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Complete Convergence of Randomly Weighted Sums of NOD Random Variables

Zhangrui ZHAO, Xudong QIAO, Wenzhi YANG*, Shuhe HU
School of Mathematical Sciences, Anhui University, Anhui 230601*, P. R. China*

Abstract In this paper, we investigate the complete convergence of double-indexed randomly weighted sums of negatively orthant dependent (NOD) random variables. Some complete moment convergence and complete convergence of this dependent sequence are presented, Marcinkiewicz-Zygmund-type strong law of large numbers is also obtained. Our results extend some corresponding ones. In addition, some simulations are illustrated to show the convergence.

Keywords complete convergence; randomly weighted; NOD sequences; strong law of large numbers

MR(2010) Subject Classification 60E15; 60F15

1. Introduction

Definition 1.1 *A finite collection of random variables* $X_1, X_2, \ldots, X_n$ *is said to be negatively upper orthant dependent (NUOD) if for all real numbers* $x_1, x_2, \ldots, x_n$,

$$P(X_1 > x_1, \ldots, X_n > x_n) \le \prod_{i=1}^{n} P(X_i > x_i),$$

and negatively lower orthant dependent (NLOD) if for all real numbers $x_1, x_2, \ldots, x_n$,

$$P(X_1 \le x_1, \ldots, X_n \le x_n) \le \prod_{i=1}^{n} P(X_i \le x_i).$$

A finite collection of random variables $X_1, X_2, \ldots, X_n$ *is said to be negatively orthant dependent (NOD) if they are both NUOD and NLOD.*

An infinite sequence $\{X_n, n \ge 1\}$ *is said to be NOD (NUOD or NLOD) if every finite subcollection is NOD (NUOD or NLOD).*

Lehmann [1] introduced the notion of NOD. Joag-Dev and Proschan [2] introduced the well-known notion negatively associated (NA) and presented many examples of NA sequences. They

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* Corresponding author
E-mail address: zruiz2958@163.com (Zhangrui ZHAO); wzyang@ahu.edu.cn (Wenzhi YANG); hushuhe@263.net (Shuhe HU)

pointed out NA sequences are NOD sequences but the converse statement cannot always be true. NA sequences are stronger than NOD sequences but they maintain many good properties of independent sequences. For more examples and limit theorems of these random fields and related systems, one can refer to Bulinski and Shaskin [3]. Many authors pay attention on the study of NOD. For the study of NOD sequence, Bozorgnia et al. [4] investigated the limit theory, Asadian et al. [5] obtained some moment inequalities, Wang et al. [6] established some exponential inequalities and inverse moments, Wang et al. [7] obtained the strong limit theory, Sung [8] investigated the moving average processes, Yang et al. [9], Wang et al. [10] and Wang and Si [11] studied the estimator of nonparametric regression with NOD errors, etc.

On the one hand, since Hsu and Robbins [12] gave the concept of complete convergence, the complete convergence has been an important basic tool in probability and statistics. Many authors investigated the complete convergence and complete moment convergence. For example, Zhang [13], Li and Zhang [14] and Yang et al. [15] studied the complete convergence and complete moment convergence for moving-average processes based on φ-mixing sequence, NA sequence and negative quadrant dependent (NQD) sequences, respectively; Wu [16,17] and Sung [18] investigated the complete convergence of weighted sums of NOD sequences; Wang et al. [19] and Wu and Volodin [20] established the complete convergence for arrays of rowwise NOD sequence; Chen and Sung [21] obtained the complete convergence and strong laws of large numbers for weighted sums of NOD sequences, etc.

On the other hand, many researchers pay attention on the study of randomly weight sums of random variables. For example, Thanh and Yin [22] established the almost sure and complete convergence of randomly weighted sums of independent random elements in Banach spaces; Thanh et al. [23] investigated the complete convergence of randomly weighted sums of $\tilde{\rho}$-mixing sequences; Han and Xiang [24] extended the results of Thanh et al. [23] to the results of complete moment convergence based on the double-indexed randomly weighted sums of $\tilde{\rho}$-mixing sequences; Cabrera et al. [25] and Shen et al. [26] investigated the conditional convergence for randomly weighted sums of dependent random variables; Yang et al. [27] and Yao and Lin [28] obtained some results of complete convergence and moment of maximum normed based on the randomly weighted sums of martingales differences, etc.

Inspired by the papers above, we investigate the complete convergence of double-indexed randomly weighted sums of NOD sequence. Some complete moment convergence and complete convergence of this dependent sequence are presented, Marcinkiewicz-Zygmund-type strong law of large numbers is also obtained. We extend the results of Thanh et al. [23] and Yang et al. [27] for the randomly weighted cases of $\tilde{\rho}$-mixing sequence and martingale differences to the one of NOD sequences. In addition, some simulations are illustrated to show the convergence. For the details, please see the results in Section 3. Some lemmas and the proofs of main results are presented in Sections 2 and 4, respectively. Throughout the paper, let $I(A)$ be the indicator function of set A, $x^+ = \max(0, x)$ and C_1, $C_2 \ldots$ denote some positive constants independent of n, which may be different in various places.

2. Some lemmas

Lemma 2.1 ([4]) *Let random variables $X_1, X_2, \ldots, X_n$ be NOD, $f_1, f_2, \ldots, f_n$ be all nondecreasing (or nonincreasing) functions. Then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are NOD.*

Remark 2.2 Let $\{X_n, n \geq 1\}$ be an NOD sequence and $\{Y_n, n \geq 1\}$ be a sequence of nonnegative and independent random variables, which is independent of $\{X_n, n \geq 1\}$. Let $Z_n = X_n Y_n$. Then, by the definition of NOD and the nonnegativity and independence of $\{Y_n, n \geq 1\}$, we have that for all real numbers $z_1, \ldots, z_n$,

$$
\begin{aligned}
P(Z_1 \leq z_1, \ldots, Z_n \leq z_n) &= P(X_1Y_1 \leq z_1, \ldots, X_nY_n \leq z_n) \\
&= \int_0^\infty \cdots \int_0^\infty P(X_1u_1 \leq z_1, \ldots, X_nu_n \leq z_n)\mathrm{d}F_{Y_1}(u_1)\cdots \mathrm{d}F_{Y_n}(u_n) \\
&\leq \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n P(X_iu_i \leq z_i)\mathrm{d}F_{Y_1}(u_1)\cdots \mathrm{d}F_{Y_n}(u_n) \\
&= \prod_{i=1}^n P(X_iY_i \leq z_i) = \prod_{i=1}^n P(Z_i \leq z_i),
\end{aligned}
$$

by using the fact that $u_1X_1, u_2X_2, \ldots, u_nX_n$ are NOD following from Lemma 2.1. Similarly, we have for all real numbers $z_1, \ldots, z_n$,

$$
P(Z_1 > z_1, \ldots, Z_n > z_n) \leq \prod_{i=1}^n P(Z_i > z_i).
$$

Thus, $\{Z_n, n \geq 1\}$ is also an NOD sequence.

Lemma 2.3 ([5]) *Let $p \geq 2$ and $\{X_n, n \geq 1\}$ be an NOD sequence such that $EX_n = 0$ and $E|X_n|^p < \infty$ for all $n \geq 1$. Then there holds*

$$
E\Big|\sum_{i=1}^n X_i\Big|^p \leq C_p\Big\{\sum_{i=1}^n E|X_i|^p + \Big(\sum_{i=1}^n EX_i^2\Big)^{p/2}\Big\},
$$

where C_p is a positive constant depending only on p.

Lemma 2.4 ([29]) *Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be sequences of random variables. Then for any $n \geq 1$, $q > 1$, $\varepsilon > 0$ and $a > 0$,*

$$
E\Big(\Big|\sum_{i=1}^n (X_i + Y_i)\Big| - \varepsilon a\Big)^+ \leq \Big(\frac{1}{\varepsilon^q} + \frac{1}{q-1}\Big)\frac{1}{a^{q-1}}E\Big|\sum_{i=1}^n X_i\Big|^q + E\Big|\sum_{i=1}^n Y_i\Big|.
$$

Lemma 2.5 ([30,31]) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a nonnegative random variable X, i.e., $\sup_{n\geq 1} P(|X_n| > t) \leq CP(X > t)$ for some positive constant C and for all $t \geq 0$. Then for any $n \geq 1$, $\alpha > 0$ and $\beta > 0$, the following two statements hold:*

$$
E[|X_n|^\alpha I(|X_n| \leq \beta)] \leq C_1\{E[X^\alpha I(X \leq \beta)] + \beta^\alpha P(X > \beta)\},
$$

$$
E[|X_n|^\alpha I(|X_n| > \beta)] \leq C_2 E[X^\alpha I(X > \beta)].
$$

Consequently, we have $E|X_n|^\alpha \le C_3 EX^\alpha$ *for all* $n \ge 1$.

3. Main results and simulation

In the following, we list some assumptions:

(A.1) Let $\{X_n, n \ge 1\}$ be a mean zero sequence of NOD random variables stochastically dominated by a nonnegative random variables X.

(A.2) For every $n \ge 1$, let $\{A_{ni}, 1 \le i \le n\}$ be a sequence of independent random variables satisfying that $\{A_{ni}, 1 \le i \le n\}$ is independent of $\{X_n, n \ge 1\}$.

Theorem 3.1 *Assume that (A.1) and (A.2) are satisfied. Let* $\alpha > 1/2$, $1 < p < 2$ *and* $EX^p < \infty$. *If*

$$\sum_{i=1}^{n} EA_{ni}^2 = O(n), \tag{1}$$

then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} E\Big(\Big|\sum_{i=1}^{n} A_{ni}X_i\Big| - \varepsilon n^\alpha\Big)^+ < \infty. \tag{2}$$

So it follows

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\Big|\sum_{i=1}^{n} A_{ni}X_i\Big| > \varepsilon n^\alpha\Big) < \infty. \tag{3}$$

Theorem 3.2 *Assume that (A.1) and (A.2) are satisfied. Let* $\alpha > 1/2$, $p \ge 2$ *and* $EX^p < \infty$. *If*

$$\sum_{i=1}^{n} E|A_{ni}|^q = O(n), \quad \text{for some } q > \frac{2(\alpha p-1)}{2\alpha-1}, \tag{4}$$

then we obtain the results of (2) and (3) for every $\varepsilon > 0$.

If $1 \le l < 2$, $p = 2l$ and $\alpha = 2/p$ in Theorem 3.2, then we establish the following result.

Theorem 3.3 *Suppose that (A.1) and (A.2) are fulfilled. Let* $1 \le l < 2$ *and* $EX^{2l} < \infty$. *If*

$$\sum_{i=1}^{n} E|A_{ni}|^q = O(n), \quad \text{for some } q > \frac{2l}{2-l}, \tag{5}$$

then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-1/l} E\Big(\Big|\sum_{i=1}^{n} A_{ni}X_i\Big| - \varepsilon n^{1/l}\Big)^+ < \infty, \tag{6}$$

and

$$\sum_{n=1}^{\infty} P\Big(\Big|\sum_{i=1}^{n} A_{ni}X_i\Big| > \varepsilon n^{1/l}\Big) < \infty. \tag{7}$$

In particular, we have the Marcinkiewicz-Zygmund-type strong law of large numbers

$$\lim_{n\to\infty} \frac{1}{n^{1/l}} \sum_{i=1}^{n} A_{ni}X_i = 0, \quad \text{a.s.} \tag{8}$$

Meanwhile, for the case $p = 1$, we have the following result.

Theorem 3.4 *Suppose that (A.1) and (A.2) are fulfilled. Let* $\alpha > 0$ *and* $E[X\log X] < \infty$. *If (1) holds, then for every* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{-2} E\Big(\Big|\sum_{i=1}^{n} A_{ni}X_i\Big| - \varepsilon n^{\alpha}\Big)^{+} < \infty. \tag{9}$$

In particular, we have

$$\sum_{n=1}^{\infty} n^{\alpha-2} P\Big(\Big|\sum_{i=1}^{n} A_{ni}X_i\Big| > \varepsilon n^{\alpha}\Big) < \infty. \tag{10}$$

Simulation 3.5 We use MATLAB software to do some simulations for the convergence of (8) in Theorem 3.3. For $n \geq 2$, let $(X_1, X_2, \dots, X_n)$ be a normal random vector such as $(X_1, X_2, \dots, X_n) \sim N_n(0, \Sigma)$, where 0 is zero vector,

$$\Sigma = \begin{bmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \vdots & \vdots & \vdots & & \vdots \\ \rho & \rho & \rho & \dots & 1 \end{bmatrix}_{n\times n},$$

and $\rho \in (-\frac{1}{n-1}, 0]$. By Joag-Dev and Proschan [2], it can be seen that $(X_1, X_2, \dots, X_n)$ is an NA vector, which implies that it is also an NOD vector. In the following, there are two cases of assumption of $\{A_{ni}, 1 \leq i \leq n, n \geq 1\}$:

(i) For all $n \geq 1$, let $\{A_{ni}, 1 \leq i \leq n\}$ be i.i.d. uniform distribution satisfying $A_{n1} \sim U(-1, 1)$, which are also independent of $\{X_n, n \geq 1\}$.

(ii) For all $n \geq 1$, let $\{A_{ni}, 1 \leq i \leq n\}$ be i.i.d. student distribution satisfying $A_{n1} \sim t(m)$ with degree of freedom $m > 0$, which are also independent of $\{X_n, n \geq 1\}$.

By MATLAB software, we make the Box plots to illustrate

$$\frac{1}{n^{1/l}} \sum_{i=1}^{n} A_{ni}X_i \to 0, \quad n \to \infty. \tag{11}$$

For $l = 1.4$ (or $l = 1$), $\rho = -\frac{1}{n}$ (or $\rho = -\frac{1}{2n}$), the uniform distribution $A_{n1} \sim U(-1, 1)$ and sample size $n = 100, 200, \dots, 1000$, we repeat the experiments 10000 times and obtain the Box plots such as Figures 1 and 2.

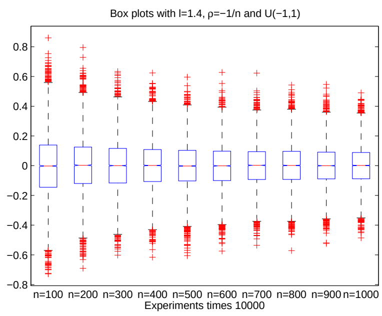

Figure 1 The first of multiple normal random variables randomly weighted by uniform random variables

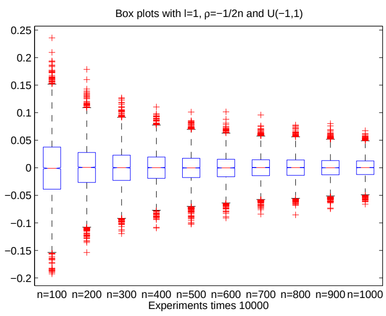

Figure 2 The second of multiple normal random variables randomly weighted by uniform random variables

In Figures 1 and 2, the label of y-axis is the value of (11) and the label of x-axis is the number of sample n, by repeating the experiments 10000 times. By Figure 1, for the case of $l=1.4$, $\rho=-\frac{1}{n}$ and $A_{n1}\sim U(-1,1)$, it can be found that the medians are close to 0 and their variation ranges slowly become smaller as the sample n increases. By the strong conditions $l=1$ and $\rho=-\frac{1}{2n}$, we can see that the medians are close to 0 and the variation ranges quickly become smaller as the sample n increases in Figure 2.

Similarly, for $l=1.2$, $\rho=-\frac{1}{n}$, student distributions $A_{n1}\sim t(5)$ and $A_{n1}\sim t(25)$, we obtain the Figures 3 and 4. By Figures 3 and 4, it can be checked that the medians are close to 0 and their variation range become smaller as the sample n increasing. Comparing Figure 3 with Figure 4, the variation range of Figure 4 is smaller than the one of Figure 3, which can be explained that the variance of $t(25)$ is smaller than the one of $t(5)$.

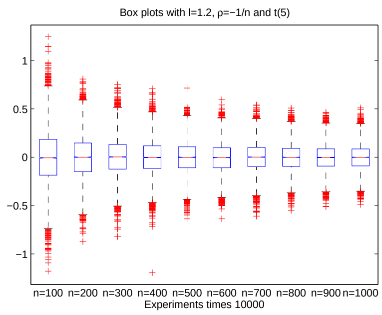

Figure 3 The third of multiple normal random variables randomly weighted by student random variables

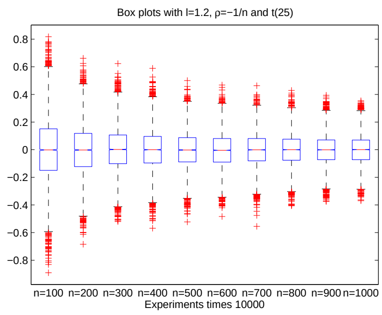

Figure 4 The fourth of multiple normal random variables randomly weighted by student random variables

4. The proofs of main results

Proof of Theorem 3.1 Since $A_{ni}X_i=A_{ni}^+X_i-A_{ni}^-X_i$, without loss of generality, we assume $A_{ni}\geq 0$ in the proof. For $n\geq 1$, let $X_{ni}=-n^\alpha I(X_i<-n^\alpha)+X_iI(|X_i|\leq n^\alpha)+n^\alpha I(X_i>n^\alpha)$, $\tilde{X}_{ni}=n^\alpha I(X_i<-n^\alpha)+X_iI(|X_i|>n^\alpha)-n^\alpha I(X_i>n^\alpha)$, $1\leq i\leq n$.

It can be found that

$$A_{ni}X_i=[A_{ni}X_{ni}-E(A_{ni}X_{ni})]+E(A_{ni}X_{ni})+A_{ni}\tilde{X}_{ni},\quad 1\leq i\leq n.$$

Therefore, by Lemma 2.4 with $a=n^\alpha$ and $q=2$, we obtain that

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha}E\Big(\Big|\sum_{i=1}^{n}A_{ni}X_i\Big|-\varepsilon n^\alpha\Big)^+\\&\leq C_1\sum_{n=1}^{\infty}n^{\alpha p-2-2\alpha}E\Big|\sum_{i=1}^{n}[A_{ni}X_{ni}-E(A_{ni}X_{ni})]\Big|^2+\\&\quad\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha}E\Big|\sum_{i=1}^{n}A_{ni}\tilde{X}_{ni}\Big|+\sum_{n=1}^{\infty}n^{\alpha p-2-\alpha}\Big|\sum_{i=1}^{n}E(A_{ni}X_{ni})\Big|\\&:=H_1+H_2+H_3.\end{aligned}\tag{12}$$

Combining (1) with Hölder's inequality, one has that

$$\sum_{i=1}^{n} E|A_{ni}| \le \Big(\sum_{i=1}^{n} EA_{ni}^2\Big)^{1/2}\Big(\sum_{i=1}^{n} 1\Big)^{1/2} = O(n). \tag{13}$$

Since that for for every $n \ge 1$, $\{A_{ni}, 1 \le i \le n\}$ is independent of the sequence $\{X_n, n \ge 1\}$, one has by Markov's inequality, Lemma 2.5, (13) and $EX^p < \infty$ $(p > 1)$ that

$$\begin{aligned}
H_2 \le & 2\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} E|A_{ni}|E|X_i|I(|X_i| > n^{\alpha}) \\
\le & C_1 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E[XI(X > n^{\alpha})] \\
= & C_1 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} \sum_{m=n}^{\infty} E[XI(m < X^{1/\alpha} \le m+1)] \\
= & C_1 \sum_{m=1}^{\infty} E[XI(m < X^{1/\alpha} \le m+1)] \sum_{n=1}^{m} n^{\alpha(p-1)-1} \\
\le & C_2 \sum_{m=1}^{\infty} m^{\alpha p-\alpha} E[XI(m < X^{1/\alpha} \le m+1)] \\
\le & C_2 \sum_{m=1}^{\infty} E[X^p I(m < X^{1/\alpha} \le m+1)] \le C_3 EX^p < \infty.
\end{aligned} \tag{14}$$

On the other hand, it can be seen that $E(A_{ni}X_i) = EA_{ni}EX_i = 0$, $1 \le i \le n$, $n \ge 1$. So, by (13), Lemma 2.5 and the proof of (14), we have

$$\begin{aligned}
H_3 = & \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \Big| \sum_{i=1}^{n} \Big[- n^{\alpha} EA_{ni} I(X_i < -n^{\alpha}) - EA_{ni}X_i I(|X_i| > n^{\alpha}) \\
& + n^{\alpha} EA_{ni} I(X_i > n^{\alpha})\Big]\Big| \\
\le & \ 2\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} \sum_{i=1}^{n} E|A_{ni}|E[|X_i|I(|X_i| > n^{\alpha})] \\
\le & \ C_1 \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E[XI(X > n^{\alpha})] \le C_2 EX^p < \infty.
\end{aligned} \tag{15}$$

From Lemma 2.1 we know that $\{X_{ni}, 1 \le i \le n\}$ are NOD random variables. Combining the assumption of $\{A_{ni}\}$ with Remark 2.2, we establish that $\{[A_{ni}X_{ni} - E(A_{ni}X_{ni})], 1 \le i \le n\}$ are mean zero NOD random variables. So, by Markov's inequality, (1), Lemma 2.3 with $p = 2$ and Lemma 2.5, we get that

$$\begin{aligned}
H_1 = & \ C_1 \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} E\Big| \sum_{i=1}^{n} [A_{ni}X_{ni} - E(A_{ni}X_{ni})]\Big|^2 \\
\le & \ C_2 \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} \sum_{i=1}^{n} E(A_{ni}X_{ni})^2 \\
\le & \ C_3 \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} E[X^2 I(X \le n^{\alpha})] + C_4 \sum_{n=1}^{\infty} n^{\alpha p-1} EI(X > n^{\alpha})
\end{aligned}$$

$$:= C_3H_{11} + C_4H_{12}. \tag{16}$$

Since $p < 2$ and $EX^p < \infty$, it can be checked that

$$\begin{aligned}H_{11} &= \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} \sum_{i=1}^{n} E[X^2 I((i-1)^\alpha < X \le i^\alpha)] \\ &= \sum_{i=1}^{\infty} E[X^2 I((i-1)^\alpha < X \le i^\alpha)] \sum_{n=i}^{\infty} n^{\alpha p-1-2\alpha} \\ &\le C_1 \sum_{i=1}^{\infty} E[X^p X^{2-p} I((i-1)^\alpha < X \le i^\alpha)] i^{\alpha p-2\alpha} \le C_1 EX^p < \infty.\end{aligned} \tag{17}$$

By the proof of (14),

$$H_{12} \le \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E[XI(X > n^\alpha)] \le CEX^p < \infty. \tag{18}$$

Combining (12) with (13)–(18), we can get (2) immediately. Moreover, by Remark 2.6 of Sung [29], (3) also holds true. □

Proof of Theorem 3.2 We use the same notation as in the proof of Theorem 3.1. Obviously, by $p \ge 2$, it is easy to see that $q > 2(\alpha p - 1)/(2\alpha - 1) \ge 2$. Consequently, for any $1 \le r \le 2$, by Hölder's inequality and condition (4), we have

$$\sum_{i=1}^{n} E|A_{ni}|^r \le \Big(\sum_{i=1}^{n} E|A_{ni}|^q\Big)^{r/q} \Big(\sum_{i=1}^{n} 1\Big)^{1-r/q} = O(n). \tag{19}$$

From (12), (14), (15) and (19), it follows that $H_2 < \infty$ and $H_3 < \infty$. So we have to prove $H_1 < \infty$ under conditions of Theorem 3.2. Since $q > 2$, similar to the proof of (16), by Lemma 2.3, it follows

$$\begin{aligned}H_1 =& C_1 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} E\Big|\sum_{i=1}^{n} [A_{ni}X_{ni} - E(A_{ni}X_{ni})]\Big|^q \\ \le& C_2 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \Big(\sum_{i=1}^{n} E[A_{ni}X_{ni} - E(A_{ni}X_{ni})]^2\Big)^{q/2} + \\ & C_2 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \sum_{i=1}^{n} E|A_{ni}X_{ni} - E(A_{ni}X_{ni})|^q \\ :=& C_2H_{11} + C_2H_{12}.\end{aligned} \tag{20}$$

Obviously, by Lemma 2.5, we have

$$\begin{aligned}&E[A_{ni}X_{ni} - E(A_{ni}X_{ni})]^2 \le CEA_{ni}^2 EX_{ni}^2 \\ &\quad \le CEA_{ni}^2 \{E[X^2 I(X \le n^\alpha)] + n^{2\alpha} P(X > n^\alpha)\} \\ &\quad \le CEA_{ni}^2 \{E[X^2 I(X \le n^\alpha)] + E[X^2 I(X > n^\alpha)]\} \\ &\quad = CEA_{ni}^2 EX^2,\ 1 \le i \le n.\end{aligned} \tag{21}$$

For $p \geq 2$, $EX^p < \infty$ implies $EX^2 < \infty$. Thus, by (19) and (21), one has that

$$H_{11} \leq C_3 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \Big(\sum_{i=1}^{n} EA_{ni}^2 EX^2 \Big)^{q/2} \leq C_4 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha+q/2} < \infty, \tag{22}$$

following from the fact that $q > 2(\alpha p - 1)/(2\alpha - 1)$. Meanwhile, by C_r inequality, Lemma 2.5 and (4),

$$\begin{aligned} H_{12} \leq & C_5 \sum_{n=1}^{\infty} n^{\alpha p-2-q\alpha} \sum_{i=1}^{n} E|A_{ni}|^q E|X_{ni}|^q \\ \leq & C_6 \sum_{n=1}^{\infty} n^{\alpha p-1-q\alpha} E[X^q I(X \leq n^\alpha)] + C_7 \sum_{n=1}^{\infty} n^{\alpha p-1} P(X > n^\alpha) \\ := & C_6 H_{12}^{*} + C_7 H_{12}^{**}. \end{aligned} \tag{23}$$

From $p \geq 2$ and $\alpha > 1/2$ it follows $2(\alpha p - 1)/(2\alpha - 1) - p \geq 0$, which implies $q > p$. So, by $EX^p < \infty$, we get that

$$\begin{aligned} H_{12}^{*} = & \sum_{n=1}^{\infty} n^{\alpha p-1-q\alpha} \sum_{i=1}^{n} E[X^q I((i-1)^\alpha < X \leq i^\alpha)] \\ = & \sum_{i=1}^{\infty} E[X^q I((i-1)^\alpha < X \leq i^\alpha)] \sum_{n=i}^{\infty} n^{\alpha p-1-q\alpha} \\ \leq & C_1 \sum_{i=1}^{\infty} E[X^p X^{q-p} I((i-1)^\alpha < X \leq i^\alpha)] i^{\alpha p - q\alpha} \\ \leq & C_1 EX^p < \infty. \end{aligned} \tag{24}$$

By the proof of (14),

$$H_{12}^{**} \leq \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E[XI(X > n^\alpha)] \leq CEX^p < \infty. \tag{25}$$

Consequently, by (20), (22)–(25), we obtain that $H_1 < \infty$. So, we obtain the result (2). Similarly, by Remark 2.6 of Sung [29], (3) also holds true. □

Proof of Theorem 3.3 By taking $p = 2l$, $\alpha = 2/p$, we have $\alpha p = 2$. On the other hand, by the fact $1 \leq l < 2$, we can see that condition (4) reduces to (5). As an application of Theorem 3.2, one gets (6) and (7) immediately. Combining (7) with Borel Cantelli lemma, we establish that $\lim_{n\to\infty} \frac{1}{n^{1/l}} \sum_{i=1}^{n} A_{ni} X_i = 0$, a.s. So, (8) holds. The proof of the theorem is completed. □

Proof of Theorem 3.4 Similarly to the proof of Theorem 3.1, by Lemma 2.4, it follows

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{-2} E\Big(\Big| \sum_{i=1}^{n} A_{ni} X_i \Big| - \varepsilon n^\alpha \Big)^{+} \\ & \leq C_1 \sum_{n=1}^{\infty} n^{-2-\alpha} E\Big| \sum_{i=1}^{n} [A_{ni} X_{ni} - E(A_{ni} X_{ni})] \Big|^2 + \\ & \quad \sum_{n=1}^{\infty} n^{-2} E\Big| \sum_{i=1}^{n} A_{ni} \tilde{X}_{ni} \Big| + \sum_{n=1}^{\infty} n^{-2} \Big| \sum_{i=1}^{n} E(A_{ni} X_{ni}) \Big| \end{aligned}$$

$$:= Q_1 + Q_2 + Q_3. \tag{26}$$

Similarly to the proof of (14), it follows

$$\begin{aligned} Q_2 \le & 3\sum_{n=1}^{\infty} n^{-1} E[XI(X > n^\alpha)] = 3\sum_{n=1}^{\infty} n^{-1} \sum_{m=n}^{\infty} E[XI(m < X^{1/\alpha} \le m+1)] \\ = & 3\sum_{m=1}^{\infty} E[XI(m < X^{1/\alpha} \le m+1)] \sum_{n=1}^{m} n^{-1} \\ \le & C_1 \sum_{m=1}^{\infty} \log m E[XI(m < X^{1/\alpha} \le m+1)] \le C_2 E[X \log X] < \infty. \end{aligned} \tag{27}$$

Similarly, by the proof of (15), we have

$$Q_3 \le C_1 \sum_{n=1}^{\infty} n^{-1} E[XI(X > n^\alpha)] \le C_2 E[X \log X] < \infty. \tag{28}$$

On the other hand, similarly to the proof of (16), we obtain that

$$\begin{aligned} Q_1 \le & C_1 \sum_{n=1}^{\infty} n^{-2-\alpha} \sum_{i=1}^{n} E(A_{ni} X_{ni})^2 = C_1 \sum_{n=1}^{\infty} n^{-2-\alpha} \sum_{i=1}^{n} E A_{ni}^2 E X_{ni}^2 \\ \le & C_2 \sum_{n=1}^{\infty} n^{-1-\alpha} E[X^2 I(X \le n^\alpha)] + C_3 \sum_{n=1}^{\infty} n^{\alpha-1} P(X > n^\alpha) \\ \le & C_2 \sum_{i=1}^{\infty} E[X^2 I((i-1)^\alpha < X \le i^\alpha)] \sum_{n=i}^{\infty} n^{-1-\alpha} + C_4 E[X \log X] \\ \le & C_5 \sum_{i=1}^{\infty} E[X^2 I((i-1)^\alpha < X \le i^\alpha)] i^{-\alpha} + C_4 E[X \log X] \\ \le & C_6 EX + C_5 E[X \log X] < \infty. \end{aligned} \tag{29}$$

So, by (26)–(29), (9) holds. On the other hand, by (3) with $p = 1$, (10) also holds under the conditions of Theorem 3.4. □

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