Journal of Mathematical Research with Applications
Sept., 2017, Vol. 37, No. 5, pp. 603–612
DOI:10.3770/j.issn:2095-2651.2017.05.010
Http://jmre.dlut.edu.cn

Lie Weak Amenability of Triangular Banach Algebra

Lin CHEN[1,*], **Fangyan LU**[2]
1. *Department of Mathematics and Physics, Anshun University, Guizhou* 561000*, P. R. China;*
2. *Department of Mathematics, Soochow University, Jiangsu* 215006*, P. R. China*

Abstract Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebra and $\mathcal{M}$ be Banach $\mathcal{A},\mathcal{B}$-module. Then $\mathcal{T}=\begin{pmatrix}\mathcal{A} & \mathcal{M}\\ & \mathcal{B}\end{pmatrix}$ becomes a triangular Banach algebra when equipped with the Banach space norm $\|\begin{pmatrix}a & m\\ & b\end{pmatrix}\|=\|a\|_{\mathcal{A}}+\|m\|_{\mathcal{M}}+\|b\|_{\mathcal{B}}$. A Banach algebra $\mathcal{T}$ is said to be Lie n-weakly amenable if all Lie derivations from $\mathcal{T}$ into its n^{th} dual space $\mathcal{T}^{(n)}$ are standard. In this paper we investigate Lie n-weak amenability of a triangular Banach algebra $\mathcal{T}$ in relation to that of the algebras $\mathcal{A},\mathcal{B}$ and their action on the module $\mathcal{M}$.

Keywords triangular Banach algebra; weak amenability; Lie derivation

MR(2010) Subject Classification 47B47; 47B49

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and suppose $\mathcal{M}$ is a Banach $\mathcal{A},\mathcal{B}$-module; that is, $\mathcal{M}$ is a Banach space, a left $\mathcal{A}$-module and a right $\mathcal{B}$-module and the action of $\mathcal{A}$ and $\mathcal{B}$ are jointly continuous. Let $\mathcal{M}^*$ be the dual space of $\mathcal{M}$. We can define a right action of $\mathcal{A}$ on $\mathcal{M}^*$ and a left action on $\mathcal{M}^*$ via

$$\langle m, f\cdot a\rangle=\langle am, f\rangle,\quad \langle m, b\cdot f\rangle=\langle mb, f\rangle$$

for all $a\in\mathcal{A}$, $b\in\mathcal{B}$ and $f\in\mathcal{M}^*$. Then $\mathcal{M}^*$ is a Banach $\mathcal{B},\mathcal{A}$-module. Similarly, the second dual $\mathcal{M}^{**}$ of $\mathcal{M}$ becomes a Banach $\mathcal{A},\mathcal{B}$-module under the actions

$$\langle f, a\cdot\phi\rangle=\langle f\cdot a,\phi\rangle,\quad \langle f,\phi\cdot b\rangle=\langle b\cdot f,\phi\rangle$$

for all $a\in\mathcal{A}$, $b\in\mathcal{B}$ and $f\in\mathcal{M}^*$ and $\phi\in\mathcal{M}^{**}$. By continuing this process to higher order dual space of $\mathcal{M}$, it is seen that for each $n\ge 1$, $\mathcal{M}^{(2n)}$ is a Banach $\mathcal{A},\mathcal{B}$-module and $\mathcal{M}^{(2n-1)}$ is a Banach $\mathcal{B},\mathcal{A}$-module. Let $\mathcal{M}$ be a Banach $\mathcal{A}$-bimodule, a derivation is a continuous linear map $\delta:\mathcal{A}\to\mathcal{M}$ such that $\delta(ab)=\delta(a)\cdot b+a\cdot\delta(b)$ holds for all $a,b\in\mathcal{A}$. If $m\in\mathcal{M}$, we define $\delta_m(a)=a\cdot x-x\cdot a$ for each $a\in\mathcal{A}$. Then δ_m is a derivation. Such derivations are called inner derivations. We use the notation $Z^1(\mathcal{A},\mathcal{M})$ for the space of all continuous derivations from $\mathcal{A}$ into $\mathcal{M}$, and $N^1(\mathcal{A},\mathcal{M})$ for the space of all inner derivations from $\mathcal{A}$ into $\mathcal{M}$. The quotient space $H^1(\mathcal{A},\mathcal{X})=Z^1(\mathcal{A},\mathcal{M})/N^1(\mathcal{A},\mathcal{M})$ is called the first Hochschild cohomology group of $\mathcal{A}$ with

Received November 2, 2016; Accepted May 17, 2017
Supported by the National Natural Science Foundation of China (Grant Nos. 11171244; 11601010).
* Corresponding author
E-mail address: linchen198112@163.com (Lin CHEN); fylu@suda.edu.cn (Fanyan LU)

coefficients in $\mathcal{M}$. The Banach algebra $\mathcal{A}$ is amenable if $H^1(\mathcal{A},\mathcal{M}^*)=\{0\}$ for every Banach $\mathcal{A}$-bimodule $\mathcal{M}$. This notion was first defined by Johnson in [1], who proved that the amenability of a locally compact group G is equivalent to amenability of the group algebra $L^1(G)$. Haagerup [2] proved that all nuclear C^*-algebras are amenable. Later, Bade, Curtis and Dales [3] introduced the concept of weak amenability, who showed that a commutative Banach algebra is weakly amenable if and only if every bounded derivation $\delta:\mathcal{A}\to\mathcal{A}^*$ is inner and hence zero. This motivated Johnson to introduce the notion of weak amenability for general Banach algebras and proved the group algebra $L^1(G)$ is weakly amenable for every locally compact group G in [4]. More recently, Dales et al. [5] defined the concept of n-weak amenability and permanent weak amenability for Banach algebra $\mathcal{A}$ as follows: for every positive integer n, Banach algebra $\mathcal{A}$ is said to be n-weakly amenable if $H^1(\mathcal{A},\mathcal{A}^{(n)})=\{0\}$ and $\mathcal{A}$ is permanently weakly amenable if it is n-weakly amenable for all $n\geq 1$, and showed every C^*-algebra is permanently weakly amenable. Zhang [6] investigated n-weak amenability of module extension Banach algebra.

Besides derivation on operator algebra, we also consider Lie derivation. A Lie derivation is a continuous linear map $\delta:\mathcal{A}\to\mathcal{M}$ such that $\delta([a,b])=[\delta(a),b]+[a,\delta(b)]$ holds for all $a,b\in\mathcal{A}$, here $[a,b]=ab-ba$ is the usual Lie product. A Lie derivation δ is standard if it can be decomposed as $\delta=d+\tau$, where d is a derivation from $\mathcal{A}$ into $\mathcal{M}$ and τ is a linear map from $\mathcal{A}$ into the center of $\mathcal{M}$ vanishing on each commutator. In [7], Alaminos, Brešar and Villena showed that all Lie derivations from a von Neumann algebra $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule are standard. Cheung [8] investigated the Lie derivation from triangular Banach algebra into itself. Lu [9] showed every Lie derivation from nest algebra into itself is standard. Naturally, for every positive integer n, we can define Lie n-weak amenability of Banach algebra $\mathcal{A}$, $\mathcal{A}$ is said to be Lie n-weakly amenable if every Lie derivation from $\mathcal{A}$ into $\mathcal{A}^{(n)}$ is standard.

Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\mathcal{M}$ be a Banach $\mathcal{A},\mathcal{B}$-module. We define

$$\mathcal{T}=\begin{pmatrix}\mathcal{A} & \mathcal{M}\\ & \mathcal{B}\end{pmatrix}.$$

Then $\mathcal{T}$ is a complex algebra with usual multiplication and addition actions in the space of 2×2 matrices. $\mathcal{T}$ becomes a Banach algebra with the following norm for every $a\in\mathcal{A}$, $b\in\mathcal{B}$ and $m\in\mathcal{M}$

$$\left\|\begin{pmatrix}a & m\\ & b\end{pmatrix}\right\|:=\|a\|_{\mathcal{A}}+\|m\|_{\mathcal{M}}+\|b\|_{\mathcal{B}}.$$

This algebra $\mathcal{T}$ is called the triangular Banach algebra. Let $t=\left(\begin{smallmatrix}a & m\\ & b\end{smallmatrix}\right)\in\mathcal{T}$ and $\tau=\left(\begin{smallmatrix}f & h\\ & g\end{smallmatrix}\right)\in\mathcal{T}^*$. Then $\mathcal{T}^*$ acts on $\mathcal{T}$ as follows: $\tau(t)=f(a)+h(m)+g(b)$. Now the left module action of $\mathcal{T}$ on $\mathcal{T}^*$ is give by $\left(\begin{smallmatrix}a & m\\ & b\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}f & h\\ & g\end{smallmatrix}\right)=\left(\begin{smallmatrix}a\cdot f+m\cdot h & b\cdot h\\ & b\cdot g\end{smallmatrix}\right)$, and the right module action of $\mathcal{T}$ on $\mathcal{T}^*$ is as follows:

$$\begin{pmatrix}a & m\\ & b\end{pmatrix}\cdot\begin{pmatrix}f & h\\ & g\end{pmatrix}=\begin{pmatrix}a\cdot f+m\cdot h & b\cdot h\\ & b\cdot g\end{pmatrix},$$

$$\begin{pmatrix}f & h\\ & g\end{pmatrix}\cdot\begin{pmatrix}a & m\\ & b\end{pmatrix}=\begin{pmatrix}f\cdot a & h\cdot a\\ & h\cdot m+g\cdot b\end{pmatrix}.$$

Thus $\mathcal{T}^*$ becomes a Banach $\mathcal{T}$-bimodule. We remove the dot for simplicity. This process may be repeated to define the module actions of $\mathcal{T}$ on $\mathcal{T}^{(n)}$ for every $n \in \mathbb{N}$ as follows [10]:

$$\begin{pmatrix} a & m \\ & b \end{pmatrix} \cdot \begin{pmatrix} a^{(2n-1)} & m^{(2n-1)} \\ & b^{(2n-1)} \end{pmatrix} = \begin{pmatrix} aa^{(2n-1)} + mm^{(2n-1)} & bm^{(2n-1)} \\ & bb^{(2n-1)} \end{pmatrix},$$

$$\begin{pmatrix} a^{(2n-1)} & m^{(2n-1)} \\ & b^{(2n-1)} \end{pmatrix} \cdot \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} a^{(2n-1)}a & m^{(2n-1)}a \\ & m^{(2n-1)}m + b^{(2n-1)}b \end{pmatrix}$$

and

$$\begin{pmatrix} a & m \\ & b \end{pmatrix} \cdot \begin{pmatrix} a^{(2n)} & m^{(2n)} \\ & b^{(2n)} \end{pmatrix} = \begin{pmatrix} aa^{(2n)} & am^{(2n)} + m^{(2n)}b \\ & bb^{(2n)} \end{pmatrix},$$

$$\begin{pmatrix} a^{(2n)} & m^{(2n)} \\ & b^{(2n)} \end{pmatrix} \cdot \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} a^{(2n)}a & a^{(2n)}m + m^{(2n)}b \\ & +b^{(2n)}b \end{pmatrix},$$

for all $\begin{pmatrix} a & m \\ & b \end{pmatrix} \in \mathcal{T}$, $\begin{pmatrix} a^{(2n-1)} & m^{(2n-1)} \\ & b^{(2n-1)} \end{pmatrix} \in \mathcal{T}^{(2n-1)}$ and $\begin{pmatrix} a^{(2n)} & m^{(2n)} \\ & b^{(2n)} \end{pmatrix} \in \mathcal{T}^{(2n)}$.

The n-weak amenability of triangular Banach algebra was first investigated by Forrest and Marcoux in [10] and it was proved that triangular Banach algebra $\mathcal{T}$ is weakly amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable. Since then, amenability of triangular Banach algebra was studied by many authors [11–14]. In [14], it was shown that the triangular Banach algebra $\mathcal{T}$ is amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ satisfy $\mathcal{M} = 0$. In this paper, we consider Lie weak amenability of triangular Banach algebra.

2. Lie $(2n-1)$-weak amenability

For triangular Banach algebra $\mathcal{T}$, the projections $\pi_{\mathcal{A}} : \mathcal{T} \to \mathcal{A}$ and $\pi_{\mathcal{B}} : \mathcal{T} \to \mathcal{B}$ are defined by $\pi_{\mathcal{A}}(x) = a$ and $\pi_{\mathcal{B}}(x) = b$, for any $x = \begin{pmatrix} a & m \\ & b \end{pmatrix} \in \mathcal{T}$. For a Banach $\mathcal{A}$-bimodule $\mathcal{X}$, the centre of $\mathcal{X}$, denoted by $Z(\mathcal{X})$, is the set

$$\{X \in \mathcal{X} : AX = XA \text{ for all } A \in \mathcal{A}\}.$$

The following lemma gives the structure of the dual of triangular Banach algebra.

Lemma 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\mathcal{M}$ be a Banach $\mathcal{A},\mathcal{B}$-module. Let $\mathcal{T} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{pmatrix}$ be the corresponding triangular Banach algebra. The centre of $\mathcal{T}^{(n)}$ is given by*

$$Z(\mathcal{T}^{(2n-1)}) = \Big\{ \begin{pmatrix} f & 0 \\ & g \end{pmatrix} : f \in Z(\mathcal{A}^{(2n-1)}),\ g \in Z(\mathcal{B}^{(2n-1)}) \Big\},$$

and

$$Z(\mathcal{T}^{(2n)}) = \Big\{ \begin{pmatrix} a & 0 \\ & b \end{pmatrix} : am = mb, \forall\, m \in \mathcal{M}^{(2n)}, a \in Z(\mathcal{A}^{(2n)}), b \in Z(\mathcal{B}^{(2n)}) \Big\}.$$

Moreover, if $\mathcal{M}$ is faithful, then there exists a unique algebra isomorphism γ from $\pi_{\mathcal{A}}(Z(\mathcal{T}^{(2n)}))$ to $\pi_{\mathcal{B}}(Z(\mathcal{T}^{(2n)}))$ such that $am = m\gamma(a)$ for every $m \in \mathcal{M}$.

Proof By the module action of $\mathcal{T}$ on $\mathcal{T}^{(2n-1)}$, it is easy to show that

$$Z(\mathcal{T}^{(2n-1)})=\Big\{\begin{pmatrix} f & 0\\ & g\end{pmatrix}: f\in Z(\mathcal{A}^{(2n-1)}),\ g\in Z(\mathcal{B}^{(2n-1)})\Big\}.$$

Suppose $a\in\mathcal{A}^{(2n)}$, $b\in\mathcal{B}^{(2n)}$ and $am=mb$ for all $m\in\mathcal{M}^{(2n)}$, it is easy to verify $\left(\begin{smallmatrix} a & 0\\ & b\end{smallmatrix}\right)\in Z(\mathcal{T}^{(2n)})$.

Conversely, if $\left(\begin{smallmatrix} a & n\\ & 0\end{smallmatrix}\right)\in Z(\mathcal{T}^{(2n)})$, we have

$$\begin{aligned}\begin{pmatrix} a & n\\ & b\end{pmatrix}&=\begin{pmatrix} I_{\mathcal{A}} & 0\\ & 0\end{pmatrix}\begin{pmatrix} a & n\\ & b\end{pmatrix}\\ &=\begin{pmatrix} a & n\\ & b\end{pmatrix}\begin{pmatrix} I_{\mathcal{A}} & 0\\ & 0\end{pmatrix}=\begin{pmatrix} a & 0\\ & 0\end{pmatrix},\end{aligned}$$

so $n=0$. Since

$$\begin{aligned}\begin{pmatrix} 0 & am\\ & b\end{pmatrix}&=\begin{pmatrix} a & 0\\ & b\end{pmatrix}\begin{pmatrix} 0 & m\\ & 0\end{pmatrix}\\ &=\begin{pmatrix} 0 & m\\ & 0\end{pmatrix}\begin{pmatrix} a & 0\\ & b\end{pmatrix}=\begin{pmatrix} 0 & mb\\ & 0\end{pmatrix},\end{aligned}$$

then $am=mb$ for any $m\in\mathcal{M}^{(2n)}$. Since

$$\begin{aligned}\begin{pmatrix} aa' & 0\\ & bb'\end{pmatrix}&=\begin{pmatrix} a & 0\\ & b\end{pmatrix}\begin{pmatrix} a' & 0\\ & b'\end{pmatrix}\\ &=\begin{pmatrix} a' & 0\\ & b'\end{pmatrix}\begin{pmatrix} a & 0\\ & b\end{pmatrix}=\begin{pmatrix} a'a & 0\\ & b'b\end{pmatrix},\end{aligned}$$

we have $aa'=a'a$ and $bb'=b'b$ for any $a'\in\mathcal{A}$ and $b'\in\mathcal{B}$, and hence $a\in Z(\mathcal{A}^{(2n)})$ and $b\in Z(\mathcal{B}^{(2n)})$.

In the following we suppose that $\mathcal{M}$ is a faithful bimodule. If $am=mb$ for any $a'\in\mathcal{A}$, we have

$$(aa'-a'a)m=a(a'm)-aa'(am)=(a'm)b-a'(mb)=0,\quad \forall m\in\mathcal{M}^{(2n)}.$$

Hence $aa'=a'a$ as $\mathcal{M}^{(2n)}$ is faithful. Therefore $a\in Z(\mathcal{A}^{(2n)})$ and similarly we have $b\in Z(\mathcal{B}^{(2n)})$. So $\left(\begin{smallmatrix} a & 0\\ & b\end{smallmatrix}\right)\in Z(\mathcal{T}^{(2n)})$. Thus for any $a\in\pi_{\mathcal{A}}(Z(\mathcal{T}^{(2n)}))\subseteq Z(\mathcal{A}^{(2n)})$, there exists $b\in\pi_{\mathcal{B}}(Z(\mathcal{T}^{(2n)}))$ such that $\left(\begin{smallmatrix} a & 0\\ & b\end{smallmatrix}\right)\in Z(\mathcal{T}^{(2n)})$. Suppose there exists another b' satisfying $\left(\begin{smallmatrix} a & 0\\ & b'\end{smallmatrix}\right)\in Z(\mathcal{T}^{(2n)})$, then we have $mb=am=mb'$ for all $m\in\mathcal{M}^{(2n)}$ and hence $b=b'$ as $\mathcal{M}^{(2n)}$ is faithful. Therefore, there exists a unique map $\gamma:\pi_{\mathcal{A}}(Z(\mathcal{T}^{(2n)}))\to\pi_{\mathcal{B}}(Z(\mathcal{T}^{(2n)}))$ satisfying $am=m\gamma(a)$ for all $a\in\mathcal{A}$ and $m\in\mathcal{T}^{(2n)}$. Next we show that γ is an algebra isomorphism. If $\gamma(a)=0$ then $am=0$ for every $m\in\mathcal{M}^{(2n)}$ and thus $a=0$. Therefore γ is injective. That γ is surjective follows from the definition of $\pi_{\mathcal{B}}(Z(\mathcal{T}^{(2n)}))$. It is clear that γ is linear. For any $a,a'\in\pi_{\mathcal{A}}(Z(\mathcal{T}^{(2n)}))$, from

$$(aa')m=a(a'm)=(a'm)\gamma(a)=a'(m\gamma(a))=m\gamma(a)\gamma(a'),$$

it follows $\gamma(aa')=\gamma(a)\gamma(a')$, proving that γ is an algebra isomorphism. $\Box$

Lemma 2.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a Banach $\mathcal{A},\mathcal{B}$-module. A continuous linear map $L:\mathcal{T}\to\mathcal{T}^*$ is a Lie derivation if and only if L is of the form*

$$L\Big(\begin{pmatrix} a & m \\ & b \end{pmatrix}\Big)=\begin{pmatrix} g_{\mathcal{A}}(a)+h_{\mathcal{B}}(b)-mm_0 & m_0a-bm_0 \\ & g_{\mathcal{B}}(b)+h_{\mathcal{A}}(a)+m_0m \end{pmatrix},$$

where $m_0\in\mathcal{M}^*$, $g_{\mathcal{A}}:\mathcal{A}\to\mathcal{A}^*$ is a Lie derivation, $g_{\mathcal{B}}:\mathcal{B}\to\mathcal{B}^*$ is a Lie derivation, $h_{\mathcal{A}}:A\to Z(\mathcal{B}^*)$ is a linear map satisfying $h_{\mathcal{A}}([a,a'])=0$ and $h_{\mathcal{B}}:B\to Z(\mathcal{A}^*)$ is a linear map satisfying $h_{\mathcal{B}}([b,b'])=0$.

Proof Suppose L is a Lie derivation on $\mathcal{T}$. Write L as

$$L\Big(\begin{pmatrix} a & m \\ & b \end{pmatrix}\Big)=\begin{pmatrix} g_{\mathcal{A}}(a)+h_{\mathcal{B}}(b)+k_1(m) & r_1(a)-r_2(b)+f(m) \\ & g_{\mathcal{B}}(b)+h_{\mathcal{A}}(a)+k_2(m) \end{pmatrix},$$

where $g_{\mathcal{A}}:\mathcal{A}\to\mathcal{A}^*$, $h_{\mathcal{B}}:\mathcal{B}\to\mathcal{A}^*$, $k_1:\mathcal{M}\to\mathcal{A}^*$, $g_{\mathcal{B}}:\mathcal{B}\to\mathcal{B}^*$, $h_{\mathcal{A}}:\mathcal{A}\to\mathcal{B}^*$, $k_2:\mathcal{M}\to\mathcal{B}^*$, $r_1:\mathcal{A}\to\mathcal{M}^*$, $r_2:\mathcal{B}\to\mathcal{M}^*$, $f:\mathcal{M}\to\mathcal{M}^*$ are all continuous linear maps. For identity $I_{\mathcal{A}}\in\mathcal{A}^*$, let $L(\left(\begin{smallmatrix} I_{\mathcal{A}} & 0 \\ & 0\end{smallmatrix}\right))=\left(\begin{smallmatrix} i & m_0 \\ & j\end{smallmatrix}\right)$ for some $i\in\mathcal{A}^*$, $n\in\mathcal{M}^*$ and $j\in\mathcal{B}^*$. Since $[\left(\begin{smallmatrix} a & 0 \\ & 0\end{smallmatrix}\right),\left(\begin{smallmatrix} I_{\mathcal{A}} & 0 \\ & 0\end{smallmatrix}\right)]=0$, we have

$$\begin{aligned}
0&=L\Big(\Big[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix},\begin{pmatrix} I_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix}\Big]\Big)\\
&=\Big[L\Big(\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\Big),\begin{pmatrix} I_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix}\Big]+\Big[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix},L\Big(\begin{pmatrix} I_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix}\Big)\Big]\\
&=\Big[\begin{pmatrix} g_{\mathcal{A}}(a) & r_1(a) \\ & h_{\mathcal{A}}(a) \end{pmatrix},\begin{pmatrix} I_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix}\Big]+\Big[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix},\begin{pmatrix} i & m_0 \\ & j \end{pmatrix}\Big]\\
&=\begin{pmatrix} ai-ia & r_1(a)-m_0a \\ & 0 \end{pmatrix}.
\end{aligned}$$

Thus, for all $a\in\mathcal{A}$, we can get $ai=ia$, $r_1(a)=m_0a$. Similarly, since $[\left(\begin{smallmatrix} 0 & 0 \\ & b\end{smallmatrix}\right),\left(\begin{smallmatrix} I_{\mathcal{A}} & 0 \\ & 0\end{smallmatrix}\right)]=0$, we can get

$$\begin{aligned}
0&=L\Big(\Big[\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix},\begin{pmatrix} I_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix}\Big]\Big)\\
&=\Big[\begin{pmatrix} h_{\mathcal{B}}(b) & -r_2(b) \\ & g_{\mathcal{B}}(b) \end{pmatrix},\begin{pmatrix} I_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix}\Big]+\Big[\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix},\begin{pmatrix} i & m_0 \\ & j \end{pmatrix}\Big]\\
&=\begin{pmatrix} 0 & -r_2(b)+bm_0 \\ & bj-jb \end{pmatrix}.
\end{aligned}$$

Thus, $bj=jb$, $r_2(b)=-bm_0$, for all $b\in\mathcal{B}$. For $m\in\mathcal{M}$, then

$$\begin{aligned}
L\Big(\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\Big)&=\begin{pmatrix} k_1(m) & f(m) \\ & k_2(m) \end{pmatrix}=L\Big(\Big[\begin{pmatrix} I_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix},\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\Big]\Big)\\
&=\Big[L\Big(\begin{pmatrix} I_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix}\Big),\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\Big]+\Big[\begin{pmatrix} I_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix},L\Big(\begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\Big)\Big]
\end{aligned}$$

$$= \Big[\begin{pmatrix} i & m_0 \\ & j \end{pmatrix}, \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}\Big] + \Big[\begin{pmatrix} I_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} k_1(m) & f(m) \\ & k_2(m) \end{pmatrix}\Big]$$

$$= \begin{pmatrix} -mm_0 & -f(m) \\ & m_0 m \end{pmatrix}.$$

Thus, $k_1(m) = -mm_0$, $k_2(m) = m_0 m$, $f(m) = 0$, for all $m \in \mathcal{M}$. So, for each $a \in \mathcal{A}$, $L(\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}) = \begin{pmatrix} g_{\mathcal{A}}(a) & m_0 a \\ & h_{\mathcal{A}}(a) \end{pmatrix}$. For $a' \in \mathcal{A}$, $L(\begin{pmatrix} [a,a'] & 0 \\ & 0 \end{pmatrix}) = \begin{pmatrix} g_{\mathcal{A}}([a,a']) & [a,a']m_0 \\ & h_{\mathcal{A}}([a,a']) \end{pmatrix}$. On the other hand,

$$0 = L\Big(\Big[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} a' & 0 \\ & 0 \end{pmatrix}\Big]\Big)$$

$$= \Big[L\Big(\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\Big), \begin{pmatrix} a' & 0 \\ & 0 \end{pmatrix}\Big] + \Big[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, L\Big(\begin{pmatrix} a' & 0 \\ & 0 \end{pmatrix}\Big)\Big]$$

$$= \Big[\begin{pmatrix} g_{\mathcal{A}}(a) & m_0 a \\ & h_{\mathcal{A}}(a) \end{pmatrix}, \begin{pmatrix} a' & 0 \\ & 0 \end{pmatrix}\Big] + \Big[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} g_{\mathcal{A}}(a') & m_0 a' \\ & h_{\mathcal{A}}(a') \end{pmatrix}\Big]$$

$$= \begin{pmatrix} [g_{\mathcal{A}}(a), a'] + [a, g_{\mathcal{A}}(a')] & [a,a']m_0 \\ & 0 \end{pmatrix}.$$

Thus, $g_{\mathcal{A}}([a,a']) = [g_{\mathcal{A}}(a), a'] + [a, g_{\mathcal{A}}(a')]$ and $h_{\mathcal{A}}([a,a']) = 0$. Similarly, we can show $g_{\mathcal{B}}([b,b']) = [g_{\mathcal{B}}(b), b'] + [b, g_{\mathcal{B}}(b')]$ and $h_{\mathcal{B}}([b,b']) = 0$. Finally, we have

$$0 = L\Big(\Big[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\Big]\Big)$$

$$= \Big[L\Big(\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\Big), \begin{pmatrix} 0 & 0 \\ & 0 \end{pmatrix}\Big] + \Big[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, L\Big(\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\Big)\Big]$$

$$= \Big[\begin{pmatrix} g_{\mathcal{A}}(a) & m_0 a \\ & h_{\mathcal{A}}(a) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\Big] + \Big[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} h_{\mathcal{B}}(b) & bm_0 \\ & g_{\mathcal{B}}(b) \end{pmatrix}\Big]$$

$$= \begin{pmatrix} [a, h_{\mathcal{B}}(b)] & 0 \\ & [h_{\mathcal{A}}(a), b] \end{pmatrix}.$$

Thus, $[a, h_{\mathcal{B}}(b)] = 0$ and $[h_{\mathcal{A}}(a), b] = 0$. Since a and b are arbitrary, we conclude that $h_{\mathcal{A}}(\mathcal{A}) \subseteq Z(\mathcal{B}^*)$ and $h_{\mathcal{B}}(\mathcal{B}) \subseteq Z(\mathcal{A}^*)$.

Conversely, if L is of the above form, it is routine to calculate and verify that L is a Lie derivation. □

Theorem 2.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a Banach $\mathcal{A},\mathcal{B}$-module. Let $\mathcal{T} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{pmatrix}$ be the corresponding triangular Banach algebra. Then $\mathcal{T}$ is Lie weakly amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are Lie weakly amenable.*

Proof Suppose $\mathcal{T}$ is Lie weakly amenable. Then Lie derivation $L : \mathcal{T} \to \mathcal{T}^*$ is written as $D + \tau$, where D is a derivation and τ is a linear map from $\mathcal{T}$ into the center of $\mathcal{T}^*$ vanishing on each commutator. By [10, Lemma 3.2], there exist derivations $\delta_1 : \mathcal{A} \to \mathcal{A}^*$, $\delta_3 : \mathcal{B} \to \mathcal{B}^*$ and an element $n_0 \in \mathcal{M}^*$ such that

$$D\Big(\begin{pmatrix} a & m \\ & b \end{pmatrix}\Big) = \begin{pmatrix} \delta_1(a) - mn_0 & n_0a - bn_0 \\ & \delta_3(b) + n_0m \end{pmatrix}.$$

By Lemma 2.1, we know $Z(\mathcal{T}^*) = \begin{pmatrix} Z(\mathcal{A}^*) & 0 \\ & Z(\mathcal{B}^*) \end{pmatrix}$. Thus, we can suppose

$$\tau\Big(\begin{pmatrix} a & m \\ & b \end{pmatrix}\Big) = \begin{pmatrix} i_1(a) + i_2(m) + i_3(b) & 0 \\ & j_1(a) + j_2(m) + j_3(b) \end{pmatrix},$$

where i_l and j_l $(l = 1, 2, 3)$ are continuous linear maps from $\mathcal{A}, \mathcal{M}, \mathcal{B}$ into $Z(\mathcal{A}^*)$ and $Z(\mathcal{B}^*)$, respectively. By Lemma 2.2, $h_{\mathcal{A}}(\mathcal{A}) \subseteq Z(\mathcal{B}^*)$ and $h_{\mathcal{B}}(\mathcal{B}) \subseteq Z(\mathcal{A}^*)$, hence we can get $h_{\mathcal{A}} = j_1$ and $h_{\mathcal{B}} = i_3$. Since $L = D + \tau$, we have $n_0 = m_0$ (by taking $a = I_{\mathcal{A}}, b = 0$ and $m = 0$), $i_2 = j_2 = 0$ (by taking $a = 0, m = m, b = 0$), and $g_{\mathcal{A}} = \delta_1 + i_1$, $g_{\mathcal{B}} = \delta_2 + j_3$ (by taking $a = a, m = 0, b = 0$). Since τ vanishes on commutators, so do i_1 and j_3. Therefore, $\mathcal{A}$ and $\mathcal{B}$ are Lie weakly amenable.

By [10, Lemma 3.2] and Lemma 2.2, it is easy to verify that if $\mathcal{A}$ and $\mathcal{B}$ are Lie weakly amenable, then $\mathcal{T}$ is Lie weakly amenable. □

Recall that the action of $\mathcal{T}$ on $\mathcal{T}^{(3)}$ is a restriction of $\mathcal{T}^{(2)}$ on $\mathcal{T}^{(3)}$. By repeating the same process as above, we can show $\mathcal{T}$ is Lie 3-weakly amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are Lie 3-weakly amenable. In fact, this argument can extend to any dual of the form $\mathcal{T}^{(2n-1)}$ with $n \geq 1$. So we have the following result.

Theorem 2.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a Banach $\mathcal{A},\mathcal{B}$-module. Let $\mathcal{T} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{pmatrix}$ be the corresponding triangular Banach algebra. Let n be a positive integer. Then $\mathcal{T}$ is Lie $(2n-1)$-weakly amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are Lie $(2n-1)$-weakly amenable.*

3. Lie $(2n)$-weak amenability

The action of $\mathcal{T}$ upon $\mathcal{T}^{(2n)}$ for each $n \geq 1$ is a restriction of the Arens product to the canonical embedding of $\mathcal{T}$ into $\mathcal{T}^{(2n)}$. We observe that the action coincide with the standard matrix multiplications. Similar to the proof of Lemma 2.2, through elementary matrix calculations, we have following lemma.

Lemma 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a Banach $\mathcal{A},\mathcal{B}$-module. A continuous linear map $L : \mathcal{T} \to \mathcal{T}^{(2n)}$ is a Lie derivation if and only if L is of the form*

$$L\Big(\begin{pmatrix} a & m \\ & b \end{pmatrix}\Big) = \begin{pmatrix} g_{\mathcal{A}}(a) + h_{\mathcal{B}}(b) & am_0 - m_0b + f(m) \\ & g_{\mathcal{B}}(b) + h_{\mathcal{A}}(a) \end{pmatrix},$$

where $m_0 \in \mathcal{M}^{(2n)}$, $g_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^{(2n)}$, $g_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^{(2n)}$, $h_{\mathcal{A}} : A \to Z(\mathcal{B}^{(2n)})$ with $h_{\mathcal{A}}([a, a']) = 0$, $h_{\mathcal{B}} : B \to Z(\mathcal{A}^{(2n)})$ with $h_{\mathcal{B}}([b, b']) = 0$ and $f : \mathcal{M} \to \mathcal{M}^{(2n)}$ are continuous linear maps satisfying

(i) $g_{\mathcal{A}}$ is a Lie derivation on $\mathcal{A}$, $f(am) = g_{\mathcal{A}}(a)m - mh_{\mathcal{A}}(a) + af(m)$.

(ii) $g_{\mathcal{B}}$ is a Lie derivation on $\mathcal{B}$, $f(mb) = mg_{\mathcal{B}}(b) - h_{\mathcal{B}}(b)m + f(m)b$.

Theorem 3.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a faithful Banach $\mathcal{A},\mathcal{B}$-*

module. Let $\mathcal{T} = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{pmatrix}$ be the corresponding triangular Banach algebra. Let n be a positive integer. Then $\mathcal{T}$ is Lie $(2n)$-weakly amenable if and only if $h_{\mathcal{A}}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{T}^{(2n)}))$ and $h_{\mathcal{B}}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{T}^{(2n)}))$.

Proof Suppose $\mathcal{T}$ is Lie $(2n)$-weakly amenable. Then Lie derivation $L : \mathcal{T} \to \mathcal{T}^*$ is written as $D + \tau$, where D is a derivation and τ is a linear map from $\mathcal{T}$ into the center of $\mathcal{T}^*$ vanishing on each commutator. By [10, Proposition 3.9], there exist continuous derivations $\delta_1 : \mathcal{A} \to \mathcal{A}^{(2n)}$, $\delta_3 : \mathcal{B} \to \mathcal{B}^{(2n)}$ and an element $n_0 \in \mathcal{M}^{(2n)}$ such that

$$D\Big(\begin{pmatrix} a & m \\ & b \end{pmatrix}\Big) = \begin{pmatrix} \delta_1(a) & an_0 - n_0 b + \rho(m) \\ & \delta_3(b) + n_0 m \end{pmatrix}$$

with $\rho(am) = \delta_1(a)m + a\rho(m)$ and $\rho(mb) = \rho(m)b + m\delta_3(b)$. By Lemma 2.1, we can assume that

$$\tau\Big(\begin{pmatrix} a & m \\ & b \end{pmatrix}\Big) = \begin{pmatrix} i_1(a) + i_2(m) + i_3(b) & 0 \\ & j_1(a) + j_2(m) + j_3(b) \end{pmatrix},$$

where i_l and j_l $(l = 1, 2, 3)$ are continuous linear maps from $\mathcal{A}, \mathcal{M}, \mathcal{B}$ into $Z(\mathcal{A}^*)$ and $Z(\mathcal{B}^*)$, respectively. Since $L = D + \tau$, we have $n_0 = m_0$ (by taking $a = I_{\mathcal{A}}, b = 0$ and $m = 0$), $i_2 = j_2 = 0$ (by taking $a = 0, m = m, b = 0$), and $g_{\mathcal{A}} = \delta_1 + i_1$, $g_{\mathcal{B}} = \delta_2 + j_3$, $h_{\mathcal{A}} = j_1$ and $h_{\mathcal{B}} = i_3$ (by taking $a = a, m = 0, b = 0$). Hence

$$h_{\mathcal{A}}(a) = j_1(a) = \pi_{\mathcal{B}}\Big(\tau\Big(\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}\Big)\Big) \in \pi_{\mathcal{B}}(Z(\mathcal{T}^{(2n)}))$$

and

$$h_{\mathcal{B}}(b) = i_3(b) = \pi_{\mathcal{A}}\Big(\tau\Big(\begin{pmatrix} 0 & 0 \\ & b \end{pmatrix}\Big)\Big) \in \pi_{\mathcal{A}}(Z(\mathcal{T}^{(2n)})).$$

Conversely, suppose that $h_{\mathcal{A}}(\mathcal{A}) \subseteq \pi_{\mathcal{B}}(Z(\mathcal{T}^{(2n)}))$ and $h_{\mathcal{B}}(\mathcal{B}) \subseteq \pi_{\mathcal{A}}(Z(\mathcal{T}^{(2n)}))$. By Lemma 2.1, there exists the unique group isomorphism γ from $\pi_{\mathcal{A}}(Z(\mathcal{T}^{(2n)}))$ to $\pi_{\mathcal{B}}(Z(\mathcal{T}^{(2n)}))$. Define

$$D\Big(\begin{pmatrix} a & m \\ & b \end{pmatrix}\Big) = \begin{pmatrix} g_{\mathcal{A}}(a) - \gamma^{-1}h_{\mathcal{A}}(a) & an_0 - n_0 b + f(m) \\ & g_{\mathcal{B}}(b) - \gamma h_{\mathcal{B}}(b) \end{pmatrix}$$

and

$$h\Big(\begin{pmatrix} a & m \\ & b \end{pmatrix}\Big) = \begin{pmatrix} \gamma^{-1}h_{\mathcal{A}}(a) + h_{\mathcal{B}}(b) & 0 \\ & h_{\mathcal{A}}(a) + \gamma h_{\mathcal{B}}(b) \end{pmatrix}.$$

It is easy to verify that $h(\begin{pmatrix} a & m \\ & b \end{pmatrix}) \in Z(\mathcal{T}^{(2n)})$ vanishing on commutators. By Lemma 3.1, we have

$$f(am) = g_{\mathcal{A}}(a)m - mh_{\mathcal{A}}(a) + af(m) = (g_{\mathcal{A}}(a) - \gamma^{-1}h_{\mathcal{A}}(a))m + af(m)$$

and similarly

$$f(mb) = mg_{\mathcal{B}}(b) - h_{\mathcal{B}}(b)m + f(m)b = m(g_{\mathcal{B}}(b) - \gamma h_{\mathcal{B}}(b)) + f(m)b.$$

Let $p_{\mathcal{A}}(a) = g_{\mathcal{A}}(a) - \gamma^{-1}h_{\mathcal{A}}(a)$. Then $f(aa'm) = p_{\mathcal{A}}(aa')m + aa'f(m)$. Also,

$$f(aa'm) = p_{\mathcal{A}}(a)a'm + af(a'm) = p_{\mathcal{A}}(a)a'm + ap_{\mathcal{A}}(a')m + aa'f(m).$$

Thus, $p_{\mathcal{A}}(aa')m = (p_{\mathcal{A}}(a)a' + ap_{\mathcal{A}}(a'))m$. Hence since $\mathcal{M}^{(2n)}$ is faithful, we have $p_{\mathcal{A}}(aa') = p_{\mathcal{A}}(a)a' + ap_{\mathcal{A}}(a')$, $p_{\mathcal{A}}$ is a derivation. Similarly, $g_{\mathcal{B}}(b) - \gamma h_{\mathcal{B}}(b)$ is also a derivation. Therefore, by [10, Proposition 3.9], D is a derivation. $\Box$

4. Lie weak amenability of finite dimension nest algebra

Recall that a nest on a Hilbert space $\mathcal{H}$ is a chain $\mathcal{N}$ of closed subspaces of $\mathcal{H}$ which contains $\{0\}$ and $\mathcal{H}$, and which is closed under the operations of taking intersections and closed spans with the corresponding nest algebra $\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}$. Our main result here is the following theorem.

Theorem 4.1 *Let $\mathcal{T}(\mathcal{N})$ be a nest algebra on a finite dimensional complex Hilbert space $\mathcal{H}$. Then $\mathcal{T}(\mathcal{N})$ is Lie permanently weakly amenable.*

Proof For each $n \geq 1$, since $\mathcal{T}(\mathcal{N})$ is finite dimensional, it follows that $\mathcal{T}(\mathcal{N})^{(2n)}$ is isomorphic to $\mathcal{T}(\mathcal{N})$ as a Banach algebra. By [9], $\mathcal{T}(\mathcal{N})$ is Lie $(2n)$-weak amenable (When the nest is trivial and $\mathcal{T}(\mathcal{N}) \simeq \mathcal{B}(\mathcal{H})$ by [15]).

As for odd weak amenability, the proof works by induction upon the dimension m of the space $\mathcal{H}$ upon which $\mathcal{T}(\mathcal{N})$ acts. When $n = 1$, $\mathcal{T}(\mathcal{N}) \simeq \mathbb{C}$. It is clear that the conclusion holds. Suppose the result holds for all nest algebras acting upon Hilbert spaces of dimension less than k. Let $\mathcal{T}(\mathcal{N})$ be a nest algebra acting on a space of dimension k. Then either the nest $\mathcal{N}$ is trivial, in which case $\mathcal{T}(\mathcal{N}) \simeq \mathcal{B}(\mathcal{H})$ is a C^*-algebra, by [7, Theorem 6.11], it is Lie amenable and hence is permanently Lie weakly amenable, or there exists an element $\{0\} \neq N \neq \mathcal{H}$ in $\mathcal{N}$ and we may decompose $\mathcal{T}(\mathcal{N})$ as $\mathcal{T}(\mathcal{N}) = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{smallmatrix}\right)$ where $\mathcal{A}$ is a nest algebra upon N, $\mathcal{B}$ is a nest algebra upon $N^{\perp}$, and $\mathcal{M}$ is the space of all operators taking $N^{\perp}$ into N. By our induction hypothesis, $\mathcal{A}$ and $\mathcal{B}$ are Lie permanently weakly amenable. By Theorem 2.4, we conclude that $\mathcal{T}(\mathcal{N})$ is Lie $(2n-1)$-weakly amenable for all $n \geq 1$.

Combining these two results yields our theorem. $\Box$

Acknowledgements The authors would like to thank the referee for his valuable comments and corrections which have improved the quality of this paper.

References

[1] B. E. JOHNSON. *Cohomology in Banach algebras.* American Mathematical Society, Providence, R.I., 1972.
[2] U. HAAGERUP. *All nuclear C^*-algebra are amenable.* Invent. Math., 1983, **74**(2): 305–319.
[3] W. G. BADE, P. C. CURTIS, H. G. DALES. *Amenability and weak amenability for beurling and Lipschitz algebras.* Proc. London Math. Soc., 1987, **55**(2): 359–377.
[4] B. E. JOHNSON. *Weak amenability of group algebras*, Bull. London. Math. Soc., 1991, **396**(23): 281–284.
[5] H. G. DALES, F. GHAHRAANI, N. GRONBAEK. *Derivations into iterated duals of Banach algebras.* Studia Math., 1998, **128**(1): 19–54.
[6] Yong ZHANG. *Weak amenability of module extensions of Banach algebras.* Trans. Amer. Math. Soc., 2002, **354**(10): 4131–4151.
[7] J. ALAMINOS, M. BREŠAR, A. R. VILLENA. *The strong degree of von Neumann algebras and the structure of Lie and Jordan derivations.* Math. Proc. Cambridge Philos. Soc., 2004, **137**(2): 441–463.
[8] W. CHEUNG. *Lie derivations of triangular algebras.* Linear Multilinear Algebra, 2003, **51**(3): 299–310.

[9] Fangyan LU. *Lie triple derivations on nest algebras*. Math. Nachr. 2007, **280**(8): 882–887.
[10] B. E. FORREST, L. W. MARCOUX. *Weak amenability of triangular Banach algebras*. Trans. Amer. Math. Soc., 2002, **354**(4): 1435–1452.
[11] G. ABBASPOUR. *Ideal amenability of triangular Banach algebras*. Int. J. Contemp. Math. Anal., 2010, **4**(26): 1285–1290.
[12] A. BODAGHI, A. JABBARI. *n-weakly module amenability of triangular Banach algebras*. Math. Slovaca, Mathematics, 2013, **65**(3): 645–666.
[13] S. ETEMAD, M. ETTEFAGH. $(2n-1)$*-ideal amenability of triangular Banach algebra*. Proc. Indian Acad. Sci. Math. Sci., 2015, **125**(2): 199–207.
[14] A. R. MEDGHALCHI, M. H. SATTARI, T. YAZDANPANAH. *Amenability and weak amenability of triangular Banach algebras*. Bull. Iranian Math. Soc., 2005, **31**(2): 57–69.
[15] C. R. MIERS. *Lie derivations of von Neumann algebras*. Duke Math. J., 1973, **40**(2): 403–409.