Journal of Mathematical Research with Applications
Nov., 2017, Vol. 37, No. 6, pp. 697–709
DOI:10.3770/j.issn:2095-2651.2017.06.006
Http://jmre.dlut.edu.cn

L_p Stability of the Truncated Hierarchical B-Spline Basis

Jianping ZHOU, Renhong WANG, Chongjun LI*
Department of Mathematics, Dalian University of Technology, Liaoning 116024*, P. R. China*

Abstract The truncated hierarchical B-spline basis has been proposed for adaptive data fitting and has already drawn a lot of attention in theory and applications. However the stability with respect to the L_p-norm, $1 \leq p < \infty$, is not clear. In this paper, we consider the L_p stability of the truncated hierarchical B-spline basis, since the L_p stability is useful for curve and surface fitting, especially for least squares fitting. We prove that this basis is weakly L_p stable. This means that the associated constants to be considered in the stability analysis are at most of polynomial growth in the number of the hierarchy depth.

Keywords hierarchical spline space; truncated hierarchical B-spline basis; stable basis; condition number of basis; partition of unity

MR(2010) Subject Classification 41A15; 65D07; 65L20

1. Introduction

The splines are a widely used tool in computer graphics, animation, modeling, CAD, CAGD, and many other related fields, but they do not have the local refinement property, which is very important in adaptive function approximation and detailed representation of geometric models [1–4]. The hierarchical spline models provide a flexible and simple framework for local refinement such that they are effective in many applications [5].

The concept of the hierarchical splines was originally introduced in [6] as an accumulation of tensor-product splines with nested knot vectors. After that the researchers pay attention to how to construct hierarchical B-spline basis. The hierarchical B-spline (HB-spline for short) basis for tensor-product meshes was constructed in [7] and was extended in [8]. This basis does not have the property of partition of unity. Later, a kind of truncated hierarchical B-spline (THB-spline for short) basis was introduced in [9]. This basis forms a convex partition of unity and has smaller support than those of the HB-spline basis. Moreover, there are some work about hierarchical splines for non tensor-product mashes: polynomials splines over hierarchical T-meshes [10,11], hierarchical Powell-Sabin splines [12], hierarchical splines on regular triangular partitions [13], truncated hierarchical generating system for Zwart-Powell elements [14], and bivariate hierarchical quartic box splines on three-directional meshes [15].

Received May 25, 2017; Accepted September 1, 2017
Supported by the National Natural Science Foundation of China (Grant Nos. 11290143; 11471066), Fundamental Research of Civil Aircraft (Grant No. MJ-F-2012-04) and the Fundamental Research Funds for the Central Universities (Grant No. DUT15LK44).
* Corresponding author
E-mail address: jp.zhou@mail.dlut.edu.cn (Jianping ZHOU); renhong@dlut.edu.cn (Renhong WANG); chongjun @dlut.edu.cn (Chongjun LI)

Throughout this paper, we use Ω to denote an open interval $(a,b)\subset\mathbb{R}$ or an open rectangle $(a,b)\times(a',b')\subset\mathbb{R}^2$. The norm of a continuous function f defined on Ω is defined by

$$\|f\|_p=\|f\|_{p,\Omega}:=\begin{cases}(\int_\Omega|f|^p\mathrm{d}\mu)^{1/p}, & 1\le p<\infty,\\ \max_\Omega|f|, & p=\infty.\end{cases}\tag{1}$$

And the norm of a vector $\mathbf{c}=\{c_i\}_{i\in I}$ is defined by

$$\|\mathbf{c}\|_p:=\begin{cases}(\sum_{i\in I}|c_i|^p)^{1/p}, & 1\le p<\infty,\\ \max_{i\in I}|c_i|, & p=\infty.\end{cases}\tag{2}$$

If a basis is suitable for numerical computations, then it should be reasonably insensitive to round-off errors. That is to say, functions with small functions values should have small coefficients and vice versa. A basis with this property is said to be well conditioned or stable and the stability is measured by the condition number of the basis.

Definition 1.1 *Let V be a normed linear space. A basis $\{\phi_i\}_{i\in I}$ with the index set I for V is said to be stable with respect to a norm $\|\cdot\|_p$, $1\le p\le\infty$, if there exist two positive constants K_1 and K_2 such that*

$$K_1^{-1}\|\mathbf{c}\|_p\le\|\sum_{i\in I}c_i\cdot\phi_i\|_p\le K_2\|\mathbf{c}\|_p,\tag{3}$$

for all sets of coefficients $\mathbf{c}=\{c_i\}_{i\in I}$. Let K_1^* and K_2^* denote the smallest possible value of K_1 and K_2 such that inequality (3) holds. The condition number of the basis $\{\phi_i\}_{i\in I}$ is then defined to be $\kappa_p=\kappa_p(\{\phi_i\}_{i\in I})=K_1^*\cdot K_2^*$.

From this definition, we know that K_1K_2 gives an upper bound for the condition number κ_p of the basis $\{\phi_i\}_{i\in I}$ and K_1, K_2 should be small. There is no unique answer to the question how small the constants K_1 and K_2 should be, but it is typically required that the constants should be independent of the dimension of V, or at least grow very slowly with it.

The univariate and bivariate tensor-product B-spline basis are L_∞ stable [16–18]. The task of finding L_∞ stable basis for the multivariate spline spaces over triangulations can only be done for general triangulations when $d\ge 3r+2$ (see [3]). As for the HB-spline and THB-spline basis, the cases are more complicated. The HB-spline basis is a weakly L_∞ stable basis, provided that the nested subdomains satisfy certain conditions [19]. The absence of the strong L_∞ stability of the HB-spline basis is implied by the missing partition of unity. The THB-spline basis is a strongly L_∞ stable basis under certain reasonable assumptions on the given knot configuration [20]. The L_p stability of a basis, $1\le p<\infty$, is useful in many applications, for example the case $p=2$ is closely related to the least squares approximations. In this paper, we consider the L_p stability of the THB-spline basis, $1\le p<\infty$. We will prove that the THB-spline basis is a weakly L_p stable basis, $1\le p<\infty$, by using some classical tools of the mathematical analysis. This means that the associated constants to be considered in the stability analysis are at most of polynomial growth in the number of hierarchy depth. An example is proposed that the constants to be considered in the L_p stability analysis depend on the number of the hierarchical depth.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries about this paper. In Section 3, we review the hierarchical spline spaces. We give the proofs of the L_p stability ($1 \le p < \infty$) of the B-spline and the truncated hierarchical B-spline basis in Section 4 and 5, respectively. We conclude this paper with a summary in Section 6.

2. Preliminaries

Let $\{B_i\}_{i\in I}$ be a sequence of B-spline basis functions defined on Ω, where I is an index set such that the functions B_i do not vanish on Ω for $i \in I$. And let $S = \text{span}\{B_i\}_{i\in I}$ be the linear space of the linear combinations of $\{B_i\}_{i\in I}$ with real coefficients $\mathbf{c} = \{c_i\}_{i\in I}$.

Lemma 2.1 ([21]) *Let f be a real univariate polynomial of degree d having no zeros in interval $(-1, 1)$, and let f be positive on $(-1, 1)$. Then*

$$\max_{-1\le x\le 1} f(x) \le \frac{1+d}{2} \cdot \int_{-1}^{1} f(x)\mathrm{d}x. \tag{4}$$

Lemma 2.2 *Let f be a real univariate polynomial of degree d having no zeros in interval (a, b). Then*

$$\|f\|_\infty \le (1 + d)(b - a)^{-1/p}\|f\|_p, \quad 1 \le p < \infty. \tag{5}$$

Proof From Lemma 2.1 and the Hölder inequality, we obtain the result. □

Lemma 2.3 *Let f be a real bivariate polynomial in two variables of degree d in the first variable and of degree d' in the second variable having no zeros in $(a, b) \times (a', b')$. Then*

$$\|f\|_\infty \le (1 + d)(1 + d')[(b - a)(b' - a')]^{-1/p}\|f\|_p, \quad 1 \le p < \infty. \tag{6}$$

Proof Firstly, we fix the second variable. Using Lemma 2.2 to the first variable, we have

$$(\max_{a\le x\le b} |f(x, y)|)^p \le (1 + d)^p(b - a)^{-1} \int_a^b |f(x, y)|^p\mathrm{d}x, \quad \forall y \in (a', b'). \tag{7}$$

Then we take the maximum on both sides. Using Lemma 2.2 to the second variable, we obtain the result. □

For a univariate polynomial f of degree d, its blossom $B[f]$ is a function of d variables.

Definition 2.4 ([22]) *Let $f(x)$ be a univariate polynomial of degree d, its blossom $B[f](x_1, \ldots, x_d)$ is a function of d variables with the following properties:*

(i) Symmetric property:

$$B[f](x_1, \ldots, x_d) = B[f](\pi(x_1, \ldots, x_d)),\ \pi \text{ is any permutation of } \{x_1, \ldots, x_d\}; \tag{8}$$

(ii) Affine property:

$$B[f](\ldots, \alpha u + \beta v, \ldots) = \alpha B[f](\ldots, u, \ldots) + \beta B[f](\ldots, v, \ldots),\ \alpha + \beta = 1; \tag{9}$$

(iii) Diagonal property:

$$B[f](x, \ldots, x) = f(x). \tag{10}$$

From the blossom method we can obtain explicit formulas for the B-spline coefficients of a univariate spline function.

Lemma 2.5 ([22]) *Let $f=\sum_{i=0}^n c_iB_i$ be a univariate spline of degree d with knot vector $\boldsymbol{t}=\{t_i\}_{i=0}^{n+d+1}$. Then its B-spline coefficients can be given by*

$$c_i=B[f|_{(t_k,t_{k+1})}](t_{i+1},\dots,t_{i+d}),\quad k=i,i+1,\dots,i+d, \tag{11}$$

where $B[f](x_1,\dots,x_d)$ is the blossom of f and $f|_{(t_k,t_{k+1})}$ is the restriction of f to the interval (t_k,t_{k+1}).

Lemma 2.6 *Let $S=\operatorname{span}\{B_i\}_{i=0}^n$ be a univariate spline space of degree d with knot vector $\boldsymbol{t}=\{t_i\}_{i=0}^{n+d+1}$ and let any $f=\sum_{i=0}^n c_iB_i\in S$. Then*

$$|c_i|\le\frac{(2d(d-1))^d}{d!}\cdot\|f\|_{\infty,(t_i,t_{i+d+1})},\quad i=0,1,\dots,n. \tag{12}$$

Proof Let $[t_{l_i},t_{l_i+1}]$ be the largest subinterval of $[t_{i+1},t_{i+d}]\subset(t_i,t_{i+d+1})=\operatorname{supp}(B_i)$ for fixed i, $i=0,1,\dots,n$. And let $\{x_{i,k}\}_{k=0}^d$ be uniformly spaced points

$$x_{i,k}=t_{l_i}+k(t_{l_i+1}-t_{l_i})/d,\quad k=0,1,\dots,d \tag{13}$$

in the interval $[t_{l_i},t_{l_i+1}]$. Since the restriction of f to $[t_{l_i},t_{l_i+1}]$ is a univariate polynomial of degree d, it can be represented both in terms of $\{B_i\}_{i=0}^n$ and the Lagrange coefficient polynomials $\{p_{i,k}\}_{k=0}^d$

$$f(x)|_{[t_{l_i},t_{l_i+1}]}=\sum_{j=0}^n c_jB_j(x)|_{[t_{l_i},t_{l_i+1}]}=\sum_{j=l_i-d}^{l_i}c_jB_j(x)=\sum_{k=0}^d f(x_{i,k})p_{i,k}(x)|_{[t_{l_i},t_{l_i+1}]}, \tag{14}$$

where

$$p_{i,k}(x)=\prod_{j=0,j\ne k}^d\frac{x-x_{i,j}}{x_{i,k}-x_{i,j}},\quad x\in[t_{l_i},t_{l_i+1}],\ k=0,1,\dots,d. \tag{15}$$

On the other hand, $p_{i,k}$ also can be represented in terms of $\{B_i\}_{i=0}^n$

$$p_{i,k}(x)|_{[t_{l_i},t_{l_i+1}]}=\sum_{j=0}^n\omega_{i,k}^jB_j(x)|_{[t_{l_i},t_{l_i+1}]}=\sum_{j=l_i-d}^{l_i}\omega_{i,k}^jB_j(x),\quad k=0,1,\dots,d. \tag{16}$$

By substituting Eq. (16) into Eq. (14), we obtain

$$\begin{aligned}f(x)|_{[t_{l_i},t_{l_i+1}]}&=\sum_{j=l_i-d}^{l_i}c_jB_j(x)=\sum_{k=1}^d f(x_{i,k})p_{i,k}(x)\\&=\sum_{k=1}^d f(x_{i,k})(\sum_{j=l_i-d}^{l_i}\omega_{i,k}^jB_j(x))=\sum_{j=l_i-d}^{l_i}(\sum_{k=0}^d\omega_{i,k}^jf(x_{i,k}))B_j(x).\end{aligned} \tag{17}$$

Since $\{B_j\}_{j=0}^n$ are B-spline basis functions, we have

$$c_i=\sum_{k=0}^d\omega_{i,k}^if(x_{i,k}). \tag{18}$$

From Lemma 2.5, we have

$$\omega_{i,k}^i = \frac{1}{d!} \sum_{(j_1,\ldots,j_d)\in\pi_d} \prod_{r=1}^{d} \frac{(t_{i+j_1}-x_{i,0})\cdots(t_{i+j_k}-x_{i,k-1})(t_{i+j_{k+1}}-x_{i,k+1})\cdots(t_{i+j_d}-x_{i,d})}{(x_{i,k}-x_{i,0})\cdots(x_{i,k}-x_{i,k-1})(x_{i,k}-x_{i,k+1})\cdots(x_{i,k}-x_{i,d})}, \tag{19}$$

for $k = 0, 1, \ldots, d$, where π_d denotes the set of all permutations of the integers $\{1, 2, \ldots, d\}$. Since $t_{i+j_1}, \ldots, t_{i+j_d}, x_{i,0}, \ldots, x_{i,k-1}, x_{i,k+1}, \ldots, x_{i,d} \in [t_{i+1}, t_{i+d}]$, we have

$$(t_{i+j_1}-x_{i,0})\cdots(t_{i+j_k}-x_{i,k-1})(t_{i+j_{k+1}}-x_{i,k+1})\cdots(t_{i+j_d}-x_{i,d}) \le (t_{i+d}-t_{i+1})^d. \tag{20}$$

From Eq. (13) we know that $x_{i,k} - x_{i,l} = (k-l)(t_{l_i+1} - t_{l_i})/d$ for $1 \le l \le d$ but with $l \ne k$. Since $[t_{l_i}, t_{l_i+1}]$ is the largest subinterval of $[t_{i+1}, t_{i+d}]$, we have

$$\begin{aligned}\prod_{l=0,l\ne k}^{d} |x_{i,k}-x_{i,l}| &= \prod_{l=0,l\ne k}^{d} \frac{|k-l|}{d}(t_{l_i+1}-t_{l_i}) = k!(d-k)!(\frac{t_{l_i+1}-t_{l_i}}{d})^d \\ &\ge k!(d-k)!(\frac{t_{i+d}-t_{i+1}}{d(d-1)})^d \end{aligned} \tag{21}$$

for all k. The sum in Eq. (19) contains $d!$ terms which means that

$$\begin{aligned}\sum_{k=0}^{d} |\omega_{i,k}| &\le \sum_{k=0}^{d} \frac{1}{d!} d! \frac{(t_{i+d}-t_{i+1})^d}{k!(d-k)!(\frac{t_{i+d}-t_{i+1}}{d(d-1)})^d} = \sum_{k=0}^{d} \frac{(d(d-1))^d}{d!} \frac{d!}{k!(d-k)!} \\ &= \frac{(d(d-1))^d}{d!} \cdot \sum_{k=0}^{d} \frac{d!}{k!(d-k)!} = \frac{(d(d-1))^d}{d!} \cdot \sum_{k=0}^{d} \binom{d}{k} \\ &= \frac{(2d(d-1))^d}{d!}. \end{aligned} \tag{22}$$

Combining Eq. (18) and inequality (22), we have

$$\begin{aligned}|c_i| &= |\sum_{k=0}^{d} \omega_{i,k}^i f(x_{i,k})| \le \sum_{k=0}^{d} |\omega_{i,k}^i| \cdot \|f\|_{\infty,(t_i,t_{i+d+1})} \\ &\le \frac{(2d(d-1))^d}{d!} \cdot \|f\|_{\infty,(t_i,t_{i+d+1})}, \quad i = 0, 1, \ldots, n. \end{aligned} \tag{23}$$

Lemma 2.7 *Let $S = \mathrm{span}(B_i \times B'_j)_{i=0,\ldots,n;j=0,\ldots,n'}$ be a bivariate tensor-product spline space in two variables of degree d in the first variable and degree d' in the second variable with knot vector $\boldsymbol{t} \times \boldsymbol{t'} = \{t_i\}_{i=0}^{n+d+1} \times \{t'_j\}_{j=0}^{n'+d'+1}$, and let any $f = \sum_{i=0}^{n} \sum_{j=0}^{n'} c_{ij} B_i B'_j \in S$. Then*

$$\begin{aligned}|c_{ij}| &\le \frac{(2d(d-1))^d (2d'(d'-1))^{d'}}{d!d'!} \|f\|_{\infty,(t_i,t_{i+d+1})\times(t'_j,t'_{j+d'+1})}, \\ &\qquad i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, n'. \end{aligned} \tag{24}$$

Proof Firstly, we fix the second variable and use Lemma 2.6 to the first variable. Then we use Lemma 2.6 to the second variable. The result follows. □

Lemma 2.8 *Let $f = \sum_{i=0}^{n} c_i B_i$ be a univariate spline function of degree d with knot vector $\boldsymbol{t} = \{t_i\}_{i=0}^{n+d+1}$, and let $h = \max_i |t_{i+1} - t_i|$ be the maximum interval between knots. Then*

$$|c_i| \le h^{-1/p} (2d(d-1))^d \frac{(1+d)^{1-1/p}}{d!} \cdot \|f\|_{p,(t_i,t_{i+d+1})}, \tag{25}$$

for all sets of coefficients $\{c_i\}_{i=1}^n$, $1 \le p < \infty$.

Proof Using Lemmas 2.2 and 2.6 we have the result. □

Lemma 2.9 *Let* $f = \sum_{i=0}^n \sum_{j=0}^{n'} c_{ij} B_i B'_j$ *be a bivariate tensor-product spline in two variables of degree* d *in the first variable and degree* d' *in the second variable, and let* $h = \max_{i,j} |t_{i+1} - t_i| \cdot |t'_{j+1} - t'_j|$ *be the maximum cell between all cells. Then*

$$|c_{ij}| \le h^{-1/p}(2d(d+1))^d(2d'(d'+1))^{d'} \frac{[(1+d)(1+d')]^{1-1/p}}{d!d'!} \cdot \|f\|_{p,(t_i,t_{i+d+1})\times(t'_j,t'_{j+d'+1})}, \quad (26)$$

for all sets of coefficients $\{c_{ij}\}_{1\le i\le n, 1\le j\le n'}$, $1 \le p < \infty$.

Proof From Lemmas 2.3 and 2.7 we have the result. □

3. Hierarchical spline spaces

In what follows, let

$$S^0 \subset S^1 \subset S^2 \subset \cdots \quad (27)$$

be an infinite sequence of nested linear spaces defined on Ω. We assume that each space S^l is spanned by a B-spline basis $B^l = \{B_k^l\}_{k\in I_l}$, where I_l is the index set such that the functions B_k^l do not vanish on Ω for $k \in I_l$. Those basis functions of B^l are defined on a partition Δ^l of Ω, where Δ^{l+1} is a refinement of Δ^l, $l = 0, 1, \ldots$. In addition, let the hierarchy $\{\Omega^l\}$ be a sequence of nested bounded domains

$$\Omega = \Omega^0 \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^l \supseteq \cdots, \quad (28)$$

where l represents the level of the hierarchy and Ω^l represents the domain selected to be refined at level l. We also suppose that the boundary $\partial\Omega^l$ is aligned with the edges of Δ^l. The hierarchical B-spline basis [7] is defined by

$$\mathcal{H} = \bigcup_{l\in\mathbb{Z}^+} \mathcal{H}^l = \bigcup_{l\in\mathbb{Z}^+} \{\beta \in B^l : \mathrm{supp}(\beta) \cap D^l \ne \emptyset\}. \quad (29)$$

However, this basis does not have the property of partition of unity and has relatively large support. In view of this, a kind of truncated hierarchical B-spline basis was presented [9]. The THB-spline basis is defined by

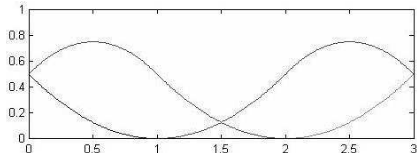
$$\mathcal{T} = \bigcup_{l\in\mathbb{Z}^+} \mathcal{T}^l = \bigcup_{l\in\mathbb{Z}^+} \{\mathrm{trunc}^{l+1}(\beta) : \beta \in B^l, \text{ and } \mathrm{supp}(\beta) \cap D^l \ne \emptyset\}, \quad (30)$$

where the truncation of $f \in S^l$ with respect to B^{l+1} on Ω^{l+1} is defined by

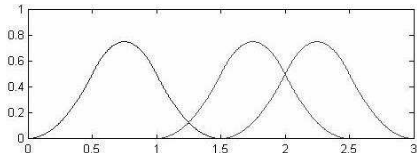
$$\mathrm{trunc}^{l+1}(f) = \sum_{\mathrm{supp}(B_k^{l+1}) \not\subseteq \Omega^{l+1}} c_k^{l+1} B_k^{l+1}, \quad (31)$$

where $f = \sum_k c_k^{l+1} B_k^{l+1}$ is its representation with respect to B^{l+1}.

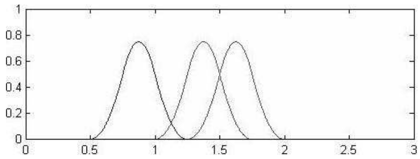
In what follows, we will use $\mathfrak{S}$ to denote the hierarchical space span$\mathcal{H}$ = span$\mathcal{T}$, N to denote the depth of the hierarchy, and $\mathcal{T} = \{\tau_{lk} : l \in \mathbb{Z}^+, k \in I_l^\tau\}$ to denote the THB-spline basis, where I_l^τ is the index set such that the functions τ_{lk} do not vanish on Ω^l for $k \in I_l^\tau$. An example of an HB-spline basis is shown in Figure 1. And an example of a THB-spline basis is shown in Figure 2. It can be seen that the supports of the THB-spline basis are smaller than those of the HB-spline basis.



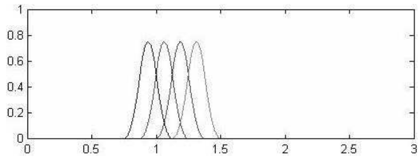
(a) HB-spline basis functions of level 0



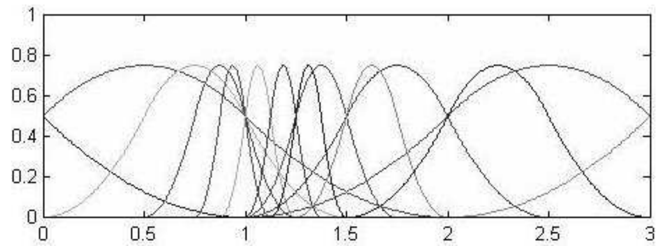
(b) HB-spline basis functions of level 1



(c) HB-spline basis functions of level 2



(d) HB-spline basis functions of level 3



(e) Combination of basis functions from the hierarchical levels 0-3

Figure 1 Univariate quadratic HB-splines defined on the hierarchical meshes

Lemma 3.1 ([20]) *The THB-spline basis* $\mathcal{T} = \{\tau_{lk} : l \in \mathbb{Z}^+, k \in I_l^\tau\}$ *has the following properties:*

(i) *Non-negativity:* $\tau_{lk} \geq 0$, $\forall \tau_{lk} \in \mathcal{T}$;

(ii) *Locally compact support:* $\forall \tau_{lk} \in \mathcal{T}$ *has locally compact support;*

(iii) *Nested property:* $\mathrm{span}\mathcal{T}^l \subseteq \mathrm{span}\mathcal{T}^{l+1}$, $l \in \mathbb{Z}^+$;

(iv) $\mathcal{T}$ *forms a convex partition of unity;*

(v) $\forall \tau_{lk} \in \mathcal{T}$ *with level* l, $\exists B_k^l \in B^l$ *so that*

$$\tau_{lk}|_{\Omega^l \backslash \Omega^{l+1}} = B_k^l|_{\Omega^l \backslash \Omega^{l+1}}. \tag{32}$$

B_k^l *is called the mother of* τ_{lk} *and indicated by* $\mathrm{mot}(\tau_{lk})$. τ_{lk} *is called the child of* B_k^l *and indicated by* $\mathrm{child}(B_k^l)$. *The level of* B_k^l *is the level of* τ_{lk};

(vi) *If the restriction of* f *to* $\pi_{l_0k_0} = (\Omega^{l_0} \backslash \Omega^{l_0+1}) \bigcap \mathrm{supp}(\tau_{l_0k_0})$, *indicated as* $f|_{\pi_{l_0k_0}}$, *can be represented both in terms of basis of* B^l *and THB-spline basis*

$$f|_{\pi_{l_0k_0}} = \sum_{B_{k_0}^{l_0} \in B^{l_0}} c_{k_0}^{l_0} B_{k_0}^{l_0}|_{\pi_{l_0k_0}} = \sum_{k_0 \in \mathbb{Z}^+} \sum_{l_0 \in I_{l_0}} c_{l_0k_0} \tau_{l_0k_0}|_{\pi_{l_0k_0}}, \tag{33}$$

where $B_{k_0}^{l_0} = \mathrm{mot}(\tau_{l_0k_0})$, *then*

$$c_{k_0}^{l_0} = c_{l_0k_0}. \tag{34}$$

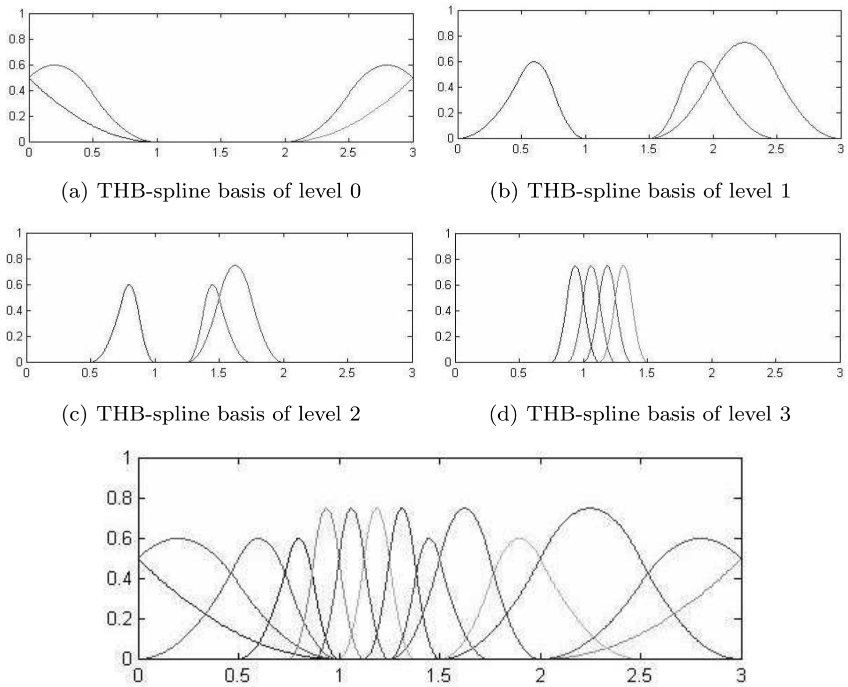
The concepts of weakly and strongly stable basis were introduced in the context of multiresolution analysis [23]. They were extended to the context of the hierarchical spline spaces [20]. The stability of the THB(or HB)-spline basis depends not only on the number of the hierarchy depth N, but also on the choice of the hierarchy. This is different from the case of the stability of wavelet expansions, which depend solely on the number N.

Definition 3.2 *A basis $\{\phi_i\}_{i\in I}$ for the hierarchical spline space $\mathfrak{S}$ is called a strongly L_p stable basis, $1 \le p \le \infty$, if there exist two constants K_1 and K_2 which are independent of the hierarchy $\{\Omega^l\}$ such that*

$$K_1^{-1}\|\boldsymbol{c}\|_p \le \|\sum_{i\in I} c_i\phi_i\|_p \le K_2\|\boldsymbol{c}\|_p, \tag{35}$$

for all sets of coefficients $\mathbf{c} = \{c_i\}_{i\in I}$. The basis $\{\phi_i\}_{i\in I}$ is called a weakly L_p stable basis if there exist two polynomials $K_1(N)$ and $K_2(N)$ which depend on the depth N of the hierarchy $\{\Omega^l\}$, such that inequality (35) holds for all sets of coefficients $\mathbf{c} = \{c_i\}_{i\in I}$.

The HB-spline basis is a weakly L_∞ stable basis, provided that the nested subdomains satisfy certain conditions [19]. The absence of the strong L_∞ stability of the HB-spline basis is implied by the missing partition of unity. The THB-spline basis is a strongly L_∞ stable basis under certain reasonable assumptions on the given knot configuration [20].



(a) THB-spline basis of level 0

(b) THB-spline basis of level 1

(c) THB-spline basis of level 2

(d) THB-spline basis of level 3

(e) Combination of basis from the hierarchical levels 0-3

Figure 2 Univariate quadratic THB-splines defined on the hierarchical meshes

4. L_p Stability of B-spline basis

It is well known that the univariate and bivariate tensor-product B-spline basis are L_∞ stable [16–18]. Now we consider the L_p stability of the univariate and bivariate tensor-product B-spline basis, where p is a real number in the interval $[1,\infty)$.

Theorem 4.1 *Let $S = \operatorname{span}\{B_i\}_{i=0}^n$ be a univariate spline space of degree d defined on $\Omega = [a,b] \subset \mathbb{R}$, and let $h = \max_i |t_{i+1} - t_i|$ be the maximum interval between knots. Then there exist two constants K_1 and K_2 such that*

$$K_1^{-1}\|\boldsymbol{c}\|_p \le \|f\|_p \le K_2\|\boldsymbol{c}\|_p, \tag{36}$$

for any $f = \sum_{i=0}^n c_i B_i \in S$ and all coefficients $\mathbf{c} = \{c_i\}_{i=0}^n$, where

$$K_1 = h^{-1/p}(1+d)\frac{(2d(d-1))^d}{d!} \tag{37}$$

and

$$K_2 = (b-a)^{1/p}. \tag{38}$$

Proof We first consider the upper inequality. Remembering q is the conjugate number of p and applying the Hölder inequality, we have

$$|f| = \Big|\sum_{i=0}^n c_i B_i\Big| \le \sum_{i=0}^n |c_i B_i| \le (\sum_{i=0}^n |c_i|^p)^{1/p}(\sum_{i=0}^n |B_i|^q)^{1/q} \le (\sum_{i=0}^n |c_i|^p)^{1/p}(\sum_{i=0}^n B_i)^{1/q}. \tag{39}$$

Raising this to the pth power, and recalling that $\{B_i\}_{i=0}^n$ have the property of partition of unity, we can obtain

$$|f|^p \le \sum_{i=0}^n |c_i|^p. \tag{40}$$

So we have

$$\|f\|_p^p = \int_a^b |f|^p \mathrm{d}x \le \int_a^b \sum_{i=0}^n |c_i|^p \mathrm{d}x = \sum_{i=0}^n |c_i|^p \cdot \int_a^b \mathrm{d}x = (b-a)\cdot\|\mathbf{c}\|_p^p. \tag{41}$$

Taking pth roots, we prove the upper inequality

$$\|f\|_p \le K_2 \cdot \|\mathbf{c}\|_p, \tag{42}$$

where $K_2 = (b-a)^{1/p}$.

Then we consider the lower inequality. Firstly we fix i. From Lemma 2.9 we have

$$|c_i| \le h^{-1/p}(2d(d-1))^d \frac{(1+d)^{1-1/p}}{d!} \cdot \|f\|_{p,(t_i,t_{i+d+1})}. \tag{43}$$

Raising this to the pth power and summing over i, we have

$$\|\mathbf{c}\|_p^p = \sum_{i=0}^n |c_i|^p \le [h^{-1/p}(2d(d-1))^d \frac{(1+d)^{1-1/p}}{d!}]^p \sum_{i=0}^n \int_{t_i}^{t_{i+d+1}} |f|^p \mathrm{d}x. \tag{44}$$

The above sum at most contains $d+1$ terms, which means that if we take pth roots, then we have

$$\|\mathbf{c}\|_p \le K_1 \cdot \|f\|_p, \tag{45}$$

where $K_1 = h^{-1/p}(1+d)(2d(d-1))^d/d!$. □

Theorem 4.2 *Let $S = \mathrm{span}\{B_i B'_j\}_{0\le i\le n, 0\le j\le n'}$ be a bivariate tensor-product spline space in two variables of degree d in the first variable and degree d' in the second variable defined on $\Omega = [a,b]\times[a',b'] \subset \mathbb{R}^2$. And let $h = \max_{i,j} |t_{i+1}-t_i|\cdot|t'_{j+1}-t'_j|$ be the maximum cell between all cells. Then there exist two constants K_1 and K_2 such that*

$$K_1^{-1}\|\mathbf{c}\|_p \le \|f\|_p \le K_2\|\mathbf{c}\|_p, \tag{46}$$

for any $f = \sum_{i=0}^{n}\sum_{j=0}^{n'} c_{ij}B_iB'_j \in S$ and all sets of coefficients $\mathbf{c} = \{c_{ij}\}$, where

$$K_1 := h^{-1/p}(1+d)(1+d')\frac{(2d(d-1))^d(2d'(d'-1))^{d'}}{d!d'!} \tag{47}$$

and

$$K_2 := [(b-a)(b'-a')]^{1/p}. \tag{48}$$

Proof This theorem can be proved by the same method as employed in Theorem 4.1. □

5. L_p Stability of the THB-spline basis

The L_p stability of a basis, $1 \le p < \infty$, is useful in many applications, for example the case $p = 2$ is closely related to the least squares approximations. In this section, we consider the L_p-stability of the THB-spline basis, $1 \le p < \infty$.

Theorem 5.1 *Let $\mathfrak{S}$ be a univariate hierarchical spline space of degree d defined on $\Omega = [a,b] \subset \mathbb{R}$ and $\mathcal{T} = \{\tau_{lk} : l \in \mathbb{Z}^+, k \in I_l^\tau\}$ be the THB-spline basis, where I_l^τ is the index set such that the functions τ_{lk} do not vanish on Ω^l for $k \in I_l^\tau$. And let h be the maximum cell between all cells. Then there exist two constants $K_1(N)$ which depends on the number of the hierarchy depth N and K_2 such that*

$$K_1(N)^{-1}\|\mathbf{c}\|_p \le \|f\|_p \le K_2\|\mathbf{c}\|_p, \tag{49}$$

for any $f = \sum_{l=0}^{N}\sum_{k\in I_l^\tau} c_{lk}\tau_{lk} \in \mathfrak{S}$ and all sets of $\mathbf{c} = \{c_{lk}\}$, where

$$K_1(N) := h^{-1/p}(1+N)^{1/p}(1+d)\frac{(2d(d-1))^d}{d!} \tag{50}$$

and

$$K_2 := (b-a)^{1/p}. \tag{51}$$

Proof We first consider the upper inequality. Remembering q is the conjugate number of p and applying the Hölder inequality, we have

$$\begin{aligned} |f| &= |\sum_{l=0}^{N}\sum_{k\in I_l^\tau} c_{lk}\tau_{lk}| \le \sum_{l=0}^{N}\sum_{k\in I_l^\tau} |c_{lk}\tau_{lk}| \\ &\le (\sum_{l=0}^{N}\sum_{k\in I_l^\tau} |c_{lk}|^p)^{1/p}(\sum_{l=0}^{N}\sum_{k\in I_l^\tau} |\ \tau_{lk}|^q)^{1/q} \\ &\le (\sum_{l=0}^{N}\sum_{k\in I_l^\tau} |c_{lk}|^p)^{1/p}(\sum_{l=0}^{N}\sum_{k\in I_l^\tau} \tau_{lk})^{1/q}, \end{aligned}$$

where $\mathcal{T} = \{\tau_{lk} : l \in \mathbb{Z}^+, k \in I_l^\tau\}$ is the THB-spline basis and I_l^τ is the index set such that the functions τ_{lk} do not vanish on Ω^l for $k \in I_l^\tau$. Raising this to the pth power, and recalling that all $\{\tau_{lk}\}$ sum to 1, we can obtain

$$| f |^p \leq \sum_{l=0}^{N} \sum_{k \in I_l^\tau} | c_{lk} |^p . \tag{53}$$

So we have

$$\begin{aligned} \|f\|_p^p &= \int_\Omega |f|^p \mathrm{d}\mu \leq \int_\Omega \sum_{l=0}^{N} \sum_{k \in I_l^\tau} |c_{lk}|^p \mathrm{d}\mu = \sum_{l=0}^{N} \sum_{k \in I_l^\tau} |c_{lk}|^p \cdot \int_\Omega \mathrm{d}\mu \\ &= (b-a) \cdot \|\mathbf{c}\|_p^p. \end{aligned} \tag{54}$$

Taking pth roots, we prove the upper inequality

$$\|f\|_p \leq K_2 \|\mathbf{c}\|_p, \tag{55}$$

where $K_2 = (b-a)^{1/p}$.

Then we consider the lower inequality. Firstly we fix l and k. From Eq. (34) and inequality (26) we have

$$|c_{lk}| \leq h^{-1/p} (2d(d-1))^d \frac{(1+d)^{1-1/p}}{d!} \cdot \|f\|_{p,\mathrm{supp}(\tau_{lk}) \bigcap (\Omega^l \backslash \Omega^{l+1})}. \tag{56}$$

Raising this to the pth power and summing over l and k, we have

$$\|\mathbf{c}\|_p^p = \sum_{l=0}^{N} \sum_{k \in I_l} | c_{lk} |^p \leq h^{-1} (2d(d-1))^{dp} \frac{(1+d)^{p-1}}{(d!)^p} \sum_{l=0}^{N} \sum_{k \in I_l^\tau} \int_{\mathrm{supp}(\tau_{lk}) \bigcap D^l} |f|^p \mathrm{d}\mu. \tag{57}$$

The above sum at most contains $(1+N)(1+d)$ terms, which means that if we take pth roots, then we have

$$\|\mathbf{c}\|_p \leq K_1(N) \cdot \|f\|_p, \tag{58}$$

where $K_1(N) = h^{-1/p}(1+N)^{1/p}(1+d)(2d(d-1))^d/d!$. $\square$

Theorem 5.2 *Let $\mathfrak{S}$ be a bivariate tensor-product hierarchical spline spaces in two variables of degree d in the first variable and degree d' in the second variable defined on $\Omega = [a,b] \times [a',b'] \subset \mathbb{R}^2$, and let $\mathcal{T} = \{\tau_{lk} : l \in \mathbb{Z}^+, k \in I_l^\tau\}$ be the THB-spline basis, where I_l^τ is the index set such that the functions τ_{lk} do not vanish on Ω^l for $k \in I_l^\tau$. And let h be the maximum cell between all cells. Then there exist two constants $K_1(N)$ which depends on the number of the hierarchy depth N and K_2 such that*

$$K_1(N)^{-1} \|\mathbf{c}\|_p \leq \|f\|_p \leq K_2 \|\mathbf{c}\|_p, \tag{59}$$

for any $f = \sum_{l=0}^{N} \sum_{k \in I_l^\tau} c_{lk} \tau_{lk} \in \mathfrak{S}$ and all sets of $\mathbf{c} = \{c_{lk}\}$, where

$$K_1(N) := h^{-1/p}(1+N)^{1/p}(1+d)(1+d')(2d(d-1))^d (2d'(d'-1))^{d'}/d!d'! \tag{60}$$

and

$$K_2 := [(b-a)(b'-a')]^{1/p}. \tag{61}$$

Proof The proof of this theorem can be completed by the method analogous to that used in Theorem 5.1. □

Remark 5.3 The constants $K_1(N)$ in inequality (49) and (59) depend on the number of the hierarchy depth. In fact, we cannot find a better constant such that inequality (49) and (59) gets stronger. To verify this remark, we will construct a univariate quadratic hierarchical spline, which will show the polynomial dependence. Let $\Omega^0 = [0,3]$ and take the grid with mesh size 1 on Ω^0. And let the grid with mesh size $1/2^l$ on Ω^l, $l \geq 1$. In each dyadic level l, we remove one B-spline, and truncate the other 3 B-splines with respect to B^{l+1} on Ω^{l+1}, and insert 4 little B-splines at the next level $l+1$.

$$\Omega^0 = [0,3],$$
$$\Omega_l = [1 - 1/2^{l-1}, 1 + 1/2^{l-2}], \quad \text{for} \quad l \geq 1.$$

The corresponding THB-spline basis functions of the first four levels are shown in Figure 2. We define f by

$$f = \sum_{l=0}^{N} \sum_{k \in I_l} 1 \cdot \tau_{lk} = 1, \tag{62}$$

where

$$c_{lk} \equiv 1, \quad \forall l, k. \tag{63}$$

So we have

$$\|f\|_p := (\int_0^3 \mid f \mid^p \,\mathrm{d}x)^{\frac{1}{p}} = 3^{\frac{1}{p}} \tag{64}$$

and

$$\|\mathbf{c}\|_p := (\sum_{l=0}^{N} \sum_{k \in I_l} \mid c_{lk} \mid^p)^{\frac{1}{p}} = (3N+5)^{\frac{1}{p}}. \tag{65}$$

These imply directly that

$$\|\mathbf{c}\|_p = (3N+5)^{\frac{1}{p}} > 3^{\frac{1}{p}} = \|f\|_p. \tag{66}$$

6. Conclusion

The hierarchical spline model provides a flexible and simple framework for adaptive function approximation and detailed representation of geometric model. They can be linearly represented by the truncated hierarchical B-spline basis. This basis forms a convex partition of unity and has the smaller supports than those of the hierarchical B-spline basis. The stability analysis of the truncated hierarchical B-spline basis is considered in this paper, since the stability of a basis is important for the computer manipulations. We prove that the THB-spline basis is a weakly L_p stable basis, $1 \leq p < \infty$, by using some classical tools of mathematical analysis. This means that the associated constants to be considered in the stability analysis are at most of polynomial growth in the number of hierarchy depth.

Acknowledgements We thank the referees for their time and comments.

References

[1] Renhong WANG, Xiquan SHI, Zhongxuan LUO, et al. *Multivariate Spline Functions and Their Applications*. Science Press, Beijing, 1994. (in Chinese)

[2] G. FARIN. *Curves and Surfaces for CAGD: a Practical Guide, 5th ed.* Morgan Kaufmann Publishers, San Francisco, 2002.

[3] Mingjun LAI, L. L. SCHUMAKER. *Spline Functions on Triangulations*. Cambidge University Press, Cambridge, 2007.

[4] Yongxia HAO, Chongjun LI. *The c1 and c2 quasi-plateau problem*. J. Computat. Appl. Math., 2018, **329**: 106–124.

[5] Xin LI, Falai CHEN, Hongmei KANG, et al. *A survey on the local refinable splines*. Science China Mathematics. (Beijing), 2016, **59**(4): 617–644.

[6] D. FORSEY, R. BARTELS. *Hierarchical B-spline refinement*. Comp. Graph., 1998, **22**: 205–212.

[7] R. KRAFT. *Adaptive and linearly independent multilevel B-splines*. Vanderbilt Univ. Press, Nashville, TN, 1997.

[8] A. V. VUONG, C. GIANNELLI, B. JÜTTLER, et al. *A hierarchical approach to adaptive local refinement in isogeometric analysis*. Comput. Meth. Appl. Mech. Eng., 2011, **200**: 3554–3567.

[9] C. GIANNELLI, B. JÜTTLER, H. SPELEERS. *Thb-splines: the truncated basis for hierarchical splines*. Comp. Aided Geom. Des., 2012, **29**: 485–498.

[10] Xin LI, Jiansong DENG, Falai CHEN. *Surface modeling with polynomial splines over hierarchical T-meshes*. Visual Computer, 2007, **23**(12): 1027–1033.

[11] Jiansong DENG, Falai CHEN, Xin LI, et al. *Polynomial splines over hierarchical T-meshes*. Graph. Models., 2008, **70**(4): 76–86.

[12] H. SPELEERS, P. DIERCKX, S. VANDEWALLE. *Quasi-hierarchical powell-sabin b-splines*. Comp. Aided Geom. Des., 2009, **26**: 174–191.

[13] Hongmei KANG, Falai CHEN, Jiansong DENG. *Hierarchical B-splines on regular triangular partitions*. Graph. Models., 2014, **76**: 289–300.

[14] U. ZORE, B. JÜTTLER. *Adaptively refined multilevel spline spaces from generating systems*. Comput. Aided Geom. Design, 2014, **31**(7-8): 545–566.

[15] V. NELLY, A. MANTZAFLARIS, B. JÜTTLER. *Characterization of bivariate hierarchical quartic box splines on a three-directional grid*. Comp. Aided Geom. Des., 2016, **41**: 47–61.

[16] C. de BOOR. *On uniform approximation by splines*. J. Approximation Theory, 1968, **1**(2): 219–235.

[17] C. de BOOR. *On Local Linear Functional which Vanish at All b-Splines But One*. Academic Press, New York, 1976.

[18] C. de BOOR. *A Practical Guide to Splines*. Springer-Verlag, New York, 1978.

[19] R. KRAFT. *Adaptive und linear unabhängige Multilevel B-Splines und ihre Anwendungen*. Ph.D. Thesis, Universitq̈t Stuttgart, 1998.

[20] C. GIANNELLI, B. JÜTTLER, H. SPELEERS. *Strongly stable bases for adaptively refined multilevel spline spaces*. Adv. Comput. Math., 2014, **40**(2): 459–490.

[21] I. DIMOVSKI, V. CAKALOV. *Integral inequalities for real polynomials*. Annuaire Univ. Sofia Fac. Math, 1964, **59**: 151–158.

[22] L. RAMSHAW. *Blossoming: A Connect the Dots Approach to Splines*. Digital Equipment Corporation, Palo Alto, 1987.

[23] M. DÆHLEN, T. LYCHE, K. MØRKEN, et al. *Multiresolution analysisover triangles based on quadratic hermite interpolation*. J. computat. Appl. math., 2000, **119**(1): 97–114.