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Local Cocycle 3-Hom-Lie Bialgebras and 3-Lie Classical Hom-Yang-Baxter Equation

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Abstract In this paper, we introduce 3-Hom-Lie bialgebras whose compatibility conditions between the multiplication and comultiplication are given by local cocycle conditions. We study a twisted 3-ary version of the Yang-Baxter Equation, called the 3-Lie classical Hom-Yang-Baxter Equation (3-Lie CHYBE), which is a general form of 3-Lie classical Yang-Baxter Equation and prove that the bialgebras induced by the solutions of 3-Lie CHYBE induce the coboundary local cocycle 3-Hom-Lie bialgebras.

Keywords local cocycle 3-Hom-Lie bialgebra; 3-Lie CHYBE; coboundary condition

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1. Introduction

3-Lie algebra as a generalization of Lie algebra, has attracted attention from several fields of mathematics and physics. In particular, the study of 3-Lie algebras plays an important role in string theory. For example, the structure of 3-Lie algebras is applied to the study of supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes [1–4]. The classical Yang-Baxter Equation (CYBE for short) has far-reaching mathematical significance. The CYBE is closely related to many topics in mathematical physics, including Hamiltonian structures, Kac-Moody algebras, Poisson-Lie groups, quantum groups, Hopf algebras, and Lie bialgebras [5,6]. There are many known solutions of the CYBE. For example, there is a 1-1 correspondence between triangular Lie bialgebras and solutions of the CYBE [7,8]. Recently, the paper "Bialgebra, the classical Yang-Baxter Equation and manin triples for 3-Lie Algebras, arXiv: 1604.05996v1 [math-ph]" written by Chengming Bai, Li Guo and Yunhe Sheng, studied local cocycle 3-Lie bialgebra and the relationship with the 3-Lie classical Yang-Baxter equation which extended the connection between Lie bialgebras and the classical Yang-Baxter equation.

Motivated by recent work on Hom-Lie bialgebras and the Hom-Yang-Baxter equation [6], 3-Lie algebra and 3-Lie classical Yang-Baxter equation, in this paper, we introduce 3-Hom-Lie bialgebras whose compatibility conditions between the multiplication and comultiplication

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are given by local cocycle condition, and is called the local cocycle 3-Hom-Lie bialgebra. We study a twisted 3-ary version of the classical Yang-Baxter Equation, called the 3-Lie classical Hom-Yang-Baxter Equation (3-Lie CHYBE for short), which is a general form of 3-Lie classical Yang-Baxter Equation, and use the solutions of 3-Lie CHYBE to induce the coboundary local cocycle 3-Hom-Lie bialgebras.

The paper is organized as follows. In Section 2, we define the 3-Hom-Lie coalgebra, the 3-Hom-Lie bialgebra with the derivation compatibility, the local cocycle 3-Hom-Lie bialgebra with cocycle compatibility, and the 3-Lie CHBYE. In Section 3, we define the coboundary local cocycle 3-Hom-Lie bialgebra and prove that it can be constructed by a multiplicative 3-Hom-Lie algebra and a solution of the 3-Lie CHBYE.

2. 3-Hom-Lie bialgebra with local cocycle condition

3-Hom-Lie algebra as a generalization of 3-Lie algebra, its derivations and representation were introduced in [3]. For $T = x_1 \otimes x_2 \otimes \cdots \otimes x_n \in L^{\otimes n}$ and $1 \leqslant i \leqslant j \leqslant n$, define the (ij)-switching operator as

$$\sigma_{ij}(T) = x_1 \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n.$$

Definition 2.1 *A* 3*-Hom-Lie algebra* $(L, [\cdot,\cdot,\cdot], \alpha)$ *is a vector space* L *endowed with a* 3*-ary linear skew-symmetry multiplication* $\mu : L \otimes L \otimes L \to L$ *satisfying*

$$\mu(1 + \sigma_{12}) = 0, \mu(1 + \sigma_{23}) = 0,$$
$$\mu(\alpha \otimes \alpha \otimes \mu)(1 - \omega_1 - \omega_2 - \omega_3) = 0,$$

where 1 *denotes the identity operator and* $\omega_i : L^{\otimes n} \to L^{\otimes n}$, $1 \le i \le 3$,

$$\omega_1(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_3 \otimes x_4 \otimes x_1 \otimes x_2 \otimes x_5, \tag{1}$$
$$\omega_2(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_4 \otimes x_5 \otimes x_1 \otimes x_2 \otimes x_3, \tag{2}$$
$$\omega_3(x_1 \otimes x_2 \otimes x_3 \otimes x_4 \otimes x_5) = x_5 \otimes x_3 \otimes x_1 \otimes x_2 \otimes x_4. \tag{3}$$

If $\alpha[\cdot,\cdot,\cdot] = [\cdot,\cdot,\cdot] \circ \alpha^{\otimes^3}$, *we say* L *is multiplicative.*

For any integer k, recall that a linear map $D : L \to L$ is called an α^k-derivation of the multiplicative 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$, if $D \circ \alpha = \alpha \circ D$ and

$$D[x, y, z] = [D(x), \alpha^k(y), \alpha^k(z)] + [\alpha^k(x), D(y), \alpha^k(z)] + [\alpha^k(x), \alpha^k(y), D(z)].$$

For any $x, y \in L$ satisfying $\alpha(x) = x$, $\alpha(y) = y$, define $ad_k(x, y) : L \to L$ by $ad_k(x, y)(z) = [x, y, \alpha^k(z)]$, $\forall z \in L$. Then it is easy to prove that $ad_k(x, y)$ is an α^{k+1}-derivation of $(L, [\cdot,\cdot,\cdot], \alpha)$, which we call an inner α^{k+1}-derivation.

Definition 2.2 *A representation* ρ *of a multiplicative* 3*-Hom-Lie algebra* $(L, [\cdot,\cdot,\cdot], \alpha)$ *on the vector space* V *with respect to* $A \in gl(V)$*, is a linear map* $\rho : L \wedge L \to gl(V)$*, such that, for any* $a, b, c, d \in L$,

(i) $\rho(\alpha(a), \alpha(b)) \circ A = A \circ \rho(a, b)$,

(ii) $\rho(\alpha(b),\alpha(c))\rho(a,d)+\rho(\alpha(c),\alpha(a))\rho(b,d)-\rho([a,b,c],\alpha(d))\circ A+\rho(\alpha(a),\alpha(b))\rho(c,d)=0,$

(iii) $\rho(\alpha(c),\alpha(d))\rho(a,b)-\rho(\alpha(a),\alpha(b))\rho(c,d)+\rho([a,b,c],\alpha(d))\circ A+\rho(\alpha(c),[a,b,d])\circ A=0.$

We often call (V,A) a Hom-module of $(L,[\cdot,\cdot,\cdot],\alpha)$.

Lemma 2.3 *Let $(L,[\cdot,\cdot,\cdot],\alpha)$ be a multiplicative 3-Hom-Lie algebra. Define $ad_1: L\wedge L\to gl(L)$ by $ad_1(x,y)(z)=[\alpha(x),\alpha(y),z]$, $\forall z\in L$. Then ad_1 is a representation of $(L,[\cdot,\cdot,\cdot],\alpha)$.*

Give a representation (ρ,V), denote by $C^p_{\alpha,A}(L,V)$ the set of p-cochains

$$C^p_{\alpha,A}(L,V):=\{\text{linear maps } f: \underbrace{L\otimes L\otimes\cdots\otimes L}_{n}\to V, A\circ f=f\circ\alpha^{\otimes^n}\}.$$

The coboundary operators associated to the Hom-module are given in [4]. For $n\geq 1$, the coboundary operator $\delta: C^n_{\alpha,A}(L,V)\to C^{n+2}_{\alpha,A}(L,V)$ is defined as follows:

For $f\in C^{2n-1}_{\alpha,A}(L,V)$,

$$\begin{aligned}&\delta^{2n-1}_{\text{hom}}f(x_1,x_2,\ldots,x_{2n+1})\\ &\quad=\rho(\alpha^{n-1}(x_{2n}),\alpha^{n-1}(x_{2n+1}))f(x_1,x_2,\ldots,x_{2n-1})-\\ &\quad\quad\rho(\alpha^{n-1}(x_{2n-1}),\alpha^{n-1}(x_{2n+1}))f(x_1,x_2,\ldots,x_{2n-2},x_{2n})+\\ &\quad\quad\sum_{k=1}^{n}(-1)^{n+k}\rho(\alpha^{n-1}(x_{2k-1}),\alpha^{n-1}(x_{2k}))f(x_1,\ldots,\hat{x_{2k-1}},\hat{x_{2k}},\ldots,x_{2n+1})+\\ &\quad\quad\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(\alpha(x_1),\ldots,\hat{x_{2k-1}},\hat{x_{2k}},\ldots,[x_{2k-1},x_{2k},x_j],\ldots,\alpha(x_{2n+1}));\end{aligned}$$

for $f\in C^{2n}_{\alpha,A}(L,V)$,

$$\begin{aligned}&\delta^{2n}_{\text{hom}}f(y,x_1,x_2,\ldots,x_{2n+1})\\ &\quad=\rho(\alpha^{n}(x_{2n}),\alpha^{n}(x_{2n+1}))f(y,x_1,x_2,\ldots,x_{2n-1})-\\ &\quad\quad\rho(\alpha^{n}(x_{2n-1}),\alpha^{n}(x_{2n+1}))f(y,x_1,x_2,\ldots,x_{2n-2},x_{2n})+\\ &\quad\quad\sum_{k=1}^{n}(-1)^{n+k}\rho(\alpha^{n}(x_{2k-1}),\alpha^{n}(x_{2k}))f(y,x_1,\ldots,\hat{x_{2k-1}},\hat{x_{2k}},\ldots,x_{2n+1})+\\ &\quad\quad\sum_{k=1}^{n}\sum_{j=2k+1}^{2n+1}(-1)^{n+k+1}f(\alpha(y),\alpha(x_1),\ldots,\hat{x_{2k-1}},\hat{x_{2k}},\ldots,[x_{2k-1},x_{2k},x_j],\ldots,\alpha(x_{2n+1})).\end{aligned}$$

Let L be a 3-Hom-Lie algebra and (ρ,V) be the representation of L. A linear map $f: L\to V$ is called a 1-cocycle on L associated to (ρ,V) if it satisfies

$$f([x,y,z])=\rho(x,y)f(z)+\rho(y,z)f(x)+\rho(z,x)f(y),\quad\forall x,y,z\in L.$$

Definition 2.4 *A 3-Hom-Lie coalgebra is a triple (L,Δ,α) consisting of a linear space L, a bilinear map $\Delta: L\to L\otimes L\otimes L$ and a linear map $\alpha: L\to L$ satisfying*

$$\begin{aligned}&\Delta+\sigma_{12}\Delta=0,\Delta+\sigma_{23}\Delta=0\quad\textit{(skew-symmetry)},\\ &(1-\omega_1-\omega_2-\omega_3)(\alpha\otimes\alpha\otimes\Delta)\Delta=0\quad\textit{(Hom-coJacobi identity)},\end{aligned}\tag{4}$$

where $1,\omega_1,\omega_2,\omega_3: L^{\otimes^5}\to L^{\otimes^5}$ satisfying identities (1), (2), (3), respectively.

We call Δ the cobracket. If $\Delta\circ\alpha=\alpha^{\otimes^3}\circ\Delta$, then we say L is comultiplicative.

Similar to the case of Lie bialgebra, suitable extensions of these conditions to the context of 3-Lie algebras are not equivalent, leading to different extensions of Hom-Lie bialgebra. In [2], the authors introduced 3-Lie bialgebra with derivation compatibility condition. According to the idea, we give the definition of 3-Hom Lie bialgebra with derivation compatibility condition.

Definition 2.5 *A 3-Hom-Lie bialgebra with derivation compatibility condition is a quadruple $(L,[\cdot,\cdot,\cdot],\Delta,\alpha)$ such that*

(i) $(L,[\cdot,\cdot,\cdot],\alpha)$ is a 3-Hom-Lie algebra,

(ii) (L,Δ,α) is a 3-Hom-Lie coalgebra,

(iii) Δ and $[\cdot,\cdot,\cdot]$ satisfy the following derivation compatibility condition:

$$\Delta[x,y,z]=ad_1^{(3)}(x,y)\Delta(z)+ad_1^{(3)}(y,z)\Delta(x)+ad_1^{(3)}(z,x)\Delta(y), \tag{5}$$

where $ad_1^{(3)}(x,y),ad_1^{(3)}(y,z),ad_1^{(3)}(z,x):L\otimes L\otimes L\to L\otimes L\otimes L$ are 3-ary linear mappings satisfying, for any $u,v,w\in L$,

$$\begin{aligned}ad_1^{(3)}(x,y)(u\otimes v\otimes w)=&(ad_1(x,y)\otimes\alpha\otimes\alpha)(u\otimes v\otimes w)+(\alpha\otimes ad_1(x,y)\otimes\alpha)(u\otimes v\otimes w)+\\&(\alpha\otimes\alpha\otimes ad_1(x,y))(u\otimes v\otimes w).\end{aligned}$$

Unfortunately, unlike the case of Hom-Lie algebra, it is difficult to develop the relations between 3-Hom-Lie bialgebra with derivation compatibility condition and Classical Hom-Yang-Equation. Next, we will introduce 3-Hom-Lie bialgebra with cocycle compatibility condition related to a natural extension of the classical Yang-Baxter equation.

Definition 2.6 *A local cocycle 3-Hom-Lie bialgebra is a quadruple $(L,[\cdot,\cdot,\cdot],\Delta,\alpha)$ which satisfies: $(L,[\cdot,\cdot,\cdot],\alpha)$ is a 3-Hom-Lie algebra; (L,Δ,α) is a 3-Hom-Lie coalgebra; $\Delta=\Delta_1+\Delta_2+\Delta_3: L\to L\otimes L\otimes L$ is a linear map satisfying the following conditions:*

(i) Δ_1 is a 1-cocycle associated to the representation $(L\otimes L\otimes L,ad_1\otimes\alpha\otimes\alpha)$,

(ii) Δ_2 is a 1-cocycle associated to the representation $(L\otimes L\otimes L,\alpha\otimes ad_1\otimes\alpha)$,

(iii) Δ_3 is a 1-cocycle associated to the representation $(L\otimes L\otimes L,\alpha\otimes\alpha\otimes ad_1)$.

For any $r=\sum_i x_i\otimes y_i\in L\otimes L$, define r_{pq} puts x_i at the p-th position, y_i at the q-th position and 1 elsewhere in an n-tensor. For example, when $n=4$, we have

$$r_{12}=\sum_i x_i\otimes y_i\otimes 1\otimes 1\in L^{\otimes^4},r_{21}=\sum_i y_i\otimes x_i\otimes 1\otimes 1\in L^{\otimes^4}.$$

Define $[[r,r,r]]^\alpha\in L\otimes^4$ by

$$\begin{aligned}[[r,r,r]]^\alpha=&[r_{12},r_{13},r_{14}]+[r_{12},r_{23},r_{24}]+[r_{13},r_{23},r_{34}]+[r_{14},r_{24},r_{34}]\\=&\sum_{i,j,k}([x_i,x_j,x_k]\otimes\alpha(y_i)\otimes\alpha(y_j)\otimes\alpha(y_k)+\alpha(x_i)\otimes[y_j,x_j,x_k]\otimes\alpha(y_j)\otimes\alpha(y_k)+\\&\alpha(x_i)\otimes\alpha(x_j)\otimes[y_i,y_j,x_k]\otimes\alpha(y_k)+\alpha(x_i)\otimes\alpha(x_j)\otimes\alpha(x_k)\otimes[y_i,y_j,y_k]).\end{aligned}\tag{6}$$

Definition 2.7 *Let L be a 3-Hom-Lie algebra and $r\in L\otimes L$. The equation*

$$[[r,r,r]]^\alpha=0$$

is called the 3*-Lie classical Hom-Yang-Baxter equation (*3*-Lie CHYBE).*

The following is the classical Yang-Baxter equation with respect to Lie algebra

$$\begin{aligned}[][r,r]]^{\alpha} &= [r_{12}, r_{13}] + [r_{12}, r_{13}] + [r_{12}, r_{13}] \\ &= \sum_{i,j}[x_i, x_j] \otimes \alpha(y_i) \otimes \alpha(y_j) + \sum_{i,k} \alpha(x_i) \otimes [y_i, x_k] \otimes \alpha(y_k) + \sum_{j,k} \alpha(x_j) \otimes \alpha(x_k) \otimes [y_j, y_k] \\ &= 0.\end{aligned}$$

From this, we can see that (6) can be regarded as a natural extension of the classical Yang-Baxter equation with respect to Lie algebra.

For any $r = \sum_i x_i \otimes y_i \in L \otimes L$, $x \in L$, define

$$\begin{cases} \Delta_1(x) = \sum_{i,j}[x, x_i, x_j] \otimes \alpha(y_j) \otimes \alpha(y_i), \\ \Delta_2(x) = \sum_{i,j} \alpha(y_i) \otimes [x, x_i, x_j] \otimes \alpha(y_j), \\ \Delta_3(x) = \sum_{i,j} \alpha(y_j) \otimes \alpha(y_j) \otimes [x, x_i, x_j]. \end{cases} \tag{7}$$

Proposition 2.8 *With the above notations and $\alpha^{\otimes^2}(r) = r$, we have*

(i) Δ_1 *is a* 1*-cocycle associated to the representation* $(L \otimes L \otimes L, ad_1 \otimes \alpha \otimes \alpha)$,

(ii) Δ_2 *is a* 1*-cocycle associated to the representation* $(L \otimes L \otimes L, \alpha \otimes ad_1 \otimes \alpha)$,

(iii) Δ_3 *is a* 1*-cocycle associated to the representation* $(L \otimes L \otimes L, \alpha \otimes \alpha \otimes ad_1)$.

Proof For all $x, y, z \in L$, we have

$$\begin{aligned}\Delta_1([x,y,z]) =& \sum_{i,j}[[x,y,z], x_i, x_j] \otimes \alpha(y_j) \otimes \alpha(y_i) \\ =& \sum_{i,j}[[x,y,z], \alpha(x_i), \alpha(x_j)] \otimes \alpha^2(y_j) \otimes \alpha^2(y_i) \\ =& \sum_{i,j}\big([[x,x_i,x_j], \alpha(y), \alpha(z)] + [[y,x_i,x_j], \alpha(z), \alpha(x)] + \\ & [[z,x_i,x_j], \alpha(x), \alpha(y)]\big) \otimes \alpha^2(y_j) \otimes \alpha^2(y_i) \\ =& (ad_1(y,z) \otimes \alpha \otimes \alpha)\Delta_1(x) + (ad_1(z,x) \otimes \alpha \otimes \alpha)\Delta_1(y) + \\ & (ad_1(x,y) \otimes \alpha \otimes \alpha)\Delta_1(z).\end{aligned}$$

Therefore, Δ_1 is a 1-cocycle associated to the representation $(L \otimes L \otimes L, ad_1 \otimes \alpha \otimes \alpha)$. The other two statements can be proved similarly. □

3. Coboundary local cocycle 3-Hom-Lie bialgebra and the 3-Lie CHBYE

Definition 3.1 *A (multiplicative) coboundary local cocycle* 3*-Hom-Lie bialgebra* $(L, [\cdot,\cdot,\cdot], \Delta, \alpha, r)$ *is a (multiplicative) local cocycle* 3*-Hom-Lie bialgebra, and there exists an element* $r = \sum_i x_i \otimes y_i \in L \otimes L$ *such that* $\alpha^{\otimes^2}(r) = r$ *and* $\Delta = \Delta_1 + \Delta_2 + \Delta_3$*, where* $\Delta_1, \Delta_2, \Delta_3$ *are induced by* r *as in (7).*

For $a \in L$ and $1 \leq i \leq 5$, define the linear map $\otimes_i L \to \otimes^5 L$ by inserting a at the i-th position. For example, for any $t = t_1 \otimes t_2 \otimes t_3 \otimes t_4$, we have $t \otimes_2 a = t_1 \otimes a \otimes t_2 \otimes t_3 \otimes t_4$. The following theorem shows how we can construct a multiplicative coboundary local cocycle

3-Hom-Lie bialgebra.

Theorem 3.2 *Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a multiplicative 3-Hom-Lie lgebra and $r \in L^{\otimes^2}$ satisfying $\alpha^{\otimes^2}(r) = r$ and*

$$\begin{aligned}
&\sum_i (ad_1(x_i, x) \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha)([[r, r, r]]_1^\alpha \otimes_2 \alpha(y_i))+\\
&\quad \sum_i (\alpha \otimes ad_1(x, x_i) \otimes \alpha \otimes \alpha \otimes \alpha)([[r, r, r]]_1^\alpha \otimes_1 \alpha(y_i))+\\
&\quad \sum_i (\alpha \otimes \alpha \otimes ad_1(x, x_i) \otimes \alpha \otimes \alpha)([[r, r, r]]_2^\alpha \otimes_5 \alpha(y_i))+\\
&\quad \sum_i (\alpha \otimes \alpha \otimes ad_1(x_i, x) \otimes \alpha \otimes \alpha)([[r, r, r]]_2^\alpha \otimes_4 \alpha(y_i))+\\
&\quad \sum_i (\alpha \otimes \alpha \otimes \alpha \otimes ad_1(x, x_i) \otimes \alpha)([[r, r, r]]_2^\alpha \otimes_3 \alpha(y_i))+\\
&\quad \sum_i (\alpha \otimes \alpha \otimes \alpha \otimes ad_1(x_i, x) \otimes \alpha)([[r, r, r]]_3^\alpha \otimes_5 \alpha(y_i))+\\
&\quad \sum_i (\alpha \otimes \alpha \otimes \alpha \otimes \alpha \otimes ad_1(x, x_i))([[r, r, r]]_3^\alpha \otimes_4 \alpha(y_i))+\\
&\quad \sum_i (\alpha \otimes \alpha \otimes \alpha \otimes \alpha \otimes ad_1(x_i, x))([[r, r, r]]_3^\alpha \otimes_3 \alpha(y_i)) = 0,
\end{aligned} \tag{8}$$

where

$$[[r, r, r]]_1^\alpha := [r_{12}, r_{13}, r_{14}] + [r_{12}, r_{23}, r_{24}] - [r_{13}, r_{32}, r_{34}] - [r_{14}, r_{42}, r_{43}],$$

$$[[r, r, r]]_2^\alpha := [r_{12}, r_{31}, r_{14}] - [r_{21}, r_{32}, r_{24}] - [r_{31}, r_{32}, r_{34}] - [r_{41}, r_{42}, r_{34}],$$

$$[[r, r, r]]_3^\alpha := -[r_{12}, r_{13}, r_{41}] + [r_{21}, r_{23}, r_{42}] - [r_{31}, r_{32}, r_{43}] - [r_{41}, r_{42}, r_{43}].$$

Let $\Delta : L \to L^{\otimes^2}$ with $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ as in (7). Then $(L, [\cdot,\cdot,\cdot], \Delta, \alpha, r)$ is a multiplicative coboundary local cocycle 3-Hom-Lie bialgebra.

Proof Let $r = \sum_i x_i \otimes y_i$. First we will prove that $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ commutes with α. For $x \in L$, using Definition 2.7, $\alpha[\cdot,\cdot,\cdot] = [\cdot,\cdot,\cdot] \circ \alpha^{\otimes^3}$ and the assumption $\alpha^{\otimes^2}(r) = r$, we have

$$\begin{aligned}
\Delta_1(\alpha(x)) &= \sum_{i,j} [\alpha(x), x_i, xj] \otimes \alpha(y_j) \otimes \alpha(y_i)\\
&= \sum_{i,j} [\alpha(x), \alpha(x_i), \alpha(xj)] \otimes \alpha^2(y_j) \otimes \alpha^2(y_i)\\
&= \alpha^{\otimes^3} \Delta_1(x).
\end{aligned}$$

Similarly, $\Delta_2(\alpha(x)) = \alpha^{\otimes^3} \Delta_2(x)$ and $\Delta_3(\alpha(x)) = \alpha^{\otimes^3} \Delta_3(x)$, then we can obtain

$$\Delta(\alpha(x)) = \Delta_1(\alpha(x)) + \Delta_2(\alpha(x)) + \Delta_3(\alpha(x)) = \alpha^{\otimes^3} \Delta(x).$$

Secondly, we show that Δ is anti-symmetric. Since

$$\sigma_{12}\Delta_1(x) = \sum_{i,j} \alpha(y_j) \otimes [x, x_i, x_j] \otimes \alpha(y_i) = \sum_{i,j} \alpha(y_i) \otimes [x, x_j, x_i] \otimes \alpha(y_j) = -\Delta_2(x),$$

$$\sigma_{12}\Delta_2(x) = \sum_{i,j}[x, x_i, x_j] \otimes \alpha(y_i) \otimes \alpha(y_j) = -\Delta_1(x),$$

$$\sigma_{12}\Delta_3(x) = \sum_{i,j} \alpha(y_i) \otimes \alpha(y_j) \otimes [x, x_i, x_j] = -\Delta_3(x).$$

Hence $\sigma_{12}\Delta(x) = -\Delta(x)$. Similarly, we have $\sigma_{23}\Delta(x) = -\Delta(x)$.

Finally, we show that the 3-Hom-co-Jacobi identity of Δ is equivalent to (8). Since each Δ contains three terms, there are 36 terms in 3-Hom-co-Jacobi identity. Let $G_i, 1 \leq i \leq 5$. Denote the sum of these terms where x is at the i-th position in the 5-tensors. Thus

$$G_1 + G_2 + G_3 + G_4 + G_5 = 0.$$

There are 6 terms in G_1 :

$$G_1 = G_{11} + G_{12} + G_{13} + G_{14} + G_{15} + G_{16},$$

where

$$\begin{aligned}
G_{11} &= \sum_{i,j,k,l} [[x, x_i, x_j], x_k, x_l] \otimes \alpha(y_l) \otimes \alpha(y_k) \otimes \alpha^2(y_j) \otimes \alpha^2(y_i),\\
G_{12} &= \sum_{i,j,k,l} [[x, x_i, x_j], x_k, x_l] \otimes \alpha(y_l) \otimes \alpha^2(y_i) \otimes \alpha(y_k) \otimes \alpha^2(y_j),\\
G_{13} &= \sum_{i,j,k,l} [[x, x_i, x_j], x_k, x_l] \otimes \alpha(y_l) \otimes \alpha^2(y_j) \otimes \alpha^2(y_i) \otimes \alpha(y_k),\\
G_{14} &= -\sum_{i,j,k,l} \alpha([x, x_i, x_j]) \otimes \alpha^2(y_j) \otimes [\alpha(y_i), x_k, x_l] \otimes \alpha(y_l) \otimes \alpha(y_k),\\
G_{15} &= -\sum_{i,j,k,l} \alpha([x, x_i, x_j]) \otimes \alpha^2(y_j) \otimes \alpha(y_k) \otimes [\alpha(y_i), x_k, x_l] \otimes \alpha(y_l),\\
G_{16} &= -\sum_{i,j,k,l} \alpha([x, x_i, x_j]) \otimes \alpha^2(y_j) \otimes \alpha(y_l) \otimes \alpha(y_l) \otimes [\alpha(y_i), x_k, x_l].
\end{aligned}$$

By Hom-Jacobi identity, we have

$$\begin{aligned}
&G_{11} + G_{12} + G_{13}\\
&\quad= \sum_{i,j,k,l} [[x_i, x_j, x_k], \alpha(x), \alpha(x_l)] \otimes \alpha^2(y_l) \otimes \alpha^2(y_k) \otimes \alpha^2(y_j) \otimes \alpha^2(y_i)\\
&\quad= \sum_{i,j,k,l} (ad_1(x_l, x) \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha)[x_k, x_j, x_i] \otimes \alpha(y_l) \otimes \alpha(y_k) \otimes \alpha(y_j) \otimes \alpha(y_i)\\
&\quad= \sum_{i} (ad_1(x_l, x) \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha)[r_{12}, r_{13}, r_{14}] \otimes_2 \alpha(y_l).
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
G_{14} &= \sum_{i,j,k,l} (ad_1(x_j, x) \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha)\alpha(x_i) \otimes \alpha(y_j) \otimes [y_i, x_l, x_k] \otimes \alpha(y_l) \otimes \alpha(y_k)\\
&= \sum_{j} (ad_1(x_j, x) \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha)[r_{12}, r_{23}, r_{24}] \otimes_2 \alpha(y_j).
\end{aligned}$$

Similarly,

$$G_{15} = -\sum_j (ad_1(x_j, x) \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha)[r_{13}, r_{32}, r_{34}] \otimes_2 \alpha(y_j),$$

$$G_{16} = \sum_j (ad_1(x_j, x) \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha)[r_{14}, r_{42}, r_{43}] \otimes_2 \alpha(y_j).$$

Therefore, we obtain

$$G_1 = \sum_i (ad_1(x_i, x) \otimes \alpha \otimes \alpha \otimes \alpha \otimes \alpha)[[r, r, r]]_1^\alpha \otimes_2 \alpha(y_i).$$

In a similar manner, we have

$$G_2 = \sum_i (\alpha \otimes ad_1(x_i, x) \otimes \alpha \otimes \alpha \otimes \alpha)[[r, r, r]]_1^\alpha \otimes_1 \alpha(y_i).$$

There are 8 terms in G_3 :

$$G_3 = G_{31} + G_{32} + G_{33} + G_{34} + G_{35} + G_{36} + G_{37} + G_{38},$$

where

$$G_{31} = \sum_{i,j,k,l} \alpha(y_l) \otimes \alpha(y_k) \otimes [[x, x_i, x_j], x_k, x_l] \otimes \alpha^2(y_j) \otimes \alpha^2(y_i),$$

$$G_{32} = -\sum_{i,j,k,l} \alpha^2(y_j) \otimes \alpha^2(y_i) \otimes [[x, x_i, x_j], x_k, x_l] \otimes \alpha(y_l) \otimes \alpha(y_k),$$

$$G_{33} = \sum_{i,j,k,l} [\alpha(y_j), x_k, x_l] \otimes \alpha(y_l) \otimes \alpha[x, x_i, x_j] \otimes \alpha(y_k) \otimes \alpha^2(y_i),$$

$$G_{34} = \sum_{i,j,k,l} \alpha(y_k) \otimes [\alpha(y_j), x_k, x_l] \otimes \alpha[x, x_i, x_j] \otimes \alpha^2(y_j) \otimes \alpha(y_k),$$

$$G_{35} = \sum_{i,j,k,l} \alpha(y_l) \otimes \alpha(y_k) \otimes \alpha[x, x_i, x_j] \otimes [\alpha(y_j), x_k, x_l] \otimes \alpha^2(y_i),$$

$$G_{36} = \sum_{i,j,k,l} [\alpha(y_i), x_k, x_l] \otimes \alpha(y_l) \otimes \alpha[x, x_i, x_j] \otimes \alpha^2(y_j) \otimes \alpha(y_k),$$

$$G_{37} = \sum_{i,j,k,l} \alpha(y_k) \otimes [\alpha(y_i), x_k, x_l] \otimes \alpha[x, x_i, x_j] \otimes \alpha^2(y_j) \otimes \alpha(y_l),$$

$$G_{38} = \sum_{i,j,k,l} \alpha(y_l) \otimes \alpha(y_k) \otimes \alpha[x, x_i, x_j] \otimes \alpha^2(y_j) \otimes [\alpha(y_i), x_k, x_l].$$

We have

$$\begin{aligned}
&G_{31} + G_{32} \\
&\quad = \sum_{i,j,k,l} \alpha^2(y_l) \otimes \alpha^2(y_k) \otimes ([\alpha(x), [x_i, x_k, x_l], \alpha(x_j)] + \\
&\quad\quad [\alpha(x), \alpha(x_i), [x_j, x_k, x_l]]) \otimes \alpha^2(y_j) \otimes \alpha^2(y_i) \\
&\quad = -\sum_{i,j,k,l} (\alpha \otimes \alpha \otimes ad_1(x_j, x) \otimes \alpha \otimes \alpha)\alpha(y_l) \otimes \alpha(y_k) \otimes [x_l, x_k, x_i] \otimes \alpha(y_j) \otimes \alpha(y_i) - \\
&\quad\quad \sum_{i,j,k,l} (\alpha \otimes \otimes\alpha \otimes ad_1(x, x_i)\alpha \otimes \alpha)\alpha(y_l) \otimes \alpha(y_k) \otimes [x_l, x_k, x_j] \otimes \alpha(y_j) \otimes \alpha(y_i)
\end{aligned}$$

$$= \sum_{j}(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)[r_{31},r_{32},r_{34}]\otimes_4\alpha(y_j)- \sum_{i}(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[r_{31},r_{32},r_{34}]\otimes_5\alpha(y_i).$$

Furthermore, we have

$$\begin{aligned} G_{33}+G_{36} &= \sum_{i,j,k,l}(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[x_l,y_j,x_k]\otimes\alpha(y_l)\otimes\alpha(y_j)\otimes \\ &\quad \alpha(y_k)\otimes\alpha(y_i)+\sum_{i,j,k,l}(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)\alpha(y_j)\otimes\alpha(y_k) \\ &= \sum_{i}(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[r_{12},r_{31},r_{14}]\otimes_5\alpha(y_i)+ \\ &\quad \sum_{j}(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)[r_{12},r_{31},r_{14}]\otimes_4\alpha(y_j), \end{aligned}$$

and similarly,

$$\begin{aligned} G_{34}+G_{37} &= -\sum_{i}(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[r_{21},r_{32},r_{24}]\otimes_5\alpha(y_i)- \\ &\quad \sum_{j}(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)[r_{21},r_{32},r_{24}]\otimes_4\alpha(y_j), \\ G_{35}+G_{38} &= -\sum_{i}(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[r_{41},r_{42},r_{34}]\otimes_5\alpha(y_i)- \\ &\quad \sum_{j}(\alpha\otimes\alpha\otimes ad_1(x_j,x)\otimes\alpha\otimes\alpha)[r_{41},r_{42},r_{34}]\otimes_4\alpha(y_j). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} G_3 &= \sum_{i}(\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)[[r,r,r]]_2^{\alpha}\otimes_5\alpha(y_i)+ \\ &\quad \sum_{i}(\alpha\otimes\alpha\otimes ad_1(x_i,x)\otimes\alpha\otimes\alpha)[[r,r,r]]_2^{\alpha}\otimes_4\alpha(y_i). \end{aligned}$$

We similarly obtain

$$\begin{aligned} G_4 &= \sum_{i}(\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha)[[r,r,r]]_2^{\alpha}\otimes_3\alpha(y_i)+ \\ &\quad \sum_{i}(\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x_i,x)\otimes\alpha)[[r,r,r]]_3^{\alpha}\otimes_5\alpha(y_i). \\ G_5 &= \sum_{i}(\alpha\otimes\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x,x_i))[[r,r,r]]_3^{\alpha}\otimes_4\alpha(y_i)+ \\ &\quad \sum_{i}(\alpha\otimes\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x_i,x))[[r,r,r]]_3^{\alpha}\otimes_3\alpha(y_i). \end{aligned}$$

This completes the proof. □

With the notations above, if r is skew-symmetric, then we can check that

$$[[r,r,r]]_1^{\alpha}=[[r,r,r]]^{\alpha},\ [[r,r,r]]_2^{\alpha}=-[[r,r,r]]^{\alpha},\ [[r,r,r]]_3^{\alpha}=[[r,r,r]]^{\alpha}.$$

We can obtain an equivalence description of (8):

$$\sum_i (ad_1(x_i,x)\otimes\alpha\otimes\alpha\otimes\alpha\otimes\alpha)([[r,r,r]]^\alpha\otimes_2\alpha(y_i))+$$
$$\sum_i (\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha\otimes\alpha)([[r,r,r]]^\alpha\otimes_1\alpha(y_i))-$$
$$\sum_i (\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha\otimes\alpha)([[r,r,r]]^\alpha\otimes_5\alpha(y_i))-$$
$$\sum_i (\alpha\otimes\alpha\otimes ad_1(x_i,x)\otimes\alpha\otimes\alpha)([[r,r,r]]^\alpha\otimes_4\alpha(y_i))-$$
$$\sum_i (\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x,x_i)\otimes\alpha)([[r,r,r]]^\alpha\otimes_3\alpha(y_i))+$$
$$\sum_i (\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x_i,x)\otimes\alpha)([[r,r,r]]^\alpha\otimes_5\alpha(y_i))+$$
$$\sum_i (\alpha\otimes\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x,x_i))([[r,r,r]]^\alpha\otimes_4\alpha(y_i))+$$
$$\sum_i (\alpha\otimes\alpha\otimes\alpha\otimes\alpha\otimes ad_1(x_i,x))([[r,r,r]]^\alpha\otimes_3\alpha(y_i))=0.$$

Summarizing the above discussion, we obtain

Corollary 3.3 *Let L be a 3-Hom-Lie algebra, $\alpha^{\otimes^2}(r)=r$ and $r\in L\otimes L$ skew-symmetric. If*

$$[[r,r,r]]^\alpha=0,$$

and $\Delta=\Delta_1+\Delta_2+\Delta_3: L\to L\otimes L\otimes L$, in which $\Delta_1,\Delta_2,\Delta_3$ are included by r as in Eq. (7). Then $(L,[\cdot,\cdot,\cdot],\Delta,\alpha)$ is a local cocycle 3-Hom-Lie bialgebra.

Corollary 3.3 can be regarded as a 3-Hom-Lie algebra analogue of the fact that $\alpha^{\otimes^2}(r)=r$ and a skew-symmetric solution of the classical Yang-Baxter equation gives a local cocycle 3-Hom-Lie bialgebra.

The next result shows that given a 3-Lie algebra endomorphism, each classical r-matrix induces an infinite family of solutions of the 3-Lie CHYBE.

Theorem 3.4 *Let L be a 3-Lie algebra, $r\in L^{\otimes^2}$ be a solution of 3-Lie CYBE, and $\alpha: L\to L$ be a 3-Lie algebra endomorphism. Then for each integer $n\geq 0$, $(\alpha^{\otimes^2})^n(r)$ is a solution of 3-Lie CHYBE in the 3-Hom-Lie algebra $L_\alpha=(L,[\cdot,\cdot,\cdot]_\alpha=\alpha\circ[\cdot,\cdot,\cdot],\alpha)$.*

Proof We can prove that L_α is a 3-Hom-Lie algebra (in fact, the Hom-Jacobi identity for $[\cdot,\cdot,\cdot]_\alpha$ is α^3 applied to the Jacobi identity of $[\cdot,\cdot,\cdot]$). It remains to show that $(\alpha^{\otimes^2})^n(r)$ satisfies the 3-Lie CHYBE in the 3-Hom-Lie algebra L_α, i.e.,

$$[[(\alpha^{\otimes^2})^n(r),(\alpha^{\otimes^2})^n(r),(\alpha^{\otimes^2})^n(r)]]^\alpha=0.$$

Write $r=\sum_i x_i\otimes y_i$. Using $\alpha([\cdot,\cdot,\cdot])=[\cdot,\cdot,\cdot]\circ\alpha^{\otimes^2}$ and the definition $[\cdot,\cdot,\cdot]_\alpha=\alpha([\cdot,\cdot,\cdot])$, we have

$$[[(\alpha^{\otimes^2})^n(r),(\alpha^{\otimes^2})^n(r),(\alpha^{\otimes^2})^n(r)]]^\alpha$$

$$= \sum_{i,j,k}([\alpha^n(x_i), \alpha^n(x_j), \alpha^n(x_k)]_\alpha \otimes \alpha(\alpha^n(y_i)) \otimes \alpha(\alpha^n(y_j)) \otimes \alpha(\alpha^n(y_k)) +$$
$$\alpha(\alpha^n(x_i)) \otimes [\alpha^n(y_i), \alpha^n(x_j), \alpha^n(x_k)]_\alpha \otimes \alpha(\alpha^n(y_j)) \otimes \alpha(\alpha^n(y_k)) +$$
$$\alpha(\alpha^n(x_i)) \otimes \alpha(\alpha^n(x_j)) \otimes [\alpha^n(y_i), \alpha^n(y_j), \alpha^n(x_k)]_\alpha \otimes \alpha(\alpha^n(y_k)) +$$
$$\alpha(\alpha^n(x_i)) \otimes \alpha(\alpha^n(x_j)) \otimes \alpha(\alpha^n(x_k)) \otimes [\alpha^n(y_i), \alpha^n(y_j), \alpha^n(y_k)]_\alpha)$$
$$= \alpha^{n+1}([[r, r, r]]) = 0. \quad \square$$

Example 3.5 Let L be the (unique) non-trivial 3-dimensional complex 3-Lie algebra whose non-zero product with respect to a basis e_1, e_2, e_3 is given by $[e_1, e_2, e_3] = e_1$. If $r = \sum_{i<j}^3 r_{ij}(e_i \otimes e_j - e_j \otimes e_i) \in L \otimes L$ is skew-symmetric and α is the 3-Lie algebra morphism given by

$$\alpha(e_1) = a_{11}e_1, \alpha(e_2) = a_{12} + a_{22}e_2 + a_{32}e_3, \alpha(e_3) = a_{13} + a_{23}e_2 + a_{33}e_3,$$
$$\forall a_{ij} \in \mathbb{C}, \quad 1 \leq i, j \leq 3$$

with

$$a_{22}a_{33} - a_{23}a_{32} = 1,$$
$$(r_{12}a_{11} - r_{23}a_{13})a_{22} + (r_{13}a_{11} + r_{23}a_{12})a_{23} = r_{12},$$
$$(r_{12}a_{11} - r_{23}a_{13})a_{32} + (r_{13}a_{11} + r_{23}a_{12})a_{33} = r_{13}.$$

Then $L_\alpha = (L, [\cdot,\cdot,\cdot]_\alpha = \alpha \circ [\cdot,\cdot,\cdot], \alpha)$ is a 3-Hom-Lie algebra and $\alpha^{\otimes^2}(r) = r$. Moreover, r is a solution of the 3-Lie CHYBE in L, the corresponding local cocycle 3-Hom-Lie bialgebra is given by

$$\Delta_i(e_1) = (-1)^i r_{23} r \otimes_i a_{11}e_1,$$
$$\Delta_i(e_2) = (-1)^{i+1} r_{13} r \otimes_i a_{11}e_1,$$
$$\Delta_i(e_3) = (-1)^i r_{12} r \otimes_i a_{11}e_1$$

and the comultiplication $\Delta : L \to \wedge^3 L$ is given by

$$\Delta(e_1) = -r_{23}^2 a_{11} e_1 \wedge e_2 \wedge e_3, \Delta(e_2) = r_{13}r_{23}a_{11}e_1 \wedge e_2 \wedge e_3, \Delta(e_3) = -r_{12}r_{23}a_{11}e_1 \wedge e_2 \wedge e_3,$$

where $e_1 \wedge e_2 \wedge e_3 = \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}$ and S_3 is the permutation group on $\{1, 2, 3\}$.

Example 3.6 Let L be the (unique) non-trivial 4-dimensional complex 3-Lie algebra whose non-zero product with respect to a basis e_1, e_2, e_3, e_4 is given by $[e_1, e_2, e_3] = e_1$. If $r = -e_3 \otimes e_4 + e_4 \otimes e_3 + \sum_{i<j}^4 (e_i \otimes e_j - e_j \otimes e_i) \in L \otimes L$ is skew-symmetric and α is the 3-Lie algebra morphism given by

$$\alpha(e_1) = a_{11}e_1,$$
$$\alpha(e_2) = a_{12}e_1 + a_{22}e_2 + e_3 + e_4,$$
$$\alpha(e_3) = a_{13}e_1 + a_{23}e_2 + 2e_3 + 2e_4,$$
$$\alpha(e_4) = 0,$$

where

$$a_{11}(a_{22}+a_{23})+a_{12}a_{23}-a_{22}a_{13}=1,$$

$$3a_{11}+2a_{12}-a_{13}=1,$$

$$2a_{22}-a_{23}=1.$$

Then $L_\alpha=(L,[\cdot,\cdot,\cdot]_\alpha=\alpha\circ[\cdot,\cdot,\cdot],\alpha)$ is a 3-Hom-Lie algebra and $\alpha^{\otimes^2}(r)=r$. Moreover, r is a solution of the 3-Lie CHYBE in L, the corresponding nontrivial local cocycle 3-Hom-Lie bialgebra is given by

$$\Delta_i(e_1)=(-1)^i r\otimes_i(a_{11}e_1),$$

$$\Delta_i(e_2)=(-1)^{i+1} r\otimes_i(a_{11}e_1),$$

$$\Delta_i(e_3)=(-1)^i r\otimes_i(a_{11}e_1),$$

where $i=1,2,3$ and the comultiplication $\Delta: L\to\wedge^3 L$ is given by

$$\Delta(e_1)=\Delta(e_3)=-\Delta(e_2),\Delta(e_4)=0,$$

$$\Delta(e_2)=a_{11}e_1\wedge e_2\wedge e_3+a_{11}e_1\wedge e_2\wedge e_4,$$

where $e_1\wedge e_2\wedge e_3=\sum_{\sigma\in S_3}\mathrm{sgn}(\sigma)e_{\sigma(1)}\otimes e_{\sigma(2)}\otimes e_{\sigma(3)}$ and S_3 is the permutation group on $\{1,2,3\}$.

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