega)} = \Big(\int_W |B(\chi_E f)(w)|^2 \mathrm{d}\Omega(w)\Big)^{1/2}.$$

By Theorem 2.1 (i) and Hölder's inequality we have

$$|B(\chi_E f)(w)| \leq 2^{1/4} e^{|w|^2/2} \|\chi_E f\|_{L^1(\mu)} \leq 2^{1/4} e^{|w|^2/2} \|f\|_{L^2(\mu)} (\mu(E))^{1/2}.$$

Thus, $\|\chi_W B(\chi_E f)\|_{L^2(\Omega)} \leq 2^{1/4}(\mu(E))^{1/2}(m(W))^{1/2}\|f\|_{L^2(\mu)}$, and

$$\|B(f)\|_{F(\mathbb{C})} \leq 2^{1/4}(\mu(E))^{1/2}(m(W))^{1/2}\|f\|_{L^2(\mu)} + (\eta + \varepsilon)\|f\|_{L^2(\mu)}.$$

By applying Theorem 2.2, we obtain $(\mu(E))^{1/2}(m(W))^{1/2} \geq 2^{-1/4}(1 - \eta - \varepsilon)$, which gives the desired result. □

Remark 5.2 Let $\Delta_1(E) = \mu(E)$, $\Delta_2(W) = m(W)$. From Theorem 5.1, we obtain a characterization of the region of Donoho-Stark's inequality (see Figure 5),

$$\big\{(\Delta_1(E), \Delta_2(W)), E \subset \mathbb{R}, W \subset \mathbb{C}\big\} \subset \big\{(x, y), x, y > 0, xy \geq \frac{1}{\sqrt{2}}(1 - \eta - \varepsilon)^2\big\}.$$

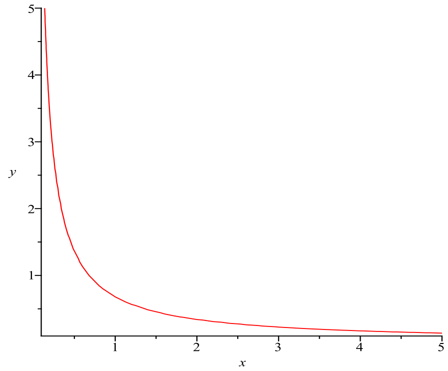

Figure 5 Region of the concentrated Donoho-Stark's inequality (5.3)

Theorem 5.3 (Donoho-Stark-type inequality) *Let E be a measurable subset of $\mathbb{R}$, W be a measurable subset of $\mathbb{C}$ and $f \in L^1 \cap L^2(\mu)$. If f is ε-concentrated on E in $L^1(\mu)$-norm, $B(f)$ is η-concentrated on W in $L^2(\Omega)$-norm and $\varepsilon, \eta < 1$, then $\mu(E)m(W) \geq \frac{1}{\sqrt{2}}(1-\varepsilon)^2(1-\eta)^2$.*

Proof Let $f \in L^1 \cap L^2(\mu)$. Assume that $\mu(E) < \infty$ and $m(W) < \infty$. From (5.1) we have

$$\begin{aligned} \|f\|_{L^1(\mu)} &\leq \varepsilon \|f\|_{L^1(\mu)} + \|\chi_E f\|_{L^1(\mu)} \\ &\leq \varepsilon \|f\|_{L^1(\mu)} + (\mu(E))^{1/2} \|f\|_{L^2(\mu)}. \end{aligned}$$

Thus

$$\|f\|_{L^1(\mu)} \leq \frac{(\mu(E))^{1/2}}{1-\varepsilon} \|f\|_{L^2(\mu)}. \tag{5.4}$$

On the other hand, from (5.2) it follows that

$$\begin{aligned} \|B(f)\|_{L^2(m)} &\leq \|B(f) - \chi_W B(f)\|_{L^2(\Omega)} + \|\chi_W B(f)\|_{L^2(\Omega)} \\ &\leq \eta \|B(f)\|_{F(\mathbb{C})} + 2^{1/4}(m(W))^{1/2} \|f\|_{L^1(\mu)}. \end{aligned}$$

Thus by Theorem 2.2,

$$\|f\|_{L^2(\mu)} \leq \frac{2^{1/4}(m(W))^{1/2}}{1-\eta} \|f\|_{L^1(\mu)}. \tag{5.5}$$

Combining (5.4) and (5.5) we obtain the result of this theorem. □

Remark 5.4 The uncertainty principles given by Theorems 5.1 and 5.3, are the analogs of the results obtained by Donoho-Stark [7] for the Fourier transform, by Kawazoe-Mejjaoli [23] for the Dunkl transform and by Soltani [24] for the Sturm-Liouville transform.

In the following we establish uncertainty inequality of Matolcsi-Szücs-type for the Segal-Bargmann transform B.

Theorem 5.5 (Matolcsi-Szücs-type inequality) *Let $f \in L^1 \cap L^2(\mu)$. If $A_f = \{x \in \mathbb{R} : f(x) \neq 0\}$ and $A_{B(f)} = \{w \in \mathbb{C} : B(f)(w) \neq 0\}$, then $\mu(A_f)m(A_{B(f)}) \geq \frac{1}{\sqrt{2}}$.*

Proof Let $f \in L^1 \cap L^2(\mu)$. We put $W = A_{B(f)}$, then by Theorem 2.1 (i) and Hölder's inequality we obtain

$$\|B(f)\|_{L^2(\Omega)} = \|\chi_W B(f)\|_{L^2(\Omega)} \leq 2^{1/4}(m(W))^{1/2} \|f\|_{L^1(\mu)}$$

$$\leq 2^{1/4}(m(W))^{1/2}(\mu(A_f))^{1/2}\|f\|_{L^2(\mu)}.$$

Then Theorem 2.2 gives the desired result. □

Remark 5.6 The uncertainty principle given by Theorem 5.5 is the analogs of the result obtained by Matolcsi-Szücs [9] and Benedicks [8] for the Fourier transform.

Conflict of Interests The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] A. BONAMI, B. DEMANGE, P. JAMING. *Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms.* Rev. Mat. Iberoamericana, 2003, **19**(1): 23–55.

[2] M. COWLING, J. F. PRICE. *Bandwidth versus time concentration: the Heisenberg-Pauli-Weyl inequality.* SIAM J. Math. Anal., 1984, **15**(1): 151–165.

[3] I. I. HIRSCHMAN. *A note on entropy.* Amer. J. Math., 1957, **79**: 152–156.

[4] W. G. FARIS. *Inequalities and uncertainty inequalities.* J. Mathematical Phys., 1978, **19**(2): 461–466.

[5] J. F. PRICE. *Inequalities and local uncertainty principles.* J. Math. Phys., 1983, **24**(7): 1711–1714.

[6] J. F. PRICE. *Sharp local uncertainty principles.* Studia Math., 1987, **85**: 37–45.

[7] D. L. DONOHO, P. B. STARK. *Uncertainty principles and signal recovery.* SIAM J. Appl. Math., 1989, **49**(3): 906–931.

[8] M. BENEDICKS. *On Fourier transforms of functions supported on sets of finite Lebesgue measure.* J. Math. Anal. Appl., 1985, **106**(1): 180–183.

[9] T. MATOLCSI, J. SZÜCS. *Intersection des mesures spectrales conjuguées.* C. R. Acad. Sci. Paris Sér. A-B, 1973, **277**: A841–A843.

[10] E. LAENG, C. MORPURGO. *An uncertainty inequality involving $L1$-norms.* Proc. Amer. Math. Soc., 1999, **127**(12): 3565–3572.

[11] C. MORPURGO. *Extremals of some uncertainty inequalities.* Bull. London Math. Soc., 2001, **33**(1): 52–58.

[12] G. B. FOLLAND, A. SITARAM. *The uncertainty principle; A Mathematical Survey.* J. Fourier Anal. Appl., 1997, **3**(3): 207–238.

[13] A. NEMRI, F. SOLTANI. *L^p uncertainty principles for the Fourier transform with numerical aspect.* Appl. Anal., 2016, **95**(5): 931–943.

[14] F. SOLTANI, A. NEMRI. *Analytical and numerical approximation formulas for the Fourier multiplier operators.* Complex Anal. Oper. Theory, 2015, **9**(1): 121–138.

[15] F. SOLTANI, A. NEMRI. *Analytical and numerical applications for the Fourier multiplier operators on $\mathbb{R}^n \times (0,\infty)$.* Appl. Anal., 2015, **94**(8): 1545–1560.

[16] C. A. BERGER, L. A. COBURN. *Toeplitz operators on the Segal-Bargmann space.* Trans. Amer. Math. Soc., 1987, **301**(2): 813–829.

[17] V. BARGMANN. *On a Hilbert space of analytic functions and an associated integral transform.* Comm. Pure Appl. Math., 1961, **14**: 187–214.

[18] F. SOLTANI. *Toeplitz and translation operators on the q-Fock spaces.* Adv. Pure Math., 2011, **1**: 325–333.

[19] F. SOLTANI. *Operators and Tikhonov regularization on the Fock space.* Int. Trans. Spec. Funct., 2014, **25**(4): 283–294.

[20] Kehe ZHU. *Uncertainty principles for the Fock space.* Preprint 2015, arXiv: 1501.02754V1.

[21] S. GHOBBER, P. JAMING. *Uncertainty principles for integral operators.* Studia Math., 2014, **220**(3): 197–220.

[22] S. GHOBBER. *Uncertainty principles involving L^1-norms for the Dunkl transform.* Int. Trans. Spec. Funct., 2013, **24**(6): 491–501.

[23] T. KAWAZOE, H. MEJJAOLI. *Uncertainty principles for the Dunkl transform.* Hiroshima Math. J., 2010, **40**(2): 241–268.

[24] F. SOLTANI. *Donoho-Stark uncertainty principle associated with a singular second-order differential operator.* Int. J. Anal. Appl., 2014, **4**(1): 1–10.