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Condition Numbers for Indefinite Least Squares Problem with Multiple Right-Hand Sides

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Abstract In this paper, we investigate the condition numbers for indefinite least squares problem with multiple right-hand sides. The normwise, mixed and componentwise condition numbers and the corresponding structured condition numbers are presented. The structured matrices under consideration include the linear structured matrices, such as the Toeplitz, Hankel, symmetric, and tridiagonal matrices, and the nonlinear structured matrices, such as the Vandermonde and Cauchy matrices. Numerical examples show that the structured condition numbers are tighter than the unstructured ones.

Keywords indefinite least squares problem; multiple right-hand sides; normwise condition number; mixed condition number; componentwise condition number; structured condition number

MR(2010) Subject Classification 65F35; 15A09; 15A12

1. Introduction

Condition number has some important applications in numerical algorithm. For example, we can estimate the forward error of a backward stable algorithm when combining the condition number of the problem and the backward error of the algorithm. Many scholars have done much research on condition numbers; see the recent book on condition [1]. The normwise condition number is the most popular and widely treated one [2]. Considering that the normwise condition number cannot accurately reflect the influence of perturbations for some small entries in the data and ignores the structure of both input and output data with respect to scaling, some scholars considered the mixed condition number which measures the errors in output using norms and the input perturbations componentwise, and the componentwise condition number which measures both the errors in output and the perturbations in input componentwise [3–5]. The explicit expressions of these two kinds of condition numbers for the matrix inverse and the linear equations were first given in [4,5]. Later, these results were generalized to the Moore-Penrose inverse and the linear least squares problem [6], which were further generalized to the weighted Moore-Penrose inverse and the weighted linear least squares problem [7]. In addition, for these two problems, Wang et al. [8] presented the normwise condition numbers.

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In the past ten years, the indefinite least squares (ILS) problem had been widely studied by scholars after Chandrasekaran et al. presenting this concept in [9]. This problem takes the form

$$\mathrm{ILS}: \min_{x\in\mathbb{R}^n}(b-Ax)^TJ(b-Ax),$$

where $A\in\mathbb{R}^{m\times n}$ $(m\geq n)$, $b\in\mathbb{R}^m$, and J is a signature matrix,

$$J=\begin{bmatrix} I_p & \\ & -I_q\end{bmatrix},\quad p+q=m.$$

In the above symbols, A^T, $\mathbb{R}^{m\times n}$, $\mathbb{R}^m$, and I_r stand for the transpose of A, the set of $m\times n$ real matrices, the real vector space of dimension m, and the identity matrix of order r, respectively.

In [9], the authors proved that the ILS problem has a unique solution if and only if

$$A^TJA>0,$$

that is, it is positive definite. In this case, $p>n$ and the unique solution to the ILS problem is

$$x=(A^TJA)^{-1}A^TJb.$$

With the QR and Cholesky factorizations, Chandrasekaran et al. [9] gave a stable and efficient algorithms for the ILS problem. Later, Bojanczyk et al. [10] provided a new algorithm based on the hyperbolic QR factorization. Recently, different methods to solve the ILS problem were proposed [11–14]. As mentioned above, the condition numbers play a significant role in numerical algorithm. So Bojanczyk et al. [10] discussed the normwise condition number for the ILS problem; Li et al. [15] studied its mixed and componentwise condition numbers.

In the present paper, we consider the condition numbers for the ILS problem with multiple right-hand sides (MRHSILS) which is a generalization of the ILS problem. The problem was first proposed by Ou and Peng [16] and its definition is

$$\min_{X\in\mathbb{R}^{n\times s}}\mathrm{tr}((B-AX)^TJ(B-AX)),$$

where tr is the trace, $A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{m\times s}$, and J is the signature matrix defined above. Like the linear systems with multiple right-hand sides [17], the linear least-squares problem with multiple right-hand sides [18], and the total least squares problems with multiple right-hand sides [19], the MRHSILS is also interesting. Moreover, the problem may have some potential applications in the robust smoothing problem of matrix form and the total least squares problem with multiple right-hand sides since, as we know, the ILS problem plays a significant role in the area of optimization known as H^∞-smoothing and in the total least squares problem [9].

In [16], the authors presented the sufficient and necessary conditions of solvability of M-RHSILS problem. However, no explicit expressions of condition numbers for the problem have been derived till now. We will focus on this problem in this paper. More specifically, we will give the normwise, mixed and componentwise condition numbers for the MRHSILS problem. Meanwhile, we will also investigate the structured condition numbers for the problem. It is worth to point out that, in the recent years, the structured condition numbers have received much attention. For example, some authors considered the structured condition numbers for the structured linear system, structured linear least squares problem, and structured matrix equations [20–26].

The rest of the paper is organized as follows. Section 2 contains some useful notations and preliminaries. In Section 3, we present the explicit expressions of the normwise, mixed and componentwise condition numbers of the MRHSILS problem. The corresponding structured condition numbers are discussed in Section 4. In Section 5, numerical examples are presented. Finally, the concluding remarks of the whole paper is given.

2. Notations and preliminaries

For a matrix $A = [a_1, \ldots, a_n] = (a_{ij}) \in \mathbb{R}^{m\times n}$ with $a_i \in \mathbb{R}^m$, define the operator vec as follows:

$$\mathrm{vec}(A) = \left[a_1^T, \ldots, a_n^T\right]^T \in \mathbb{R}^{mn}. \tag{1}$$

Given another matrix $B \in \mathbb{R}^{p\times q}$, their Kronecker product [27] is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}.$$

Some properties about Kronecker product and vec can be found in [27]:

$$\mathrm{vec}(AXB) = \left(B^T \otimes A\right)\mathrm{vec}(X), \tag{2}$$

$$\mathrm{vec}\left(A^T\right) = \Pi\mathrm{vec}(A), \tag{3}$$

$$\|A\|_{\max} = \|\mathrm{vec}(A)\|_\infty, \tag{4}$$

$$\|A\|_F = \|\mathrm{vec}(A)\|_2, \tag{5}$$

where the matrices A, B, C, D and X are of suitable orders, and Π is the well-known vec-permutation matrix. If $A = (a_{ij}) \in \mathbb{R}^{m\times n}$, we write the vec-permutation matrix Π as Π_{mn} and define $\|A\|_{\max}$ as $\max_{i,j}|a_{ij}|$. In addition, in the above notations, $\|\cdot\|_F$ denotes the Frobenius norm of a matrix, $\|\cdot\|_2$ denotes the spectral norm of a matrix or the Euclidean vector norm of a vector, and $\|\cdot\|_\infty$ denotes the infinity norm of a matrix or a vector.

In the following, we introduce the definitions of the three condition numbers mentioned in Section 1. To this end, we need the following notations. The first one is the entry-wise division [3,6,28] for the vectors $x = [x_1, \ldots, x_m] \in \mathbb{R}^m$ and $a = [a_1, \ldots, a_m] \in \mathbb{R}^m$ which is defined as $\frac{x}{a} = (\eta_1, \ldots, \eta_m)$, where

$$\eta_i = \begin{cases} \dfrac{x_i}{a_i}, & \text{if } a_i \neq 0, \\ x_i, & \text{if } a_i = 0. \end{cases}$$

In addition, for $\varepsilon > 0$, we denote $B^\circ(a,\varepsilon) = \{x | d(x,a) \leqslant \varepsilon\}$ and $B(a,\varepsilon) = \{x | \|x-a\|_2 \leqslant \varepsilon\|a\|_2\}$.

Definition 2.1 ([3,6,28]) *Let $F : \mathbb{R}^p \to \mathbb{R}^q$ be a continuous mapping defined on an open set $D_F \subset \mathbb{R}^p$, and $a \in D_F,\ a \neq 0$ such that $F(a) \neq 0$.*

(i) The normwise condition number of F at a is defined by

$$\kappa(F,a) = \lim_{\varepsilon\to 0} \sup_{\substack{x\in B(a,\varepsilon)\\ x\neq a}} \Big(\frac{\|F(x)-F(a)\|_2}{\|F(a)\|_2} \Big/ \frac{\|x-a\|_2}{\|a\|_2}\Big).$$

(ii) *The mixed condition number of F at a is defined by*

$$m(F,a)=\lim_{\varepsilon\to 0}\sup_{\substack{x\in B^o(a,\varepsilon)\\ x\neq a}}\frac{\|F(x)-F(a)\|_\infty}{\|F(a)\|_\infty}\frac{1}{d(x,a)}.$$

(iii) *The componentwise condition number of F at a is defined by*

$$c(F,a)=\lim_{\varepsilon\to 0}\sup_{\substack{x\in B^o(a,\varepsilon)\\ x\neq a}}\frac{d\big(F(x),F(a)\big)}{d(x,a)}.$$

With the Fréchet derivative, the following lemma gives the explicit expressions of these three condition numbers.

Lemma 2.2 ([3,6,28]) *With the same assumptions as Definition 2.1, and supposing that F is Fréchet differentiable at a, then*

$$\kappa(F,a)=\frac{\|DF(a)\|_2\|a\|_2}{\|F(a)\|_2}, \tag{6}$$

$$m(F,a)=\frac{\||DF(a)||a|\|_\infty}{\|F(a)\|_\infty}, \tag{7}$$

$$c(F,a)=\Big\|\frac{|DF(a)||a|}{|F(a)|}\Big\|_\infty, \tag{8}$$

where $DF(a)$ denotes the Fréchet derivative of F at a.

3. Condition numbers for MRHSILS problem

Although the authors presented the sufficient and necessary conditions of solvability of M-RHSILS problem in [16], they did not provide the unique solution and its sufficient and necessary condition. We first consider this problem since the uniqueness of the solution is the precondition of the study on condition numbers.

Theorem 3.1 *The MRHSILS problem has a unique solution if and only if*

$$A^TJA>0,$$

and the unique solution is

$$X=\left(A^TJA\right)^{-1}A^TJB. \tag{9}$$

Proof Since the tr and vec operators are connected by the following formula [27]:

$$\operatorname{tr}\left(A^TB\right)=\operatorname{vec}^T(A)\operatorname{vec}(B),$$

we have

$$\min_{X\in\mathbb{R}^{n\times s}}\operatorname{tr}\left((B-AX)^TJ(B-AX)\right)=\min_{X\in\mathbb{R}^{n\times s}}\operatorname{vec}^T(B-AX)\operatorname{vec}[J(B-AX)].$$

From (2), it follows that

$$\operatorname{vec}[J(B-AX)]=(I_s\otimes J)\operatorname{vec}(B-AX)=(I_s\otimes J)\left(\operatorname{vec}(B)-(I_s\otimes A)\operatorname{vec}(X)\right).$$

Thus, the MRHSILS problem is equivalent to

$$\min_{X\in\mathbb{R}^{n\times s}} \left[\mathrm{vec}(B)-(I_s\otimes A)\mathrm{vec}(X)\right]^T \mathrm{diag}(J,\dots,J)\left[\mathrm{vec}(B)-(I_s\otimes A)\mathrm{vec}(X)\right].$$

According to the condition on the unique solution of the ILS problem, we have that the MRHSILS problem has a unique solution if and only if

$$(I_s\otimes A)^T\widetilde{J}(I_s\otimes A)>0,$$

where $\widetilde{J}=\mathrm{diag}(J,\dots,J)$. It is easy to see that

$$(I_s\otimes A)^T\widetilde{J}(I_s\otimes A)=\mathrm{diag}\left(A^TJA,\dots,A^TJA\right)$$

and

$$\mathrm{diag}\left(A^TJA,\dots,A^TJA\right)>0 \text{ is equivalent to } A^TJA>0.$$

Then the MRHSILS problem has a unique solution if and only if $A^TJA>0$.

When the MRHSILS problem has a unique solution, using the result of the ILS problem, we have

$$\begin{aligned}
\mathrm{vec}(X)&=\left[(I_s\otimes A)^T\widetilde{J}(I_s\otimes A)\right]^{-1}(I_s\otimes A)^T\widetilde{J}\mathrm{vec}(B)\\
&=\mathrm{diag}\left(\left(A^TJA\right)^{-1}A^TJ,\dots,\left(A^TJA\right)^{-1}A^TJ\right)\mathrm{vec}(B)\\
&=\begin{bmatrix}\left(A^TJA\right)^{-1}A^TJB_1\\ \vdots\\ \left(A^TJA\right)^{-1}A^TJB_s\end{bmatrix}\\
&=\mathrm{vec}\left(\left(A^TJA\right)^{-1}A^TJB\right),
\end{aligned}$$

where B_j is the j-th column of B with $j=1,\dots,s$. To derive the above result, we have used (1) and (2). Applying the inverse operator of vec to the above equation implies

$$X=\left(A^TJA\right)^{-1}A^TJB.\ \square$$

Now, we consider the condition numbers for the MRHSILS problem. We first, according to Definition 2.1, state the specific definition of the condition numbers for the MRHSILS problem.

Let $\Delta A\in\mathbb{R}^{m\times n}$ and $\Delta B\in\mathbb{R}^{m\times s}$, and ΔA be sufficiently small such that $(A+\Delta A)^TJ(A+\Delta A)$ is positive definite. Thus the perturbed MRHSILS problem

$$\min_{X+\Delta X\in\mathbb{R}^{n\times s}} \mathrm{tr}\big(((B+\Delta B)-(A+\Delta A)(X+\Delta X))^TJ((B+\Delta B)-(A+\Delta A)(X+\Delta X))\big)$$

has a unique solution $X+\Delta X$:

$$X+\Delta X=\left[(A+\Delta A)^TJ(A+\Delta A)\right]^{-1}(A+\Delta A)^TJ(B+\Delta B). \tag{10}$$

Then the normwise condition number $\kappa(A,B)$, the mixed condition number $m(A,B)$ and the componentwise condition number $c(A,B)$ of the MRHSILS problem are defined as:

$$\kappa(A,B)=\lim_{\varepsilon\to 0}\sup_{\|[\Delta A\ \ \Delta B]\|\le\varepsilon\|[A\ \ B]\|}\frac{\|\Delta X\|_F}{\varepsilon\|X\|_F},$$

$$m(A,B)=\lim_{\varepsilon\to 0}\sup_{\substack{|\Delta A|\le\varepsilon|A|\\|\Delta B|\le\varepsilon|B|}}\frac{\|\Delta X\|_{\max}}{\varepsilon\|X\|_{\max}},$$

$$c(A,B)=\lim_{\varepsilon\to 0}\sup_{\substack{|\Delta A|\le\varepsilon|A|\\|\Delta B|\le\varepsilon|B|}}\frac{1}{\varepsilon}\|\frac{\Delta X}{X}\|_{\max},$$

where $\frac{\Delta X}{X}$ is the entry-wise division defined by unvec($\frac{\mathrm{vec}(\Delta X)}{\mathrm{vec}(X)}$). Here, unvec is inverse operator of vec.

Also, we need the following lemma.

Lemma 3.2 ([15]) *Denoting*

$$A^{[\dagger]}=\left(A^TJA\right)^{-1}A^TJ,$$

and

$$(A+\Delta A)^{[\dagger]}=[(A+\Delta A)^TJ(A+\Delta A)]^{-1}(A+\Delta A)^TJ,$$

we have

$$(A+\Delta A)^{[\dagger]}-A^{[\dagger]}=\left(A^TJA\right)^{-1}(\Delta A)^TJ(I_m-AA^{[\dagger]})-A^{[\dagger]}\Delta AA^{[\dagger]}+h.o.t.,\tag{11}$$

where h.o.t. denotes the higher order terms.

Theorem 3.3 *Let $A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{m\times s}$, and assume that $A^TJA>0$. Then the explicit expressions of the normwise, mixed and componentwise condition numbers for the MRHSILS problem can be given as follows:*

$$\kappa(A,B)=\frac{\|(JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]},\ I_s\otimes A^{[\dagger]}\|_2\|[A,\ B]\|_F}{\|X\|_F},\tag{12}$$

$$m(A,B)=\frac{\left\|\left|(JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\right|\mathrm{vec}(|A|)+\left|I_s\otimes A^{[\dagger]}\right|\mathrm{vec}(|B|)\right\|_\infty}{\|X\|_{\max}},\tag{13}$$

$$c(A,B)=\|\frac{\left|(JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\right|\mathrm{vec}(|A|)+\left|I_s\otimes A^{[\dagger]}\right|\mathrm{vec}(|B|)}{\mathrm{vec}(|X|)}\|_\infty,\tag{14}$$

where $K=A^TJA$ and $R=B-AX$.

Proof Combining (9) and (10) with (11), we have

$$\begin{aligned}\Delta X&=(A+\Delta A)^{[\dagger]}(B+\Delta B)-A^{[\dagger]}B=\left((A+\Delta A)^{[\dagger]}-A^{[\dagger]}\right)B+(A+\Delta A)^{[\dagger]}\Delta B\\&=K^{-1}\Delta A^TJ(B-AX)-A^{[\dagger]}\Delta AX+A^{[\dagger]}\Delta B+\text{h.o.t.}\\&=K^{-1}\Delta A^TJR-A^{[\dagger]}\Delta AX+A^{[\dagger]}\Delta B+\text{h.o.t.}.\end{aligned}\tag{15}$$

Applying vec to (15) and using (2) and (3), we have

$$\begin{aligned}\mathrm{vec}(\Delta X)&=\left[(JR)^T\otimes K^{-1}\right]\Pi_{mn}\mathrm{vec}(\Delta A)-X^T\otimes A^{[\dagger]}\mathrm{vec}(\Delta A)+A^{[\dagger]}\Delta B+\text{h.o.t.}\\&=[\,(JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]},\quad I_s\otimes A^{[\dagger]}\,]\begin{bmatrix}\mathrm{vec}(\Delta A)\\\mathrm{vec}(\Delta B)\end{bmatrix}+\text{h.o.t.}.\end{aligned}\tag{16}$$

Now define a mapping $\varphi : \mathbb{R}^{mn+ms} \mapsto \mathbb{R}^{ns}$ by $[\mathrm{vec}(A); \mathrm{vec}(B)] \mapsto \varphi([\mathrm{vec}(A); \mathrm{vec}(B)]) = \mathrm{vec}(X)$. Thus

$$D\varphi([\mathrm{vec}(A); \mathrm{vec}(B)]) = [\,(JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}, \quad I_s \otimes A^{[\dagger]}\,], \tag{17}$$

which together with (5) and (6) yields the normwise condition number of MRHSILS problem

$$\begin{aligned}\kappa(A,B) &= \frac{\|[\,(JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}, \quad I_s \otimes A^{[\dagger]}\,]\|_2 \left\|\begin{bmatrix}\mathrm{vec}(A)\\ \mathrm{vec}(B)\end{bmatrix}\right\|_2}{\|\mathrm{vec}(X)\|_2} \\ &= \frac{\|[\,(JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}, \quad I_s \otimes A^{[\dagger]}\,]\|_2\,\|[\,A, \quad B\,]\|_F}{\|X\|_F}.\end{aligned}$$

With (4), (7) and (17), we deduce the mixed condition number of MRHSILS problem

$$\begin{aligned}m(A,B) &= \frac{\left\||[\,(JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}, \quad I_s \otimes A^{[\dagger]}\,]| \begin{bmatrix}\mathrm{vec}(|A|)\\ \mathrm{vec}(|B|)\end{bmatrix}\right\|_\infty}{\|\mathrm{vec}(X)\|_\infty} \\ &= \frac{\left\|\left|(JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}\right| \mathrm{vec}(|A|) + \left|I_s \otimes A^{[\dagger]}\right| \mathrm{vec}(|B|)\right\|_\infty}{\|X\|_{\max}}.\end{aligned}$$

Similarly, following from (8) and (17), we can obtain the expression of $c(A,B)$,

$$\begin{aligned}c(A,B) &= \left\|\frac{|[\,(JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}, \quad I_s \otimes A^{[\dagger]}\,]| \begin{bmatrix}\mathrm{vec}(|A|)\\ \mathrm{vec}(|B|)\end{bmatrix}}{\mathrm{vec}(|X|)}\right\|_\infty \\ &= \left\|\frac{\left|(JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}\right| \mathrm{vec}(|A|) + \left|I_s \otimes A^{[\dagger]}\right| \mathrm{vec}(|B|)}{\mathrm{vec}(|X|)}\right\|_\infty. \quad \square\end{aligned}$$

Using the following lemma, we can obtain the easily computable upper bounds for the condition numbers $m(A,B)$ and $c(A,B)$, respectively.

Lemma 3.4 ([6]) *For any matrices U, V, C, D, R and S with property dimensions making the following well defined*

$$[U \otimes V + (C \otimes D)\Pi_{mn}]\mathrm{vec}(R),$$

$$\frac{[U \otimes V + (C \otimes D)\Pi_{mn}]\mathrm{vec}(R)}{S},$$

$$VRU^T \text{ and } DR^TC^T,$$

we have

$$\||[U \otimes V + (C \otimes D)\Pi_{mn}]|\mathrm{vec}(|R|)\|_\infty \leqslant \left\|\mathrm{vec}(|V||R||U|^T + |D||R|^T|C|^T)\right\|_\infty$$

and

$$\left\|\frac{|[U \otimes V + (C \otimes D)\Pi_{mn}]|\mathrm{vec}(|R|)}{|S|}\right\|_\infty \leqslant \left\|\frac{\mathrm{vec}(|V||R||U|^T + |D||R|^T|C|^T)}{|S|}\right\|_\infty.$$

Corollary 3.5 *Under the assumptions of Theorem 3.3, we have*

$$m(A,B) \leqslant m(A,B)^{\mathrm{upper}} = \frac{\left\||K^{-1}||A^T||JR| + |A^{[\dagger]}||A||X| + |A^{[\dagger]}||B|\right\|_{\max}}{\|X\|_{\max}}, \tag{18}$$

$$c(A,B) \leqslant c(A,B)^{\mathrm{upper}} = \max_{i,j} \frac{\left(|K^{-1}||A^T||JR| + |A^{[\dagger]}||A||X| + |A^{[\dagger]}||B|\right)_{ij}}{(|X|)_{ij}}. \tag{19}$$

Proof Applying Lemma 3.4 to (13) and (14) gives

$$\begin{aligned} m(A,B) &\leq \frac{\left\|\mathrm{vec}(|K^{-1}||A^T||JR|) + \mathrm{vec}(|A^{[\dagger]}||A||X|) + \mathrm{vec}(|A^{[\dagger]}||B|)\right\|_\infty}{\|X\|_{\max}} \\ &= \frac{\left\||K^{-1}||A^T||JR| + |A^{[\dagger]}||A||X| + |A^{[\dagger]}||B|\right\|_{\max}}{\|X\|_{\max}}, \\ c(A,B) &\leqslant \left\|\frac{\mathrm{vec}(|K^{-1}||A^T||JR|) + \mathrm{vec}(|A^{[\dagger]}||A||X|) + \mathrm{vec}(|A^{[\dagger]}||B|)}{\mathrm{vec}(|X|)}\right\|_\infty \\ &= \max_{i,j} \frac{\left(|K^{-1}||A^T||JR| + |A^{[\dagger]}||A||X| + |A^{[\dagger]}||B|\right)_{ij}}{(|X|)_{ij}}. \end{aligned}$$

Hence the proof is completed. □

Remark 3.6 Setting $B = b \in \mathbb{R}^m$ in (12)–(14), (18), and (19), we can get the corresponding results of the ILS problem in [15].

4. Structured condition numbers for MRHSILS problem

We first, similar to the definition of structured condition numbers in [26], define the structured condition numbers for the MRHSILS problem.

Definition 4.1 *Let $\mathcal{S}_1 \subseteq \mathbb{R}^{m\times n}$ and $\mathcal{S}_2 \subseteq \mathbb{R}^{m\times s}$ be the sets of structured matrices, $a \in \mathbb{R}^{k_1}$ and $b \in \mathbb{R}^{k_2}$ be the vector representing the structured matrices $A \in \mathcal{S}_1$ and $B \in \mathcal{S}_2$, respectively, and define the following mapping*

$$\psi : [a;b] \in \mathbb{R}^{(k_1+k_2)} \mapsto \mathrm{vec}(X) = \mathrm{vec}\Big(\left(A^TJA\right)^{-1} A^TJB\Big) \in \mathbb{R}^{ns}. \tag{20}$$

Let Δa be the perturbation on a, Δb be the perturbation on b, and the perturbed solution $X + \Delta X$ is the unique solution to the structured perturbed MRHSILS problem. Then the structured normwise, mixed and componentwise condition numbers for the structured MRHSILS problem are defined as

$$\begin{aligned} \kappa_S(a,b) &= k(\psi,[a;b]) = \lim_{\varepsilon\to 0} \sup_{\|[\Delta a;\Delta b]\|\leqslant\varepsilon\|[a;b]\|} \frac{\|\Delta X\|_F}{\varepsilon\|X\|_F}, \\ m_S(a,b) &= m(\psi,[a;b]) = \lim_{\varepsilon\to 0} \sup_{\substack{|\Delta a|\leqslant\varepsilon|a| \\ |\Delta b|\leqslant\varepsilon|b|}} \frac{\|\Delta X\|_{\max}}{\varepsilon\|X\|_{\max}}, \\ c_S(a,b) &= c(\psi,[a;b]) = \lim_{\varepsilon\to 0} \sup_{\substack{|\Delta a|\leqslant\varepsilon|a| \\ |\Delta b|\leqslant\varepsilon|b|}} \frac{1}{\varepsilon}\Big\|\frac{\Delta X}{X}\Big\|_{\max}. \end{aligned}$$

In the following, we consider the expressions of the structured condition numbers for the MRHSILS problem for two classes of structured matrices: linear structured and nonlinear structured matrices. The former contains the famous Toeplitz, Hankel, symmetric, and persymmetric matrices. The well-known examples of the latter are the Cauchy and Vandermonde matrices.

4.1. Linear structures

We consider two classes $\mathcal{L}_1$ and $\mathcal{L}_2$ of structured matrices, which are linear subspaces of $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{m\times s}$, respectively. Suppose that $\dim(\mathcal{L}_1) = k_1$ and $\dim(\mathcal{L}_2) = k_2$. Then, from [21,26], we know that, for any $A \in \mathcal{L}_1$, there exist a fixed basis $\ell_1, \dots, \ell_{k_1}$ for $\mathcal{L}_1$ and a unique vector $a = [a_1, \dots, a_{k_1}]^T \in \mathbb{R}^{k_1}$ such that

$$A = \sum_{i=1}^{k_1} a_i \ell_i.$$

Similarly, for any $B \in \mathcal{L}_2$, there exist a fixed basis $\jmath_1, \dots, \jmath_{k_2}$ for $\mathcal{L}_2$ and a unique vector $b = [b_1, \dots, b_{k_2}]^T \in \mathbb{R}^{k_2}$ such that

$$B = \sum_{j=1}^{k_2} b_j \jmath_j.$$

To obtain the explicit expressions of the structured condition numbers for the MRHSILS problem with linear structured matrices, we first present the Fréchet derivative $D\psi([a;b])$ of function ψ defined in (20).

Lemma 4.2 *The Fréchet derivative $D\psi([a;b])$ of function ψ defined in (20) is given by*

$$D\psi([a;b]) = [\mathrm{vec}(K^{-1}\ell_1^T JR – A^{[\dagger]}\ell_1 X), \dots, \mathrm{vec}(K^{-1}\ell_{k_1}^T JR – A^{[\dagger]}\ell_{k_1} X), \mathrm{vec}(A^{[\dagger]}\jmath_1), \dots, \mathrm{vec}(A^{[\dagger]}\jmath_{k_2})],$$

where $\ell_1, \dots, \ell_{k_1}$ and $\jmath_1, \dots, \jmath_{k_2}$ are the fixed bases of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

Proof Let the perturbations on a and b be $\Delta a = [\Delta a_1, \dots, \Delta a_{k_1}]^T \in \mathbb{R}^{k_1}$ and $\Delta b = [\Delta b_1, \dots, \Delta b_{k_2}]^T \in \mathbb{R}^{k_2}$, respectively, which imply that the perturbations on A and B are $\Delta A = \sum_{i=1}^{k_1} \Delta a_i \ell_i$ and $\Delta B = \sum_{j=1}^{k_2} \Delta b_j \jmath_j$. Using the operator vec, we have

$$\mathrm{vec}(\Delta A) = [\mathrm{vec}(\ell_1), \dots, \mathrm{vec}(\ell_{k_1})]\Delta a, \ \mathrm{vec}(\Delta B) = [\mathrm{vec}(\jmath_1), \dots, \mathrm{vec}(\jmath_{k_2})]\Delta b,$$

which combined with (16) gives

$$\begin{aligned}\mathrm{vec}(\Delta X) =& [(JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}]\,[\mathrm{vec}(\ell_1), \dots, \mathrm{vec}(\ell_{k_1})]\,\Delta a + \\ & (I_s \otimes A^{[\dagger]})\,[\mathrm{vec}(\jmath_1), \dots, \mathrm{vec}(\jmath_{k_2})]\,\Delta b + \text{h.o.t.} \\ =& [\mathrm{vec}(K^{-1}\ell_1^T JR - A^{[\dagger]}\ell_1 X), \dots, \mathrm{vec}(K^{-1}\ell_{k_1}^T JR - A^{[\dagger]}\ell_{k_1} X), \mathrm{vec}(A^{[\dagger]}\jmath_1), \dots, \mathrm{vec}(A^{[\dagger]}\jmath_{k_2})] \\ & \begin{bmatrix}\Delta a \\ \Delta b\end{bmatrix} + \text{h.o.t.}.\end{aligned}$$

On the other hand, according to the definition of (20), we have

$$\psi\left([a + \Delta a; b + \Delta b]\right) - \psi([a;b]) = \mathrm{vec}(\Delta X).$$

Thus, the desired result follows from the definition of Fréchet derivative.

Theorem 4.3 *Suppose that $\mathcal{L}_1 \subseteq \mathbb{R}^{m\times n}$ and $\mathcal{L}_2 \subseteq \mathbb{R}^{m\times s}$ are two linear subspaces consisting of structured matrices mentioned above. Let $A \in \mathcal{L}_1, B \in \mathcal{L}_2$ and $X = A^{[\dagger]}B$ be the unique solution of the MRHSILS problem. Then the structured normwise, mixed and componentwise* *condition numbers for the MRHSILS problem are*

$$\kappa_{\mathcal{L}}(A,B)=\frac{\left\|\left[\mathrm{vec}(K^{-1}\ell_1^T JR-A^{[\dagger]}\ell_1 X),\ldots,\mathrm{vec}(K^{-1}\ell_{k_1}^T JR-A^{[\dagger]}\ell_{k_1} X),\mathrm{vec}(A^{[\dagger]}\jmath_1),\ldots,\mathrm{vec}(A^{[\dagger]}\jmath_{k_2})\right]\right\|_2\left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\|_2}{\|X\|_F}, \tag{21}$$

$$m_{\mathcal{L}}(A,B)=\frac{\|\sum_{i=1}^{k_1}|a_i||(K^{-1}\ell_i^T JR-A^{[\dagger]}\ell_i X)|+\sum_{j=1}^{k_2}|b_j||A^{[\dagger]}\jmath_j|\|_{\max}}{\|X\|_{\max}}, \tag{22}$$

$$c_{\mathcal{L}}(A,B)=\left\|\frac{\sum_{i=1}^{k_1}|a_i|\left|(K^{-1}\ell_i^T JR-A^{[\dagger]}\ell_i X)\right|+\sum_{j=1}^{k_2}|b_j|\left|A^{[\dagger]}\jmath_j\right|}{|X|}\right\|_{\max}. \tag{23}$$

Proof The expression (21) is the immediate result of Lemma 4.2 and (6). Considering the fact

$$\frac{\||[\mathrm{vec}(K^{-1}\ell_1^T JR-A^{[\dagger]}\ell_1 X),\ldots,\mathrm{vec}(K^{-1}\ell_{k_1}^T JR-A^{[\dagger]}\ell_{k_1} X)]||a|+|\mathrm{vec}(A^{[\dagger]}\jmath_1),\ldots,\mathrm{vec}(A^{[\dagger]}\jmath_{k_2})]||b|\|_\infty}{\|X\|_{\max}}$$
$$=\frac{\|\sum_{i=1}^{k_1}|a_i||(K^{-1}\ell_i^T JR-A^{[\dagger]}\ell_i X)|+\sum_{j=1}^{k_2}|b_j||A^{[\dagger]}\jmath_j|\|_{\max}}{\|X\|_{\max}},$$

Lemma (4.2) and (7), we have (22). Similarly, from the fact

$$\left\|\frac{|[\mathrm{vec}(K^{-1}\ell_1^T JR-A^{[\dagger]}\ell_1 X),\ldots,\mathrm{vec}(K^{-1}\ell_{k_1}^T JR-A^{[\dagger]}\ell_{k_1} X)]||a|+|\mathrm{vec}(A^{[\dagger]}\jmath_1),\ldots,\mathrm{vec}(A^{[\dagger]}\jmath_{k_2})]||b|}{\mathrm{vec}(|X|)}\right\|_\infty$$
$$=\left\|\frac{\sum_{i=1}^{k_1}|a_i||(K^{-1}\ell_i^T JR-A^{[\dagger]}\ell_i X)|+\sum_{j=1}^{k_2}|b_j||A^{[\dagger]}\jmath_j|}{|X|}\right\|_{\max},$$

Lemma 4.2 and (8), it follows that (23) holds.

Remark 4.4 When there are no structures on the involved matrices, that is, $\mathcal{L}_1$ and $\mathcal{L}_2$ are respectively $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{m\times s}$, the above structured condition numbers reduce to the unstructured ones given in Theorem 3.3.

In the following, we compare the structured condition numbers with the unstructured counterparts.

Theorem 4.5 *For the structured normwise condition number, when $A\in\mathcal{L}_1$ and $B\in\mathcal{L}_2$, we have*

$$\kappa_{\mathcal{L}}(A,B)\leqslant\max\left\{\sqrt{k_1},\sqrt{k_2}\right\}\max\left\{\left\|\left[\mathrm{vec}(\ell_1),\ldots,\mathrm{vec}(\ell_{k_1})\right]\right\|_2,\left\|\left[\mathrm{vec}(\jmath_1),\ldots,\mathrm{vec}(\jmath_{k_2})\right]\right\|_2\right\}\kappa(A,B), \tag{24}$$

where k_1 and k_2 are constants depending on the specific structured matrices.

Proof Note that

$$\begin{aligned}
&\left\|\left[\mathrm{vec}(K^{-1}\ell_1^T JR-A^{[\dagger]}\ell_1 X),\ldots,\mathrm{vec}(K^{-1}\ell_{k_1}^T JR-A^{[\dagger]}\ell_{k_1} X),\mathrm{vec}(A^{[\dagger]}\jmath_1),\ldots,\mathrm{vec}(A^{[\dagger]}\jmath_{k_2})\right]\right\|_2\\
=&\left\|\left[\left((JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\right)\left[\mathrm{vec}(\ell_1),\ldots,\mathrm{vec}(\ell_{k_1})\right],(I_s\otimes A^{[\dagger]})\left[\mathrm{vec}(\jmath_1),\ldots,\mathrm{vec}(\jmath_{k_2})\right]\right]\right\|_2\\
=&\left\|\left[\left((JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\right),(I_s\otimes A^{[\dagger]})\right]\begin{bmatrix}[\mathrm{vec}(\ell_1),\ldots,\mathrm{vec}(\ell_{k_1})] & 0\\ 0 & [\mathrm{vec}(\jmath_1),\ldots,\mathrm{vec}(\jmath_{k_2})]\end{bmatrix}\right\|_2\\
\leqslant&\left\|\left[\left((JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\right),(I_s\otimes A^{[\dagger]})\right]\right\|_2\max\left\{\left\|\left[\mathrm{vec}(\ell_1),\ldots,\mathrm{vec}(\ell_{k_1})\right]\right\|_2,\left\|\left[\mathrm{vec}(\jmath_1),\ldots,\mathrm{vec}(\jmath_{k_2})\right]\right\|_2\right\}
\end{aligned}$$

and

$$\|a\|_2 \leqslant \sqrt{k_1}\|A\|_F, \quad \|b\|_2 \leqslant \sqrt{k_2}\|B\|_F,$$

where the latter are summarized from the conclusions in [21] and imply that

$$\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|_2 \leqslant \max\left\{\sqrt{k_1}, \sqrt{k_2}\right\} \|[\, A \quad B \,]\|_F .$$

These facts together with (12) and (21) lead to (24).

Remark 4.6 When the linear structures of the matrices A and B are specific, we can provide more information on k_1 and k_2. For example, when both A and B are Toeplitz or Hankel matrices, $k_1 = k_2 = 2$ [21]. Moreover, in this case, it is easy to see that the fixed basis for the Toeplitz matrix subspace or the Hankel matrix subspace is orthogonal under the inner product $\langle A_1, A_2\rangle = \mathrm{tr}(A_1^T A_2) = [\mathrm{vec}(A_1)]^T \mathrm{vec}(A_2)$ for $A_1, A_2 \in \mathbb{R}^{m\times n}$. Thus,

$$\begin{aligned} &\max\left\{\left\|\left[\mathrm{vec}(\ell_1), \dots, \mathrm{vec}(\ell_{k_1})\right]\right\|_2, \left\|\left[\mathrm{vec}(\jmath_1), \dots, \mathrm{vec}(\jmath_{k_2})\right]\right\|_2\right\} \\ &\quad = \max\left\{\max_{i=1,\dots,k_1} \|\ell_i\|_F, \max_{j=1,\dots,k_2} \|\jmath_j\|_F\right\}. \end{aligned}$$

As a result,

$$\kappa_{\mathcal{L}}(A, B) \leqslant \sqrt{2} \max\left\{\max_{i=1,\dots,k_1} \|\ell_i\|_F, \max_{j=1,\dots,k_2} \|\jmath_j\|_F\right\}\kappa(A, B).$$

Theorem 4.7 *If the fixed bases $\ell_1, \dots, \ell_{k_1}$ for $\mathcal{L}_1$ and $\jmath_1, \dots, \jmath_{k_2}$ for $\mathcal{L}_2$ satisfy $|A| = \sum_{i=1}^{k_1} |a_i||\ell_i|$ for $A \in \mathcal{L}_1$ and $|B| = \sum_{j=1}^{k_2} |b_j||\jmath_j|$ for $B \in \mathcal{L}_2$, respectively, then, for the structured mixed and componentwise condition numbers, we have*

$$m_{\mathcal{L}}(A, B) \leqslant m(A, B), \tag{25}$$

$$c_{\mathcal{L}}(A, B) \leqslant c(A, B). \tag{26}$$

Proof Based on Theorem 4.3, we by the monotonicity of the infinity norm have

$$\begin{aligned} &\Big\| \sum_{i=1}^{k_1} |a_i|\big|(K^{-1}\ell_i^T JR - A^{[\dagger]}\ell_i X)\big| + \sum_{j=1}^{k_2} |b_j||A^{[\dagger]}\jmath_j|\Big\|_{\max} \\ &\quad = \Big\|\big|\big((JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}\big)[\mathrm{vec}(\ell_1), \dots, \mathrm{vec}(\ell_{k_1})]\big||a| + \\ &\qquad \big|(I_s \otimes A^{[\dagger]})[\mathrm{vec}(\jmath_1), \dots, \mathrm{vec}(\jmath_{k_2})]\big||b|\Big\|_\infty \\ &\quad \leqslant \Big\|\big|\big((JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}\big)\big| \sum_{i=1}^{k_1} |a_i||\mathrm{vec}(\ell_i)| + \big|(I_s \otimes A^{[\dagger]})\big| \sum_{j=1}^{k_2} |b_j||\mathrm{vec}(\jmath_j)|\Big\|_\infty \\ &\quad = \Big\|\big|\big((JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}\big)\big|\mathrm{vec}(|A|) + \big|(I_s \otimes A^{[\dagger]})\big|\mathrm{vec}(|B|)\Big\|_\infty . \end{aligned}$$

Using the above inequality and the expressions of (13), (14), (22) and (23), it is easy to check that (25) and (26) hold.

Remark 4.8 When both A and B are Toeplitz or Hankel matrices, the conditions on bases in Theorem 4.7 are satisfied naturally.

4.2. Nonlinear structures

We only take the famous Vandermonde matrices and Cauchy matrices as examples to present the explicit expressions of the structured condition numbers for the MRHSILS problem with nonlinear structured matrices.

4.2.1. Vandermonde matrices

Suppose that $\mathcal{V}_1$ and $\mathcal{V}_2$ are the sets of $m\times n$ and $m\times s$ Vandermonde matrices, respectively. If $A=[a_{ij}]\in\mathcal{V}_1$ and $B=[b_{ij}]\in\mathcal{V}_2$, then there exist vectors $a=[a_0,\dots,a_{n-1}]^T\in\mathbb{R}^n$ and $b=[b_0,\dots,b_{s-1}]^T\in\mathbb{R}^s$ such that $a_{ij}=a_j^i$ for $i=0,\dots,m-1$, $j=0,\dots,n-1$, and $b_{ij}=b_j^i$ for $i=0,\dots,m-1$, $j=0,\dots,s-1$. Let $\Delta a=[\Delta a_0,\dots,\Delta a_{n-1}]^T\in\mathbb{R}^n$ and $\Delta b=[\Delta b_0,\dots,\Delta b_{s-1})]^T\in\mathbb{R}^s$ be the perturbation on a and b, respectively.

Before discussing the condition numbers, we first cite the following lemma.

Lemma 4.9 ([25]) *Let ΔA and ΔB be the perturbations of A and B caused by the perturbations of a and b. Then the explicit expressions of ΔA and ΔB are*

$$\Delta A=A_1\mathrm{diag}(\Delta a)+\text{h.o.t.},\ \ \Delta B=B_1\mathrm{diag}(\Delta b)+\text{h.o.t.},$$

where $A_1=\mathrm{diag}(c)\begin{bmatrix}\boldsymbol{0}\\ A(1:m-1,:)\end{bmatrix}$ *and* $B_1=\mathrm{diag}(c)\begin{bmatrix}\boldsymbol{0}\\ B(1:m-1,:)\end{bmatrix}$ *with* $c=[0,\dots,m-1]^T$. *Besides,*

$$\mathrm{vec}(\Delta A)=(I_n\otimes A_1)\mathrm{vec}(\mathrm{diag}(\Delta a))+\text{h.o.t.}=(I_n\otimes A_1)\mathcal{E}_1(\Delta a)+\text{h.o.t.},$$

$$\mathrm{vec}(\Delta B)=(I_s\otimes B_1)\mathrm{vec}(\mathrm{diag}(\Delta b))+\text{h.o.t.}=(I_s\otimes B_1)\mathcal{E}_2(\Delta b)+\text{h.o.t.},$$

where

$$\mathcal{E}_1=(\mathrm{vec}(E_{11}^{(n\times n)}),\dots,\mathrm{vec}(E_{nn}^{(n\times n)})),\ \ \mathcal{E}_2=(\mathrm{vec}(E_{11}^{(s\times s)}),\dots,\mathrm{vec}(E_{ss}^{(s\times s)}))$$

with $E_{ij}^{(n\times n)}=e_i^n(e_j^n)^T\in\mathbb{R}^{n\times n}$ being the (i,j)-th elementary matrix and e_i^n being the i-th column of I_n.

Now we present the Fréchet derivative $D\psi([a;b])$.

Lemma 4.10 *The Fréchet derivative $D\psi([a;b])$ of function ψ defined in (20) is given by*

$$D\psi([a;b])=\big[((JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\big)(I_n\otimes A_1)\mathcal{E}_1,(I_s\otimes A^{[\dagger]})(I_s\otimes B_1)\mathcal{E}_2\big].$$

Proof Following from (16) and Lemma 4.9, we have

$$\begin{aligned}\mathrm{vec}(\Delta X)&=\big((JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\big)\mathrm{vec}(\Delta A)+(I_s\otimes A^{[\dagger]})\mathrm{vec}(\Delta B)+\text{h.o.t.}\\&=\big((JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\big)(I_n\otimes A_1)\mathcal{E}_1(\Delta a)+(I_s\otimes A^{[\dagger]})(I_s\otimes B_1)\mathcal{E}_2(\Delta b)+\text{h.o.t.}\\&=\big[((JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]})(I_n\otimes A_1)\mathcal{E}_1,(I_s\otimes A^{[\dagger]})(I_s\otimes B_1)\mathcal{E}_2\big]\begin{bmatrix}\Delta a\\ \Delta b\end{bmatrix}+\text{h.o.t.},\end{aligned}$$

from which and the definition of Fréchet derivative we have the desired result.

This lemma together with Lemma 2.2 yields the following structured condition numbers for the MRHSILS problem when the involved matrices are the Vandermonde matrices.

Theorem 4.11 *Let $A\in\mathcal{V}_1$, $B\in\mathcal{V}_2$ and $X=A^{[\dagger]}B$ be the unique solution to the MRHSILS* *problem. Then the structured condition numbers for this problem are:*

$$\kappa_{\mathcal{V}}(A,B) = \frac{\left\|\left[\left((JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}\right)(I_n \otimes A_1)\mathcal{E}_1, (I_s \otimes A^{[\dagger]})(I_s \otimes B_1)\mathcal{E}_2\right]\right\|_2 \left\|\begin{bmatrix} a \\ b \end{bmatrix}\right\|_2}{\|X\|_F}, \tag{27}$$

$$m_{\mathcal{V}}(A,B) = \frac{\left\|\left|\left((JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}\right)(I_n \otimes A_1)\mathcal{E}_1\right||a| + \left|(I_s \otimes A^{[\dagger]})(I_s \otimes B_1)\mathcal{E}_2\right||b|\right\|_\infty}{\|X\|_{\max}}, \tag{28}$$

$$c_{\mathcal{V}}(A,B) = \left\|\frac{\left|\left((JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}\right)(I_n \otimes A_1)\mathcal{E}_1\right||a| + \left|(I_s \otimes A^{[\dagger]})(I_s \otimes B_1)\mathcal{E}_2\right||b|}{\mathrm{vec}(|X|)}\right\|_\infty. \tag{29}$$

4.2.2. Cauchy matrices

Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two classes of $m\times n$ and $m \times s$ Cauchy matrices. Let $A = [a_{ij}] \in \mathcal{C}_1$ and $B = [b_{ij}] \in \mathcal{C}_2$. Then there exist $u = [u_1, \dots, u_m]^T \in \mathbb{R}^m$ and $v = [v_1, \dots, v_n]^T \in \mathbb{R}^n$ with $u_i \neq v_j$ for $i = 1, \dots, m$, $j = 1, \dots, n$, and $r = [r_1, \dots, r_m]^T \in \mathbb{R}^m$ and $t = [t_1, \dots, t_s]^T \in \mathbb{R}^s$ with $r_i \neq t_j$ for $i = 1, \dots, m$, $j = 1, \dots, s$ such that

$$a_{ij} = \frac{1}{u_i - v_j}, \quad b_{ij} = \frac{1}{r_i - t_j}.$$

Let $\Delta w = [\Delta u; \Delta v] = [\Delta u_1, \dots, \Delta u_m, \Delta v_1, \dots, \Delta v_n]^T \in \mathbb{R}^{m+n}$ be the perturbation on $w = [u; v] \in \mathbb{R}^{m+n}$, and $\Delta z = [\Delta r; \Delta t] = [\Delta r_1, \dots, \Delta r_m, \Delta t_1, \dots, \Delta t_s]^T \in \mathbb{R}^{m+s}$ be the perturbation on $z = [r; t] \in \mathbb{R}^{m+s}$. Then the perturbation ΔA on A and the perturbation ΔB on B are given by [25]:

$$\Delta A = \left[\frac{\Delta u_i - \Delta v_j}{(u_i - v_j)^2}\right] + \text{h.o.t.} = \mathrm{diag}(\Delta u)A_1 - A_1\mathrm{diag}(\Delta v) + \text{h.o.t.} \in \mathbb{R}^{m\times n},$$

$$\Delta B = \left[\frac{\Delta r_i - \Delta t_j}{(r_i - t_j)^2}\right] + \text{h.o.t.} = \mathrm{diag}(\Delta r)B_1 - B_1\mathrm{diag}(\Delta t) + \text{h.o.t.} \in \mathbb{R}^{m\times s},$$

where $A_1 = [\frac{1}{(u_i - v_j)^2}] \in \mathbb{R}^{m\times n}$ and $B_1 = [\frac{1}{(r_i - t_j)^2}] \in \mathbb{R}^{m\times s}$. Further, they can be written as [25]

$$\mathrm{vec}(\Delta A) = \mathrm{diag}(\mathrm{vec}(A_1))P_1\Delta w + \text{h.o.t.}, \quad \mathrm{vec}(\Delta B) = \mathrm{diag}(\mathrm{vec}(B_1))P_2\Delta z + \text{h.o.t.}, \tag{30}$$

where $P_1 = \begin{bmatrix} -I_m, & e_m e_1^{(n)^T} \\ -I_m, & e_m e_2^{(n)^T} \\ \vdots & \vdots \\ -I_m, & e_m e_n^{(n)^T} \end{bmatrix}$ and $P_2 = \begin{bmatrix} -I_m, & e_m e_1^{(s)^T} \\ -I_m, & e_m e_2^{(s)^T} \\ \vdots & \vdots \\ -I_m, & e_m e_s^{(s)^T} \end{bmatrix}$. With the above results, we have the Fréchet derivative $D\psi([w; z])$ of the function ψ.

Lemma 4.12 *The Fréchet derivative $D\psi([w;z])$ of function ψ defined in (20) is*

$$D\psi([w;z]) = \left[\left[(JR)^T \otimes K^{-1}\Pi_{mn} - X^T \otimes A^{[\dagger]}\right]\mathrm{diag}(\mathrm{vec}(A_1))P_1,\ (I_s \otimes A^{[\dagger]})\mathrm{diag}(\mathrm{vec}(B_1))P_2\right].$$

Proof From (16) and (30), we have

$$
\begin{aligned}
&\mathrm{vec}(\Delta X)\\
&=\big\{\big[(JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\big]\big[\mathrm{diag}(\mathrm{vec}(A_1))P_1\Delta w\big]+(I_s\otimes A^{[\dagger]})\big[\mathrm{diag}(\mathrm{vec}(B_1))P_2\Delta z\big]\big\}+\text{h.o.t.}\\
&=\big[\big[(JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\big]\mathrm{diag}(\mathrm{vec}(A_1))P_1,\ (I_s\otimes A^{[\dagger]})\mathrm{diag}(\mathrm{vec}(B_1))P_2\big]\begin{bmatrix}\Delta w\\ \Delta z\end{bmatrix}+\text{h.o.t.}
\end{aligned}
$$

which together with the definition of Fréchet derivative gives the desired result.

As done above, following from Lemmas 2.2 and 4.12, it is easy to deduce the following theorem.

Theorem 4.13 *Suppose that $A\in\mathcal{C}_1$ with $A^TJA>0$, $B\in\mathcal{C}_2, X=A^{[\dagger]}B$ is the solution of the MRHSILS problem. Then the Cauchy structured condition numbers are:*

$$
\kappa_{\mathcal{C}}(A,B)=\frac{\big\|\big[\big[(JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\big]\mathrm{diag}(\mathrm{vec}(A_1))P_1,\ (I_s\otimes A^{[\dagger]})\mathrm{diag}(\mathrm{vec}(B_1))P_2\big]\big\|_2\left\|\begin{bmatrix}w\\ z\end{bmatrix}\right\|_2}{\|X\|_F}, \tag{31}
$$

$$
m_{\mathcal{C}}(A,B)=\frac{\Big\|\big|\big[(JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\big]\mathrm{diag}(\mathrm{vec}(A_1))P_1\big||w|+\big|(I_s\otimes A^{[\dagger]})\mathrm{diag}(\mathrm{vec}(B_1))P_2\big||z|\Big\|_\infty}{\|X\|_{\max}}, \tag{32}
$$

$$
c_{\mathcal{C}}(A,B)=\Big\|\frac{\big|\big[(JR)^T\otimes K^{-1}\Pi_{mn}-X^T\otimes A^{[\dagger]}\big]\mathrm{diag}(\mathrm{vec}(A_1))P_1\big||w|+\big|(I_s\otimes A^{[\dagger]})\mathrm{diag}(\mathrm{vec}(B_1))P_2\big||z|}{\mathrm{vec}(|X|)}\Big\|_\infty. \tag{33}
$$

Remark 4.14 In this section, we only consider the case where A and B have the same structures. The case where the structures of A and B are different can be discussed in a similar way.

5. Numerical examples

In this section, four examples are provided, where the first three examples are used to illuminate the differences between the structured condition numbers and the corresponding unstructured ones for the MRHSILS problem and the fourth one is used to compare the true errors of the solutions of MRHSILS problems with the bounds based on the condition numbers. All computations are carried out in MATLAB R2009a with the machine precision 2.2×10^{-16}.

We first consider the case where the involved matrices are Toeplitz matrices, that is, they are linear structured.

Example 5.1 The test matrices A and B are generated by the Matlab function $A=\textbf{toeplitz}(c,r)$ with $c=\textbf{randn}(m,1)$ and $r=\textbf{randn}(n,1)$, and $B=\textbf{toeplitz}(u,v)$ with $u=\textbf{randn}(s,1)$ and $v=\textbf{randn}(t,1)$. Meanwhile, we suppose

$$
J=\begin{bmatrix}I_{m-1} & \\ & -1\end{bmatrix}.
$$

In the specific experiments, we set $m=20, n=10, s=20, t=8$ and generate 500 pairs un-symmetric Toeplitz matrices A and B. Since $p=19$ and $n=10$, in this case, $A^TJA>0$, that is, the matrix A satisfies the solvability condition of the MRHSILS problems. We use (12), (13), (14), (18), and (19) to compute the unstructured condition numbers and the upper bound of the mixed and componentwise condition numbers, and use (21), (22) and (23) to compute the structured condition numbers, where the solution is computed by $X = (A^T JA)^{-1} A^T JB$ using MATLAB. The numerical results on the mean value of these condition numbers and upper bounds are shown in Table 5.1.

Condition number	Mean
$\kappa(A,B)$	135.1850
$\kappa_{\mathcal{L}}(A,B)$	69.3038
$m(A,B)$	69.0179
$m^{\text{upper}}(A,B)$	89.7973
$m_{\mathcal{L}}(A,B)$	46.2828
$c(A,B)$	1.3867×10^4
$c^{\text{upper}}(A,B)$	1.7630×10^4
$c_{\mathcal{L}}(A,B)$	8.5691×10^3

Table 5.1 Comparisons of the structured condition numbers and the unstructured ones for unsymmetric Toeplitz matrices

Example 5.2 Let A be a 25×10 Vandermonde matrix whose (i,j) -entry is $A_{ij} = (\frac{j}{10})^{i-1}$ and B be a 25×4 Vandermonde matrix whose (i,j)-entry is $B_{ij} = (\frac{j}{4})^{i-1}$. Then $a = [\frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10}, 1]^T$, $b = [\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1]^T$. Furthermore, let $J = \begin{bmatrix} I_{24} & \\ & -1 \end{bmatrix}$.

The numerical results on the mean value of the condition numbers and upper bounds are shown in Table 5.2.

Condition number	Mean
$\kappa(A,B)$	2.2970×10^6
$\kappa_{\mathcal{V}}(A,B)$	2.5375×10^5
$m(A,B)$	3.8169×10^5
$m^{\text{upper}}(A,B)$	3.8169×10^5
$m_{\mathcal{V}}(A,B)$	1.6096×10^5
$c(A,B)$	7.3181×10^{11}
$c^{\text{upper}}(A,B)$	7.3181×10^{11}
$c_{\mathcal{V}}(A,B)$	2.7326×10^{10}

Table 5.2 Comparisons of the structured condition numbers and the unstructured ones for Vandermonde matrix

Example 5.3 Let A be a 10×8 Cauchy matrix whose (i,j)-entry is $a_{ij} = \frac{1}{i+j-1}$, that is, $u_i = i,\ v_j = 1-j$, and let B be a 10×4 Cauchy matrix whose (i,j)-entry is $b_{ij} = \frac{1}{i+j-1}$, that is, $r_i = i,\ t_j = 1-j$. Furthermore, let $J = \begin{bmatrix} I_9 & \\ & -1 \end{bmatrix}$.

The numerical results on the mean value of the condition numbers and upper bounds are given in Table 5.3.

Condition number	Mean
$\kappa(A,B)$	1.9829×10^8
$\kappa_{\mathcal{C}}(A,B)$	3.3742×10^7
$m(A,B)$	5.6591×10^7
$m^{\text{upper}}(A,B)$	6.2373×10^7
$m_{\mathcal{C}}(A,B)$	1.4963×10^7
$c(A,B)$	4.4755×10^8
$c^{\text{upper}}(A,B)$	4.6026×10^8
$c_{\mathcal{C}}(A,B)$	2.7835×10^8

Table 5.3 Comparisons of the structured condition numbers and the unstructured ones for Cauchy matrix

From the numerical results in the above three examples, i.e., Tables 5.1, 5.2, and 5.3, we can find that the upper bounds for unstructured condition numbers are sharp and the structured condition numbers for the MRHSILS problem are always smaller than the unstructured ones, which confirm the analysis in Section 4. However, it should be noted that, unlike the linear system and the least squares problem [24,25], the differences between the structured condition numbers and the unstructured ones are not so remarkable.

Example 5.4 In this example, we compare the true errors of the solutions of MRHSILS problems with the bounds based on the condition numbers. We first let

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \text{ with } A_1 = \begin{bmatrix} 101 & & & \\ 102 & 102 & & \\ \vdots & \vdots & \ddots & \\ n+100 & n+100 & n+100 & n+100 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$J = \begin{bmatrix} I_{n+1} & \\ & -I_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & \cdots & s \\ 1 & 2 & \cdots & s \\ \vdots & \vdots & \cdots & s \\ 1 & 2 & \cdots & s \end{bmatrix} \in \mathbb{R}^{n\times s},$$

and generate some random perturbations ΔA and ΔB of matrices A and B, which satisfy the following requirements

$$\|\Delta A\|_F = \gamma \|A\|_F \text{ and } \|\Delta B\|_F = \gamma \|B\|_F$$

with $\gamma = 10^{-4}, 10^{-6}$ or 10^{-8}. Then, we compute the relative perturbation errors $\frac{\|\Delta X\|_F}{\|X\|_F}$, $\frac{\|\Delta X\|_{\max}}{\|X\|_{\max}}$ and $\|\frac{\Delta X}{X}\|_{\max}$, and the first order perturbation bounds given by the three different condition numbers

$$\frac{\|[\Delta A, \Delta B]\|_F}{\|[A,B]\|_F}\kappa(A,B), \|\frac{[\Delta A, \Delta B]}{[A,B]}\|_{\max} m(A,B) \text{ and } \|\frac{[\Delta A, \Delta B]}{[A,B]}\|_{\max} c(A,B).$$

In specific numerical experiments, we set $n = 20$ and $s = 5$. In this case, $A^TJA > 0$, which guarantees the MRHSILS problem has the unique solution. The numerical results are presented in Table 5.4

	$\gamma = 10^{-4}$	$\gamma = 10^{-6}$	$\gamma = 10^{-8}$
$\kappa(A, B)$	497.3134	497.3134	497.3134
$m(A, B)$	47.9587	47.9587	47.9587
$c(A, B)$	47.9587	47.9587	47.9587
$\frac{\\|\Delta X\\|_F}{\\|X\\|_F}$	0.0011	1.2112×10^{-5}	1.6984×10^{-7}
$\frac{\\|[\Delta A, \Delta B]\\|_F}{\\|[A,B]\\|_F}\kappa(A, B)$	0.0497	4.9728×10^{-4}	4.9728×10^{-6}
$\frac{\\|[\Delta A, \Delta B]\\|_F}{\\|[A,B]\\|_F}$	9.9993×10^{-5}	9.9993×10^{-7}	9.9993×10^{-9}
$\left\\|\frac{\Delta X}{X}\right\\|_{\max}$	4.1118×10^{-4}	5.6165×10^{-6}	8.4098×10^{-8}
$\left\\|\frac{[\Delta A, \Delta B]}{[A,B]}\right\\|_{\max} m(A, B)$	0.4956	0.0049	4.9084×10^{-5}
$\left\\|\frac{[\Delta A, \Delta B]}{[A,B]}\right\\|_{\max}$	0.0103	1.0240×10^{-4}	1.0235×10^{-6}
$\left\\|\frac{\Delta X}{X}\right\\|_{\max}$	0.1459	0.0010	8.5834×10^{-6}
$\left\\|\frac{[\Delta A, \Delta B]}{[A,B]}\right\\|_{\max} c(A, B)$	0.4956	0.0049	4.9084×10^{-5}
$\left\\|\frac{[\Delta A, \Delta B]}{[A,B]}\right\\|_{\max}$	0.0103	1.0240×10^{-4}	1.0235×10^{-6}

Table 5.4 Comparisons of true errors and bounds based on condition numbers

From Table 5.4, we see that the differences between the relative true errors and the corresponding first order perturbation bounds based on condition numbers are not so significant.

6. Concluding remarks

In this paper, we first present a condition under which the MRHSILS problem has the unique solution. Based on this condition, we obtain the explicit expressions of the normwise, mixed, componentwise condition numbers for the MRHSILS problem. We also deduce the structured condition numbers for two main classes of structured matrices: linear and nonlinear structured matrices. However, it is easy to see that it is expensive to compute these condition numbers. To tackle this problem, we can use the probabilistic spectral norm estimator [29] and the statistical condition estimation method [30]. This topic will be considered in the future. Three numerical examples are given to test the differences between the structured condition numbers and the corresponding unstructured ones. The numerical results show that the former are always smaller than the latter. But the differences are not so remarkable, which is unlike the case for linear system and the least squares problem given in [24,25]. Furthermore, we also compare the true errors of the solutions of MRHSILS problems with the corresponding first order perturbation bounds based on condition numbers.

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