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Hypergraphs with Spectral Radius between Two Limit Points

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Abstract In this paper, we set $\rho_r = \sqrt[r]{4}$ and $\rho_r' = \beta^{-1/r}$, where $\beta = -\frac{1}{6} \cdot (100 + 12 \cdot \sqrt{69})^{\frac{1}{3}} - \frac{2}{3 \cdot (100+12 \cdot \sqrt{69})^{\frac{1}{3}}} + \frac{4}{3} \approx 0.2451223338$. We consider connected r-uniform hypergraphs with spectral radius between ρ_r and ρ_r' and give a description of such hypergraphs.

Keywords r-uniform hypergraphs; spectral radius; α-normal

MR(2010) Subject Classification 05C50; 05C35; 05C65

1. Introduction

In 1970, Smith classified all connected graphs with the spectral radius at most 2 in [1]. Here the spectral radius of a graph is the largest eigenvalue of its adjacency matrix. In our previous paper [2], we generalized Smith's theorem to hypergraphs and classified all connected r-uniform hypergraphs with the spectral radius at most $\rho_r = \sqrt[r]{4}$.

Let us review some basic notation about hypergraphs. An r-uniform hypergraph H is a pair (V, E) where V is the set of vertices and $E \subset \binom{V}{r}$ is the set of edges. The degree of vertex v, denoted by d_v, is the number of edges incident to v. If $d_v = 1$, we call v a pendant vertex or a leaf vertex in a tree. An edge e is called a branching edge if every vertex of e is not a leaf vertex. A walk on hypergraph H is a sequence of vertices and edges: $v_0 e_1 v_1 e_2 \ldots v_l$ satisfying that both v_{i-1} and v_i are incident to e_i for $1 \le i \le l$. The vertices v_0 and v_l are called the ends of the walk. The length of a walk is the number of edges on the walk. A walk is called a path if all vertices and edges on the walk are distinct. The walk is closed if $v_l = v_0$. A closed walk is called a cycle if all vertices and edges in the walk are distinct. A hypergraph H is called connected if for any pair of vertex (u, v), there is a path connecting u and v. A hypergraph H is called a hypertree if it is connected, and acyclic. A hypergraph H is called simple if every pair of edges intersects at most one vertex, and a simple hypergraph is usually called a linear hypergraph. In fact, any non-simple hypergraph contains at least a 2-cycle: $v_1 F_1 v_2 F_2 v_1$, i.e., $v_1, v_2 \in F_1 \cap F_2$. A hypertree is always simple.

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From [2], given an r-uniform hypergraph H, the polynomial form of H is a function $P_H(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ defined for any vector $\mathbf{x} := (x_1, \ldots, x_n) \in R^n$ as

$$P_H(\mathbf{x}) = r \sum_{\{i_1,i_2,\ldots,i_r\}\in E(H)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

For any $p \geq 1$, the largest p-eigenvalue of H is defined as

$$\lambda_p(H) = \max_{|\mathbf{x}|_p=1} P_H(x).$$

In this paper as in [2], we define the spectral radius of an r-uniform hypergraph H to be $\rho(H) = \lambda_r(H)$. Equivalently, we have

$$\rho(H) = r \max_{\substack{\mathbf{x}\in\mathbb{R}^n_{\geq 0}\\ \mathbf{x}\neq 0}} \frac{\sum_{\{i_1,i_2,\ldots,i_r\}\in E(H)} x_{i_1} x_{i_2} \cdots x_{i_r}}{\sum_{i=1}^n x_i^r}. \tag{1.1}$$

Here $\mathbb{R}^n_{\geq 0}$ denotes the closed orthant in $\mathbb{R}^n$ while $\mathbb{R}^n_{>0}$ denotes the open orthant. This is a special case of p-spectral norm for $p = r$. The general p-spectral norm has been considered by various authors [3–6].

As we have considered the hypergraphs with spectral radius at most $\rho_r = \sqrt[r]{4}$ in [2], it is natural to ask what the hypergraphs look like with spectral radius slightly greater than ρ_r. Such question has been answered for graphs. Cvetković et al. [7] gave a nearly complete description of all graphs G with $2 < \rho(G) < \sqrt{2+\sqrt{5}}$. Their description was completed by Brouwer and Neumaier [8]. Namely, the graphs that satisfy this condition are isomorphic to $E(1,b,c)$ for $b \geq 2$, $E(2,2,c)$ for $c \geq 3$, and $G_{1,a:b:1,c}$ for $a \geq 3$, $c \geq 2$, $b > a + c$.

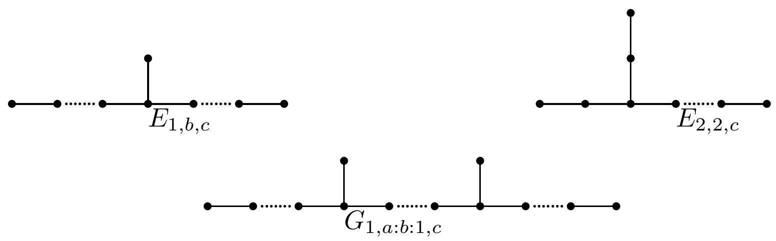


Figure 1 The graphs with spectral radius between 2 and $\sqrt{2+\sqrt{5}}$.

Observe that $\sqrt{2+\sqrt{5}}$ is the limit of the spectral radii of the sequence of graphs $E_{1,b,c}$ as b, c go to infinity. This motivates us to consider the limit of the spectral radii of the sequence of the following 3-uniform hypergraphs $F^{(3)}_{1,b,c}$.

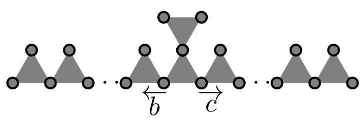


Figure 2 Hypergraphs $F^{(3)}_{1,b,c}$

In hypergraphs $F^{(3)}_{1,b,c}$, there is one branching edge, and above the branching edge there is one edge. On the left of the branching edge there are b edges, while on the right of the branching edge there are c edges. Let $\rho_3' = \lim_{b,c\to\infty} \rho(F^{(3)}_{1,b,c})$. It turns out (see Lemma 2.12) that $\rho_3' = 2\beta^{-1/3}$

where $\beta = -\frac{1}{6}\cdot(100+12\cdot\sqrt{69})^{\frac{1}{3}} - \frac{2}{3\cdot(100+12\cdot\sqrt{69})^{\frac{1}{3}}} + \frac{4}{3} \approx 0.2451223338$ is the real root of the cubic equation $x^3-4x^2+5x-1=0$. By this cubic equation, we list other remarkable identities satisfied by β with proofs left to the readers. All these identities are sometimes useful for computing the spectral radius.

$$\frac{\beta}{(1-\beta)^2} = \frac{1-\sqrt{1-4\beta}}{2}, \tag{1.2}$$

$$\frac{\beta}{(1-\beta)} = (\frac{1+\sqrt{1-4\beta}}{2})^2, \tag{1.3}$$

$$1-\sqrt{\beta(1-\beta)} = \sqrt{\frac{\beta}{(1-\beta)}}, \tag{1.4}$$

$$(1-\beta)^5 = \beta. \tag{1.5}$$

In this paper, for $r \geq 3$, setting $\rho'_r = \beta^{-1/r}$, we classify almost all r-uniform hypergraphs with spectral radius in (ρ_r, ρ'_r). The paper is organized as follows. In Section 2, we introduce the notation and some important lemmas for computing the spectral radius. In Section 3, we classify all connected 3-uniform hypergraphs with the spectral radius between ρ_3 and ρ'_3. In Section 4, by the methods of reduction and extension we classify all connected r-uniform hypergraphs with the spectral radius between ρ_r and ρ'_r.

2. Notation and lemmas

2.1. Some lemmas of finite hypergraphs

The following lemma has been proved in several papers.

Lemma 2.1 ([4–6]) *If G is a connected r-uniform hypergraph, and H is a proper subgraph of G, then*

$$\rho(H) < \rho(G).$$

In our previous paper [2], we discovered an efficient way to compute the spectral radius $\rho(H)$, in particular when H is a hypertree. We give the following definitions and lemmas from [2] for the reader's convenience.

Definition 2.2 ([2]) *A weighted incidence matrix B of a hypergraph $H=(V,E)$ is a $|V|\times|E|$ matrix such that for any vertex v and any edge e, the entry $B(v,e)>0$ if $v\in e$ and $B(v,e)=0$ if $v\notin e$.*

Definition 2.3 ([2]) *A hypergraph H is called α-normal if there exists a weighted incidence matrix B satisfying*

(1) $\sum_{e:\, v\in e} B(v,e)=1$, *for any $v\in V(H)$.*

(2) $\prod_{v\in e} B(v,e)=\alpha$, *for any $e\in E(H)$.*

Moreover, the incidence matrix B is called consistent if for any cycle $v_0e_1v_1e_2\ldots v_l$ $(v_l=v_0)$

$$\prod_{i=1}^{l}\frac{B(v_i,e_i)}{B(v_{i-1},e_i)} = 1.$$

In this case, we call H consistently α-normal.

The following important lemma was proved in [2].

Lemma 2.4 ([2, Lemma 3]) *Let H be a connected r-uniform hypergraph. Then the spectral radius of H is $\rho(H)$ if and only if H is consistently α-normal with $\alpha = (/\rho(H))^r$.*

Often we need compare the spectral radius with a particular value.

Definition 2.5 ([2]) *A hypergraph H is called α-subnormal if there exists a weighted incidence matrix B satisfying*

(1) $\sum_{e:\ v\in e} B(v,e) \le 1$, *for any* $v \in V(H)$.

(2) $\prod_{v\in e} B(v,e) \ge \alpha$, *for any* $e \in E(H)$.

Moreover, H is called strictly α-subnormal if it is α-subnormal but not α-normal.

We have the following lemma.

Lemma 2.6 ([2, Lemma 4]) *Let H be an r-uniform hypergraph. If H is α-subnormal, then the spectral radius of H satisfies $\rho(H) \le \alpha^{-\frac{1}{r}}$. Moreover, if H is strictly α-subnormal, then $\rho(H) < \alpha^{-\frac{1}{r}}$.*

Definition 2.7 ([2]) *A hypergraph H is called α-supernormal if there exists a weighted incidence matrix B satisfying*

(1) $\sum_{e:\ v\in e} B(v,e) \ge 1$, *for any* $v \in V(H)$.

(2) $\prod_{v\in e} B(v,e) \le \alpha$, *for any* $e \in E(H)$.

Moreover, H is called strictly α-supernormal if it is α-supernormal but not α-normal.

Lemma 2.8 ([2, Lemma 5]) *Let H be an r-uniform hypergraph. If H is strictly and consistently α-supernormal, then the spectral radius of H satisfies $\rho(H) > \alpha^{-\frac{1}{r}}$.*

2.2. Spectral radius of infinite hypergraphs with bounded degrees

Often, we need to consider the spectral radius of infinite hypergraph on countably many vertices. An infinite and connected r-uniform hypergraph H is said to have bounded-degree if there exists an M such that $d_v \le M$ for any vertex v. Given a bounded-degree r-uniform hypergraph H with countably many vertices, we can order the vertices $v_1, v_2, \ldots$, so that the induced graph $H_n := H[v_1, v_2, \ldots, v_n]$ is still connected. Notice that $\rho(H_n)$ is an increasing function of n and is bounded by M. Thus the limit $\lim_{n\to\infty} \rho(H_n)$ always exists, and is called the spectral radius of H.

Lemma 2.9 *For any connected infinite r-uniform hypergraph H with bounded degree, the definition of the spectral radius above is independent of the choice of the order of the vertices.*

Proof Suppose $v_1, v_2, \ldots$ and $v'_1, v'_2, \ldots$, are the lists of two orderings of the vertices. There are two injective maps $\phi_i \colon \mathbb{N} \to \mathbb{N}$ (for $i = 1, 2$) such that

$$\{v_1, v_2, \ldots, v_n\} \subset \{v'_1, v'_2, \ldots, v'_{\phi_1(n)}\};$$

$$\{v'_1, v'_2, \ldots, v'_n\} \subset \{v_1, v_2, \ldots, v_{\phi_2(n)}\}.$$

Let $H_n := H[v_1, v_2, \ldots, v_n]$ and $H'_n = H[v'_1, v'_2, \ldots, v'_n]$. We have $H_n \subseteq H'_{\phi_1(n)}$ and $H'_n \subseteq H_{\phi_2(n)}$. This implies that $\rho(H_n) \le \rho(H'_{\phi(n)})$ and $\rho(H'_n) \le \rho(H_{\phi(n)})$. Thus $\lim_{n\to\infty}\rho(H_n) \le \lim_{n\to\infty}\rho(H'_n)$ and $\lim_{n\to\infty}\rho(H'_n) \le \lim_{n\to\infty}\rho(H_n)$. Hence the two limits are equal. □

We can extend the definition of α-normal labellings to infinite hypergraphs H.

Lemma 2.10 *Suppose* $0 < \beta < \frac{1}{4}$, *let* $f(x) = \frac{\beta}{1-x}$ *and* $f_n(x) = f(f_{n-1}(x))$ *for* $n \ge 2$.

(1) If $0 < x \le \frac{1-\sqrt{1-4\beta}}{2}$, *then* $f_n(x)$ *is increasing with respect to* n, *and* $\lim_{n\to\infty} f_n(x) = \frac{1-\sqrt{1-4\beta}}{2}$. *Moreover, when* $x = \frac{1-\sqrt{1-4\beta}}{2}$, $f_n(x) = \frac{1-\sqrt{1-4\beta}}{2}$, $\forall n \ge 1$.

(2) If $\frac{1-\sqrt{1-4\beta}}{2} < x < \frac{1+\sqrt{1-4\beta}}{2}$, *then* $f_n(x)$ *is decreasing with respect to* n, *and*

$$\lim_{n\to\infty} f_n(x) = \frac{1-\sqrt{1-4\beta}}{2}.$$

Proof We first prove item (1). Since $0 < x \le \frac{1-\sqrt{1-4\beta}}{2}$, the function $f(x) = \frac{\beta}{1-x}$ attains its maximum when $x = \frac{1-\sqrt{1-4\beta}}{2}$. So, $0 < f(x) \le \frac{1-\sqrt{1-4\beta}}{2}$. Similarly, $f_2(x) = \frac{\beta}{1-f(x)}$ attains its maximum when $f(x) = \frac{1-\sqrt{1-4\beta}}{2}$, so we get $0 < f_2(x) \le \frac{1-\sqrt{1-4\beta}}{2}$. In the same way, we get $0 < f_n(x) \le \frac{1-\sqrt{1-4\beta}}{2}$, for all $n \ge 3$. On the other hand, if $0 < f_{n-1}(x) \le \frac{1-\sqrt{1-4\beta}}{2}$, we can see $\beta - f_{n-1}(x) + (f_{n-1}(x))^2$ attains its minimum when $f_{n-1}(x) = \frac{1-\sqrt{1-4\beta}}{2}$, and thus $\beta - f_{n-1}(x) + (f_{n-1}(x))^2 > 0$. So when $0 < f_n(x) < \frac{1-\sqrt{1-4\beta}}{2}$, we can get that $f_n(x) - f_{n-1}(x) = \frac{\beta}{1-f_{n-1}(x)} - f_{n-1}(x) = \frac{\beta - f_{n-1}(x) + (f_{n-1}(x))^2}{1-f_{n-1}(x)} > 0$ for all $n \ge 2$. So, $f_{n-1}(x) < f_n(x)$ for all $n \ge 2$. Thus, we let $\lim_{n\to\infty} f_n(x) = f_0(x)$, and since $f_n(x) = \frac{\beta}{1-f_{n-1}(x)}$, we get $f_0(x) = \frac{1-\sqrt{1-4\beta}}{2}$. The proof of item (2) is very similar to the proof of item (1), so we omit the proof here. □

Lemma 2.11 *Let the following graph denote* $D_{1,1,m}F^{(3)}_{1,n}$,

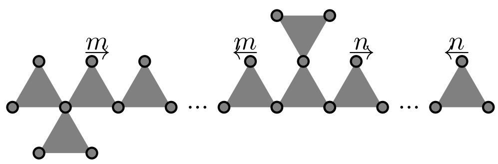


Figure 3 Hypergraphs $D_{1,1,m}F^{(3)}_{1,n}$

and the spectral radius of $D_{1,1,m}F^{(3)}_{1,n}$ *be* $\rho(D_{1,1,m}F^{(3)}_{1,n})$. *Then,*

(1) If $\rho(D_{1,1,m}F^{(3)}_{1,n}) > \rho'_3$, *then* $\rho(D_{1,1,m+1}F^{(3)}_{1,n+1}) > \rho'_3$;

(2) If $\rho(D_{1,1,m}F^{(3)}_{1,n}) < \rho'_3$, *then* $\rho(D_{1,1,m+1}F^{(3)}_{1,n+1}) < \rho'_3$.

Proof First, we prove the fact that if $0 < x \cdot y < \frac{\beta}{1-\beta}$, then $(1 - \frac{\beta}{x})(1 - \frac{\beta}{y}) < \frac{\beta}{1-\beta}$. In fact,

$$(1 - \frac{\beta}{x})(1 - \frac{\beta}{y}) = 1 - \beta \cdot \frac{x+y}{xy} + \frac{\beta^2}{xy} \le (1 - \frac{\beta}{\sqrt{xy}})^2.$$

Since $0 < x \cdot y < \frac{\beta}{1-\beta}$, we can easily check that $-\sqrt{\frac{\beta}{1-\beta}} < 1 - \frac{\beta}{\sqrt{xy}} < 1 - \sqrt{\beta(1-\beta)} = \sqrt{\frac{\beta}{1-\beta}}$. So, we have $(1 - \frac{\beta}{x})(1 - \frac{\beta}{y}) < \frac{\beta}{1-\beta}$.

We label this hypergraph as follows

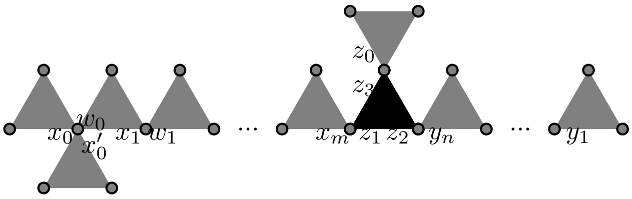


Figure 4 The labelling of hypergraphs $D_{1,1,m}F^{(3)}_{1,n}$

In order to guarantee all the edges except for the black one satisfy Definition 2.3, we set $x_0 = x_0' = y_1 = z_0 = \beta$, $w_0 = 1 - x_0 - x_0' = 1 - 2\beta$, $w_1 = 1 - x_1$, $z_3 = 1 - \beta$, $x_1 = f(2\beta)$, $x_m = f_m(2\beta)$, $y_n = f_{n-1}(\beta)$. $z_1 = 1 - f_m(2\beta)$, $z_2 = 1 - f_{n-1}(\beta)$. Setting $x = z_1 = 1 - f_m(2\beta)$, $y = z_2 = 1 - f_{n-1}(\beta)$, if $xy = (1 - f_m(2\beta))(1 - f_{n-1}(\beta)) < \frac{\beta}{1-\beta}$, then we get $(1 - \frac{\beta}{x})(1 - \frac{\beta}{y}) = (1 - f_{m+1}(2\beta))(1 - f_n(\beta)) < \frac{\beta}{1-\beta}$, that is, if $\rho(D_{1,1,m}F^{(3)}_{1,n}) < \rho_3'$, then $\rho(D_{1,1,m+1}F^{(3)}_{1,n+1}) < \rho_3'$. The proof of item (2) is very similar to the proof of item (1), so we omit the proof here. $\square$

Finally, we show that ρ_r' is the limit value of the spectral radii of $F^{(3)}_{1,n,m}$.

Lemma 2.12 *Let the following graph denote* $F^{(r)}_{1,n,m}$,

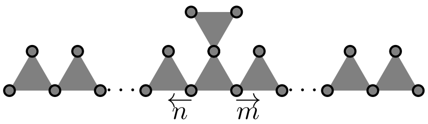


Figure 5 Hypergraphs $F^{(r)}_{1,n,m}$

and the spectral radius of $F^{(r)}_{1,n,m}$ *be* $\rho(F^{(r)}_{1,n,m})$. *Then, when* $n, m \to \infty$, $\lim_{n,m\to\infty} \rho(F^{(r)}_{1,n,m}) = \rho_r' = \beta^{-\frac{1}{r}}$, *where* $\beta = -\frac{1}{6}\cdot(100 + 12\cdot\sqrt{69})^{\frac{1}{3}} - \frac{2}{3\cdot(100+12\cdot\sqrt{69})^{\frac{1}{3}}} + \frac{4}{3} \approx 0.2451223338$.

Proof We label this graph as follows

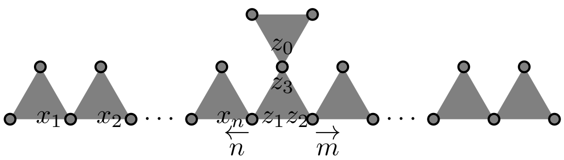


Figure 6 The labelling of hypergraphs $F^{(r)}_{1,n,m}$

Set $x_1 = z_0 = \beta$, $z_3 = 1 - \beta$, $x_2 = f(\beta)$, $x_n = f_{n-1}(\beta)$, $z_1 = 1 - x_n = 1 - f_{n-1}(\beta)$. By the symmetry, we set $z_2 = 1 - f^{m-1}(\beta)$. When $m, n \to \infty$, by the first item of Lemma 2.10, we have $z_1 = z_2 = 1 - f_{n-1}(\beta) = \frac{1+\sqrt{1-4\beta}}{2}$. Set $z_1 \cdot z_2 \cdot z_3 = \beta$, that is $(\frac{1+\sqrt{1-4\beta}}{2})^2 \cdot (1 - \beta) = \beta$. Solving this equality with maple program, we get $\beta = -\frac{1}{6}\cdot(100 + 12\cdot\sqrt{69})^{\frac{1}{3}} - \frac{2}{3\cdot(100+12\cdot\sqrt{69})^{\frac{1}{3}}} + \frac{4}{3} \approx 0.2451223338$. By Lemma 2.4, we get $\lim_{n,m\to\infty} \rho(F^{(r)}_{1,n,m}) = \rho_r' = \beta^{-\frac{1}{r}}$. $\square$

3. 3-uniform hypergraphs

Set $\beta = -\frac{1}{6}\cdot(100 + 12\cdot\sqrt{69})^{\frac{1}{3}} - \frac{2}{3\cdot(100+12\cdot\sqrt{69})^{\frac{1}{3}}} + \frac{4}{3}$, $\rho_3' = 2\beta^{-\frac{1}{3}}$ and $\rho_3 = 2\sqrt[3]{4}$. Then we have the following theorem.

Theorem 3.1 *Let $\rho(H)$ be the spectral radius of a connected 3-uniform hypergraph H. If $\rho_3 < \rho(H) \le \rho_3'$, then H must be one of the following hypergraphs:*

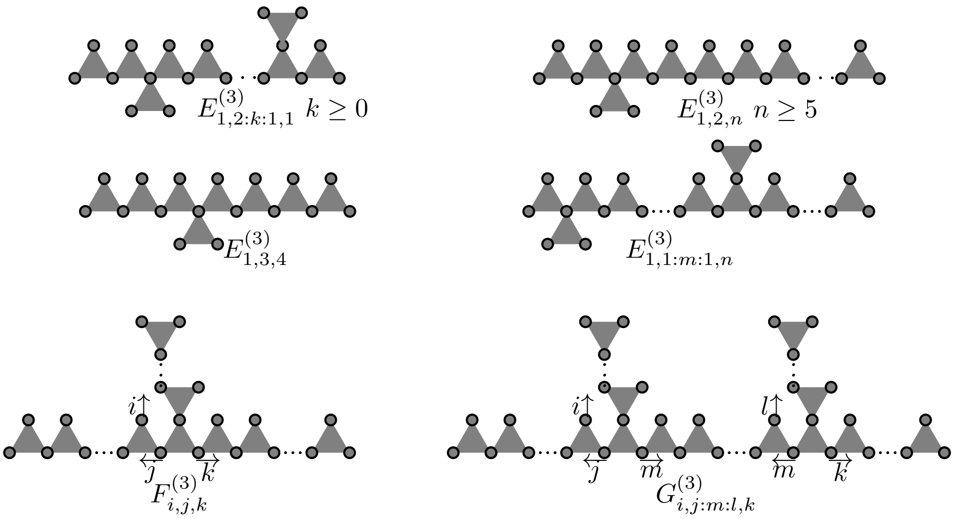


Figure 7 The hypergraphs with spectral radius between ρ_3 and ρ_3'

In $E^{(3)}_{1,1:m:1,n}$, (m,n) can be of the following types:
$(0,3)$, $(1,3)$, $(2,3)$, $(2,4)$, $(3,3)$, $(3,4)$, $(4,n)$ $(3 \le n \le 5)$, $(5,n)$ $(3 \le n \le 6)$, (m,n) $(m \ge 6, 3 \le n \le m)$.

In $F^{(3)}_{i,j,k}$, (i,j,k) can be of the following types:
$(1,j,\infty)$ $(j \ge 1)$, $(2,2,k)$ $(k \ge 8)$, $(2,3,k)$ $(k \ge 5)$, $(2,4,k)$ $(3 \le k \le 6)$, $(3,3,k)$ $(k = 3,4)$.

In $G^{(3)}_{i,j:m:l,k}$, when $i \ne 1$ and $l \ne 1$, (i,j,m,l,k) can have the following choices:
$(2,3,m,1,1)$ $(m \ge 0)$, $(2,2,m,2,2)$ $(0 \le m < 9)$, $(2,2,m,1,3)$ $(0 \le m < 9)$, $(2,2,m,1,2)$ $(m \ge 2)$.

When $i = 1$ and $l = 1$, there is not obvious rule.

Proof We first show that all 3-graphs listed in Theorem 3.1 have spectral radius $\rho(H)$ satisfying $\rho_3 < \rho(H) \le \rho_3'$. We will first show that they are β-normal or β-subnormal.

Suppose that H is a hypergraph with spectral radius $\rho(H)$ satisfying $\rho_3 < \rho(H) \le \rho_3'$, but not on the list.

Case 1 If H contains the following graph $C_2^{(3+)}$ or $C_2^{'(3+)}$

Figure 8 Hypergraphs $C_2^{(3+)}$ and $C_2^{'(3+)}$

we label the above two graphs as follows

Figure 9 The labellings of hypergraphs $C_2^{(3+)}$ and $C_2^{'(3+)}$

In graph $C_2^{(3)+}$, we set $x_1 = \beta$, $x_2 = 1-\beta$, $x_3 = x_6 = \sqrt{\frac{\beta}{1-\beta}}$, $x_4 = x_5 = \sqrt{\beta}$. In graph $C_2^{'(3)+}$, we set $y_1 = \beta$, $y_2 = y_3 = \frac{1-\beta}{2}$, $y_4 = x_5 = \frac{2\beta}{1-\beta}$. We can check that $x_3 + x_4 \approx 1.0649 > 1$ and $y_4 + y_5 \approx 1.2989 > 1$. So, by Lemma 2.8, we get $\rho(C_2^{(3)+}) > \rho_3'$ and $\rho(C_2^{'(3)+}) > \rho_3'$. If H contains $C_2^{(3)+}$ or $C_2^{'(3)+}$, by Lemma 2.1, we get $\rho(H) > \rho_3'$. Therefore, if $\rho(H) \leq \rho_3'$, then in graph H, any two edges intersect at most one vertex. Thus H must be a simple hypergraph.

Case 2 If H has a cycle, and H contains the following graph $C_n^{(3+)}$ or $C_n^{'(3+)}$

Figure 10 Hypergraphs $C_n^{(3)+}$ and $C_n^{'(3)+}$

we label this graph as follows

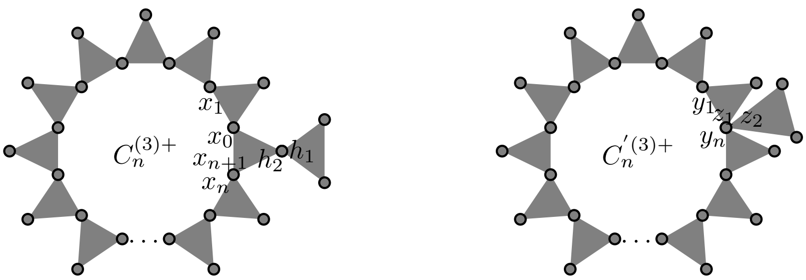


Figure 11 The labellings of $C_n^{(3)+}$ and $C_n^{'(3)+}$

Set $h_1 = x_0 = z_2 = y_1 = \beta$, $h_2 = 1-\beta$. By Lemma 2.10, we set $x_1 = f(\beta)$. By the first item of Lemma 2.10, we have $x_n = f_n(\beta) = \frac{1-\sqrt{1-4\beta}}{2} - \varepsilon$. In the same way, we get $y_n = \frac{1-\sqrt{1-4\beta}}{2} - \varepsilon$. Setting $x_{n+1} = 1 - x_n = \frac{1+\sqrt{1-4\beta}}{2} + \varepsilon$, $z_1 = 1 - y_n - z_2 = \frac{1+\sqrt{1-4\beta}}{2} - \beta + \varepsilon$, we can check $x_0 \cdot h_2 \cdot x_{n+1} < x_0 \cdot h_2 \cdot \frac{1+\sqrt{1-4\beta}}{2} \approx 0.1054 < \beta$, and $y_1 \cdot z_1 \cdot 1 < \beta \cdot (\frac{1+\sqrt{1-4\beta}}{2} - \beta) \approx 0.0796 < \beta$ So, by Lemma 2.8, we get $\rho(C_n^{(3)+}) > \rho_3'$ and $\rho(C_n^{'(3)+}) > \rho_3'$. Therefore, if H contains $C_n^{(3)+}$ or $C_n^{'(3)+}$, by Lemma 2.1, we get $\rho(H) > \rho_3'$. Thus, we can assume that H is a hypertree.

Case 3 If $\exists\, v \in V(H)$, such that $d_v \geq 5$, then H contains $S_5^{(3)}$.

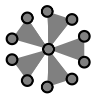


Figure 12 Hypergraph $S_5^{(3)}$

We label this graph as follows

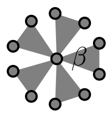


Figure 13 The partial labelling of $S_5^{(3)}$

By the symmetry, we only label one branching. We can check $5\beta \approx 1.225611667 > 1$, so, by Lemmas 2.1 and 2.8, we get $\rho(H) > \rho_3'$. Thus we can assume that every vertex in H has degree at most 4.

Case 4 If $\exists\ v \in V(H)$, such that $d_v = 4$, and H contains the following graph $S_4^{(3)+}$.

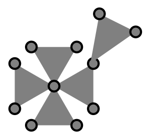


Figure 14 Hypergraph $S_4^{(3)+}$

We label this graph as follows

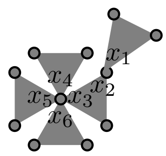


Figure 15 The labelling of $S_4^{(3)+}$

where $x_1 = \beta$, $x_2 = 1 - \beta$, $x_3 = \frac{\beta}{1-\beta}$, $x_4 = x_5 = x_6 = \beta$. We can check that $x_3 + x_4 + x_5 + x_6 \approx 1.0601 > 1$, so, by Lemmas 2.1 and 2.8, we get $\rho(H) > \rho(S_4^{(3)+}) > \rho_3'$. Thus, since $\rho(S_4^{(3)}) = \rho_3$ and $\rho(S_4^{(3)+}) > \rho_3'$, we can assume that every vertex in H has degree at most 3.

Case 5 If there exists at least three vertexes v_i, such that $d_{v_i} = 3, i = 1, 2, 3$, then H contains the following graph $D_{1,1:k:1,2}^{(3)}$ as a subgraph.

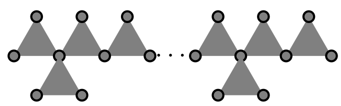


Figure 16 Hypergraphs $D_{1,1:k:1,2}^{(3)}$

We label the graph as follows.

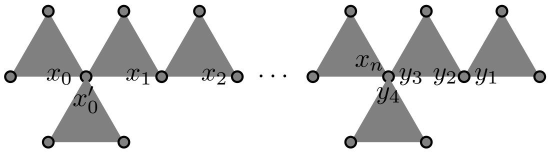


Figure 17 The labelling of $D^{(3)}_{1,1:k:1,2}$

Set $x_0 = x_0' = \beta$. Since $\frac{1-\sqrt{1-4\beta}}{2} < 2\beta < \frac{1+\sqrt{1-4\beta}}{2}$, by the second item of Lemma 2.10, we set $x_1 = f(2\beta)$, and get $x_n = f_n(2\beta) = \frac{1-\sqrt{1-4\beta}}{2} + \varepsilon$. We also set $y_1 = y_4 = \beta$, $y_2 = 1-\beta$, $y_3 = \frac{\beta}{1-\beta}$. We can check that $y_3 + y_4 + x_n = 1.0 + \varepsilon > 1$. So, by Lemmas 2.1 and 2.8, we get $\rho(H) > \rho_3'$.

If there exist at least two vertexes v_i, such that $d_{v_i} = 3, i = 1, 2$, and H contains the graph $D^{(3)}_{1,1:k:1,2}$ or the following graph as a subgraph,

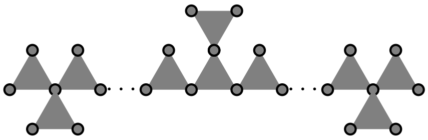


Figure 18 Hypergraphs $D^{(3)}_{1,1:h:1:w:1,1}$

we label the graph as follows.

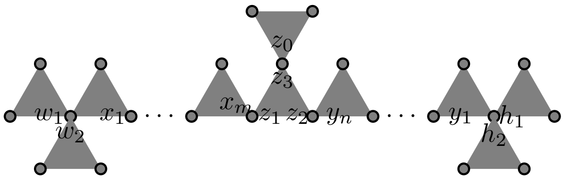


Figure 19 The labelling of $D^{(3)}_{1,1:h:1:w:1,1}$

Let $w_1 = w_2 = z_0 = \beta$, $z_3 = 1 - \beta$. Since $\frac{1-\sqrt{1-4\beta}}{2} < 2\beta < \frac{1+\sqrt{1-4\beta}}{2}$. By the second item of Lemma 2.10, we set $x_1 = f(2\beta)$, and get $x_m = f_m(2\beta) = \frac{1-\sqrt{1-4\beta}}{2} + \varepsilon$. So, $z_1 = 1 - x_m = \frac{1+\sqrt{1-4\beta}}{2} - \varepsilon$. By the symmetry, $z_2 = \frac{1+\sqrt{1-4\beta}}{2} - \varepsilon$. We can check that $z_1 \cdot z_2 \cdot z_3 < (1-\beta) \cdot (\frac{1+\sqrt{1-4\beta}}{2})^2 = \beta$. So, by Lemmas 2.1 and 2.8, we get $\rho(H) > \rho_3'$. Thus, we can assume that there exists at most one vertex v with degree $d_v = 3$.

Case 6 Suppose that v is the unique vertex with degree 3 and all other vertices have degree at most 2. We denote by $E^{(3)}_{i,j,k}$ the 3-uniform hypergraphs obtained by attaching three paths of length i, j, k to the vertex v.

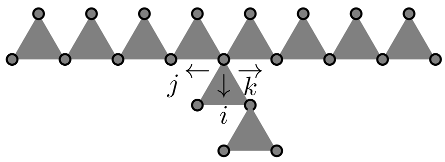


Figure 20 Hypergraphs $E^{(3)}_{i,j,k}$

Consider the three branches attached to v.

(1) Since $\rho(E^{(3)}_{2,2,2}) = \rho_3$, we consider $E^{(3)}_{2,2,3}$.

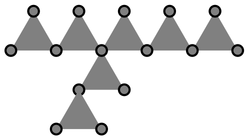


Figure 21 Hypergraphs $E^{(3)}_{2,2,3}$

We label the above graphs as follows

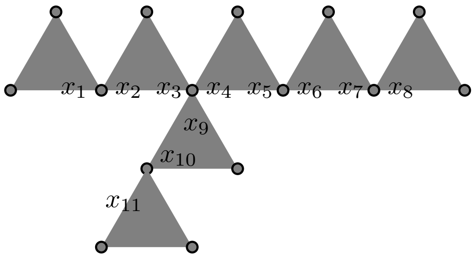


Figure 22 The labelling of $E^{(3)}_{2,2,3}$

Setting $x_1 = x_8 = x_{11} = \beta$, $x_2 = x_7 = x_{10} = 1-\beta$, $x_3 = x_6 = x_9 = \frac{\beta}{1-\beta}$, $x_5 = \frac{1-2\beta}{1-\beta}$, $x_4 = \frac{\beta(1-\beta)}{1-2\beta}$, we can check that $x_3 + x_4 + x_9 \approx 1.0124 > 1$. So, by Lemmas 2.1 and 2.8, if H contains $E^{(3)}_{2,2,3}$, we get $\rho(H) > \rho'_3$. Thus we can assume that the first branch consists of only one edge.

(2) An edge e is called a branching edge if every vertex of e is not a leaf vertex. When $i = 1, j = 2$ and the third branch consists of a branching edge, then H consists of a subgraph $D_{1,2,k}F^{(3)}_{1,1}$ shown below.

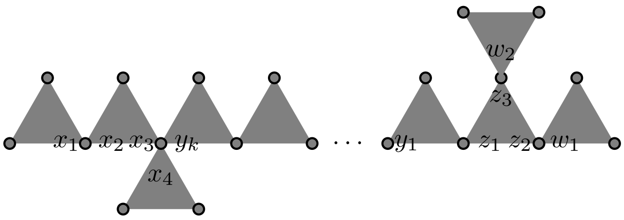


Figure 23 Hypergraphs $D_{1,2,k}F^{(3)}_{1,1}$ and the labellings

Setting $x_1 = x_4 = w_1 = w_2 = \beta$, $x_2 = z_2 = z_3 = 1-\beta$, $x_3 = \frac{\beta}{1-\beta}$, $z_1 = \frac{\beta}{(1-\beta)^2}$, we can prove $\frac{\beta}{(1-\beta)^2} = \frac{1-\sqrt{1-4\beta}}{2}$. Let $y_1 = \frac{\beta}{1-z_1} = \frac{1-\sqrt{1-4\beta}}{2}$. By Lemma 2.10, we get $y_k = \frac{1-\sqrt{1-4\beta}}{2}$. We can check that $x_3 + x_4 + y_k = 1$. So, $D_{1,2,k}F^{(3)}_{1,1}$ has a β-normal labeling, and we have $\rho(D_{1,2,k}F^{(3)}_{1,1}) = \rho'_3$. Therefore, when $i \geq 1, j \geq 2, k \geq 2$ and there is at least one branching edge in graph H, by Lemma 2.1, we get $\rho(H) \geq \rho'_3$, and the equality holds iff H is the same as $D_{1,2,k}F^{(3)}_{1,1}$, $k \geq 0$.

(3) When $i = 1, j = 2, k = n+1$, and there is no branching edge, then, the graph $H = E^{(3)}_{1,2,k}$ is as follows.

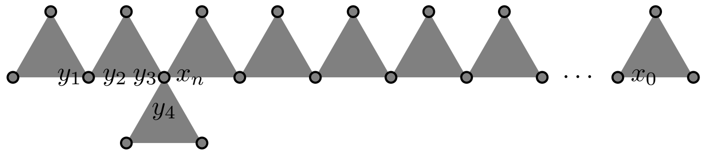


Figure 24 Hypergraphs $E^{(3)}_{1,2,k}$ and the labellings

Setting $x_0=\beta$, $n=k-1$, then when $n\to\infty$, by Lemma 2.10, we get $x_n=f_n(\beta)=\frac{1-\sqrt{1-4\beta}}{2}-\varepsilon$. Set $y_1=y_4=\beta$, $y_2=1-\beta$, $y_3=\frac{\beta}{1-\beta}$. We can check that $y_3+y_4+x_n=1-\varepsilon<1$. So, we get $\rho(H)<\rho_3'$. Since $\rho(E^3_{1,2,5})=\rho_3$, thus, when $n>5$, $\rho_3<\rho(H)<\rho_3'$.

(4) The first branch consists of only one edge, while the second branch consists of three edges, and the third branch consists of k edges. There is no branching edge. When $k=3$, $\rho(H)=\rho_3$. When $k=4$, the graph is as follows.

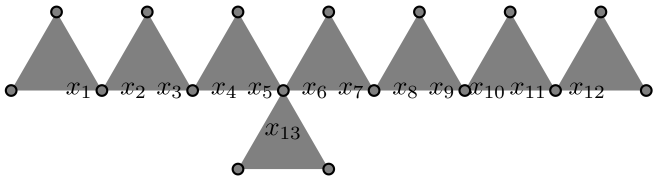


Figure 25 Hypergraphs $E^{(3)}_{1,3,4}$ and the labellings

Setting $x_1=x_{12}=x_{13}=\beta$, $x_2=x_{11}=1-\beta$, $x_3=x_{10}=\frac{\beta}{1-\beta}$, $x_4=x_9=\frac{1-2\beta}{1-\beta}$, $x_5=x_8=\frac{\beta(1-\beta)}{1-2\beta}$, $x_7=\frac{\beta^2-3\beta+1}{1-2\beta}$, $x_6=\frac{\beta(1-2\beta)}{\beta^2-3\beta+1}$, we can check that $x_5+x_6+x_{13}\approx 0.9929<1$. So, $\rho_3<\rho(E^{(3)}_{1,3,4})<\rho_3'$. When $k=5$, the graph is as follows.

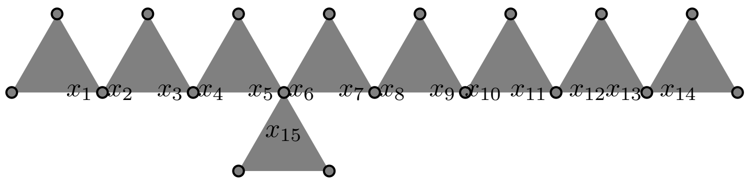


Figure 26 Hypergraphs $E^{(3)}_{1,3,5}$ and the labellings

Setting $x_1=x_{14}=x_{15}=\beta$, $x_2=x_{13}=1-\beta$, $x_3=x_{12}=\frac{\beta}{1-\beta}$, $x_4=x_{11}=\frac{1-2\beta}{1-\beta}$, $x_5=x_{10}=\frac{\beta(1-\beta)}{1-2\beta}$, $x_6=\frac{\beta(\beta^2-3\beta+1)}{3\beta^2-4\beta+1}$, we can check that $x_5+x_6+x_{15}\approx 1.0066>1$. So, $\rho(E^{(3)}_{1,3,5})>\rho_3'$.

(5) The first branch consists of only one edge, while the second and the third branches each consist of at least four edges. There is no branching edge. The graph $E^{(3)}_{1,4,4}$ is as follows.

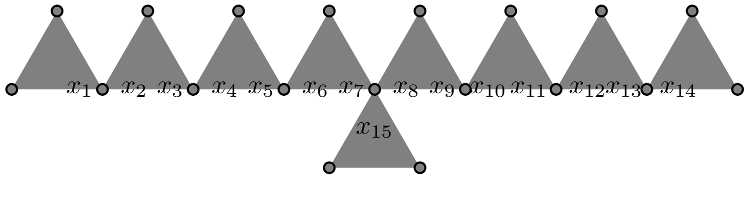


Figure 27 Hypergraphs $E^{(3)}_{1,4,4}$ and the labellings

Setting $x_1 = x_{14} = x_{15} = \beta$, $x_2 = x_{13} = 1-\beta$, $x_3 = x_{12} = \frac{\beta}{1-\beta}$, $x_4 = x_{11} = \frac{1-2\beta}{1-\beta}$, $x_5 = x_{10} = \frac{\beta(1-\beta)}{1-2\beta}$, $x_6 = \frac{\beta(\beta^2-3\beta+1)}{3\beta^2-4\beta+1}$, we can check that $x_7 + x_8 + x_{15} \approx 1.0147 > 1$. So, $\rho(E^{(3)}_{1,4,4}) > \rho'_3$. Thus, if H contains $E^{(3)}_{1,4,4}$ as a proper subgraph, $\rho(H) > \rho(E^{(3)}_{1,4,4}) > \rho'_3$.

(6) The first branch and the second branch each consist of only one edge, while the third branch consists of k edges. When there is no branching edge in the third branch, for any $k \geq 1$, $\rho(H) < \rho_3$. When there are two branching edges in the third branch, and the graph $D_{1,1,m}G_{0,1:n:1,1}$ is as follows.

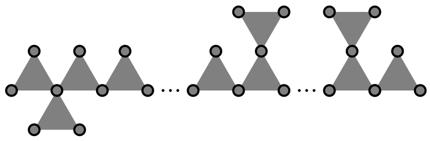


Figure 28 Hypergraphs $D_{1,1,m}G_{0,1:n:1,1}$

We label this graph as follows.

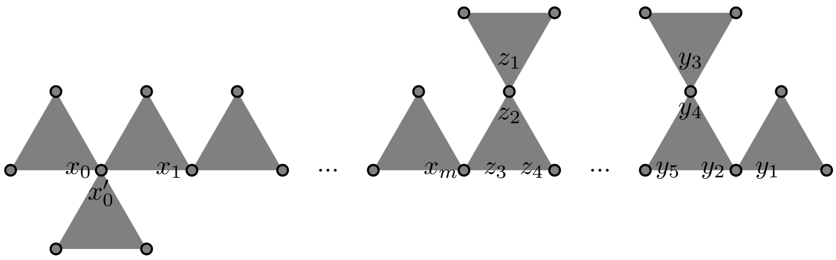


Figure 29 The labelling of $D_{1,1,m}G_{0,1:n:1,1}$

Set $x_0 = x'_0 = y_1 = y_3 = z_1 = \beta$, $z_2 = y_2 = y_4 = 1-\beta$, $x_1 = f(2\beta)$, $x_m = f_m(2\beta)$, $z_3 = 1 - f_m(2\beta)$, $y_5 = \frac{\beta}{(1-\beta)^2}$. Since $\frac{1-\sqrt{1-4\beta}}{2} < 2\beta < \frac{1+\sqrt{1-4\beta}}{2}$, by Lemma 2.10, we get that x_m is decreasing with respect to m. Thus, $z_3 = 1 - x_m$ is increasing with respect to m. So, when $m \to \infty$, we get $x_m = \frac{1-\sqrt{1-4\beta}}{2} + \varepsilon$ and $z_3 = 1 - x_m = \frac{1+\sqrt{1-4\beta}}{2} - \varepsilon$. Since $\frac{\beta}{(1-\beta)^2} = \frac{1-\sqrt{1-4\beta}}{2}$, we get $z_4 = \frac{1-\sqrt{1-4\beta}}{2}$. We can check that $z_2 \cdot z_3 \cdot z_4 < \beta \cdot (1-\beta) < \beta$, so $D_{1,1,m}G_{0,1:n:1,1}$ is β-supernormal and we have $\rho(D_{1,1,m}G_{0,1:n:1,1}) > \rho'_3$. Therefore, if H contains at least two branching edges in the third branch, then H contains graph $D_{1,1,m}G_{0,1:n:1,1}$ as a subgraph, thus $\rho(H) > \rho(D_{1,1,m}G_{0,1:n:1,1}) > \rho'_3$. So, we may assume that there is only one branching edge in the third branch in graph H.

When there is one branching edge in the third branch, and the graph $D_{1,1,m}F^{(3)}_{1,n}$ is as follows.

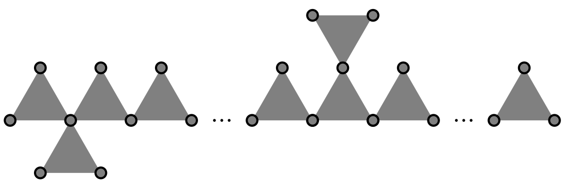


Figure 30 Hypergraphs $D_{1,1,m}F^{(3)}_{1,n}$

We label this graph as follows.

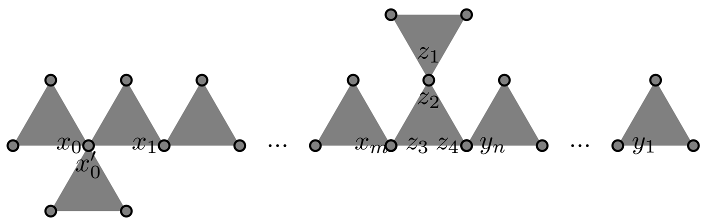


Figure 31 The labelling of $D_{1,1,m}F^{(3)}_{1,n}$

Set $x_0 = x'_0 = y_1 = z_1 = \beta$, $z_2 = 1-\beta$, $x_1 = f(2\beta)$, $x_m = f_m(2\beta)$, $y_n = f_{n-1}(\beta)$. $z_3 = 1 - f_m(2\beta)$, $z_4 = 1 - f_{n-1}(\beta)$. Since $\frac{1-\sqrt{1-4\beta}}{2} < 2\beta < \frac{1+\sqrt{1-4\beta}}{2}$, $0 < \beta < \frac{1-\sqrt{1-4\beta}}{2}$, by Lemma 2.10, we get that x_m is decreasing with respect to m, and y_n is increasing with respect to n. Thus, $z_3 = 1 - x_m$ is increasing with respect to m, and $z_4 = 1 - y_n$ is decreasing with respect to n. Moreover, we have $\rho(\widetilde{BD}^3_n) = \rho(D_{1,1,m}F^{(3)}_{1,2}) = \rho_3$, $\rho(D_{1,1,m}F^{(3)}_{1,1}) < \rho_3$, where $m \geq 0$, so if $\rho_3 < \rho(D_{1,1,m}F^{(3)}_{1,n}) \leq \rho'_3$, we only need consider $n \geq 3$. Let us consider the following cases:

(a) When $m = 0, n = 3$, we set $z_3 = 1 - 2\beta$, $z_4 = 1 - f_2(\beta) = 1 - \frac{\beta(1-\beta)}{1-2\beta}$. We can check that $z_2 \cdot z_3 \cdot z_4 = \beta$, so $\rho(D_{1,1,0}F^{(3)}_{1,3}) = \rho'_3$. When $n > 3$, since $z_4 = 1 - y_n$ is decreasing with respect to n, we have $z_2 \cdot z_3 \cdot z_4 < \beta$, so we get $\rho(D_{1,1,0}F^{(3)}_{1,n}) > \rho'_3$.

(b) When $m = 1$, if $n = 3$, we can check that $z_2 \cdot z_3 \cdot z_4 \approx 0.2496 > \beta$, so $D_{1,1,1}F^{(3)}_{1,3}$ is strictly β-subnormal; if $n = 4$, we can check that $z_2 \cdot z_3 \cdot z_4 \approx 0.2410 < \beta$, so $D_{1,1,1}F^{(3)}_{1,3}$ is strictly β-supernormal. Therefore, when $m = 1, n = 3$, $\rho_3 < \rho(D_{1,1,1}F^{(3)}_{1,3}) < \rho'_3$; when $n \geq 4$, $\rho(D_{1,1,1}F^{(3)}_{1,n}) > \rho'_3$.

(c) When $m = 2, n = 4$, we can check that $z_2 \cdot z_3 \cdot z_4 = \beta$, so $D_{1,1,2}F^{(3)}_{1,4}$ is β-normal and we have $\rho(D_{1,1,2}F^{(3)}_{1,4}) = \rho'_3$. By Lemma 2.1, when $n = 3$, we get $\rho_3 < \rho(D_{1,1,2}F^{(3)}_{1,3}) < \rho'_3$; when $n > 4$, $\rho(D_{1,1,2}F^{(3)}_{1,n}) > \rho'_3$.

(d) When $m = 3$, if $n = 4$, since $D_{1,1,2}F^{(3)}_{1,4}$ is β-normal, by Lemma 2.10, we get $D_{1,1,3}F^{(3)}_{1,4}$ is strictly β-subnormal, so, $\rho_3 < \rho(D_{1,1,3}F^{(3)}_{1,4}) < \rho'_3$; if $n = 5$, we can check that $z_2 \cdot z_3 \cdot z_4 \approx 0.2432$, so $D_{1,1,3}F^{(3)}_{1,5}$ is β-supernormal and we have $\rho(D_{1,1,3}F^{(3)}_{1,5}) > \rho'_3$. So, when $m = 3$, $n = 3, 4$, we get $\rho_3 < \rho(D_{1,1,2}F^{(3)}_{1,n}) < \rho'_3$; when $n \geq 5$, $(D_{1,1,2}F^{(3)}_{1,n}) > \rho'_3$.

(e) When $m = 4$, if $n = 5$, we can check that $z_2 \cdot z_3 \cdot z_4 \approx 0.2462$, so $D_{1,1,4}F^{(3)}_{1,5}$ is β-subnormal and we have $\rho(D_{1,1,4}F^{(3)}_{1,5}) < \rho'_3$; if $n = 6$, we can check that $z_2 \cdot z_3 \cdot z_4 \approx 0.2425$, so $D_{1,1,4}F^{(3)}_{1,5}$ is β-supernormal and we have $\rho(D_{1,1,4}F^{(3)}_{1,6}) > \rho'_3$. So, by Lemmas 2.6 and 2.1, when $m = 4, n = 3, 4, 5$, we get $\rho_3 < \rho(D_{1,1,4}F^{(3)}_{1,n}) < \rho'_3$; when $n \geq 6$, we have $\rho(D_{1,1,4}F^{(3)}_{1,n}) > \rho'_3$.

(f) When $m = 5$, if $n = 6$, we can check that $z_2 \cdot z_3 \cdot z_4 = \beta$, so $D_{1,1,5}F^{(3)}_{1,6}$ is β-normal and we have $\rho(D_{1,1,5}F^{(3)}_{1,6}) = \rho'_3$; if $n = 3, 4, 5$, by Lemma 2.1, $\rho_3 < \rho(D_{1,1,4}F^{(3)}_{1,n}) < \rho'_3$; if $n \geq 7$, by Lemma 2.1, we have $\rho(D_{1,1,5}F^{(3)}_{1,n}) > \rho'_3$.

(g) When $m = 6$, if $n = 6$, since $D_{1,1,5}F^{(3)}_{1,6}$ is β-normal, by Lemma 2.10 we get $D_{1,1,6}F^{(3)}_{1,6}$ is strictly β-subnormal, so, $\rho_3 < \rho(D_{1,1,6}F^{(3)}_{1,6}) < \rho'_3$; if $n = 7$, we can check that $z_2 \cdot z_3 \cdot z_4 \approx 0.2246$, so $D_{1,1,6}F^{(3)}_{1,7}$ is β-supernormal and we have $\rho(D_{1,1,6}F^{(3)}_{1,7}) > \rho'_3$. Therefore, by Lemma 2.1, when $m = 6$, if $3 \leq n \leq 6$, we have $\rho_3 < \rho(D_{1,1,6}F^{(3)}_{1,n}) < \rho'_3$; if $n \geq 7$, we have $\rho(D_{1,1,6}F^{(3)}_{1,n}) > \rho'_3$.

(h) For any $m = n \geq 7$, since $\rho_3 < \rho(D_{1,1,6}F_{1,6}^{(3)}) < \rho_3'$, by Lemma 2.11, we have $\rho_3 < \rho(D_{1,1,n}F_{1,n}^{(3)}) < \rho_3'$. By Lemma 2.1, when $m \geq 7$ and $3 \leq n < m$, we have $\rho_3 < \rho(D_{1,1,n}F_{1,n}^{(3)}) < \rho_3'$. On the other hand, since $\rho(D_{1,1,6}F_{1,7}^{(3)}) > \rho_3'$, by Lemmas 2.11 and 2.1, when $m \geq 7$ and $n > m$, we have $\rho(D_{1,1,m}F_{1,n}^{(3)}) > \rho_3'$.

Case 7 We denote by $F_{i,j,k}^{(3)}$ the 3-uniform hypergraphs obtained by attaching three paths of length i, j, k to each vertex of one edge.

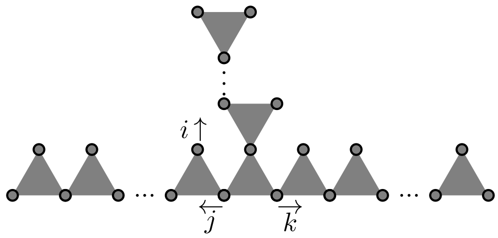


Figure 32 Hypergraphs $F_{i,j,k}^{(3)}$

We denote by $G_{i,j:m:l,k}^{(3)}$ the 3-uniform hypergraphs obtained by attaching four paths of length i, j, l, k to four ending vertices of path of length $m+2$ as shown in the following figure:

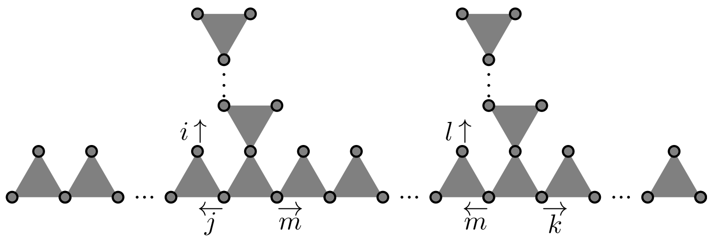


Figure 33 Hypergraphs $G_{i,j:m:l,k}^{(3)}$

Now we can assume that vertices in H have degrees at most 2. We will divide it into the following subcases according to the number of branching edges.

(1) If H has no branching edge, then H is a path, we have $\rho(H) < \rho_3$.

(2) If H has exactly one branching edge, then $H = F_{i,j,k}^{(3)}$. We label this graph as follows.

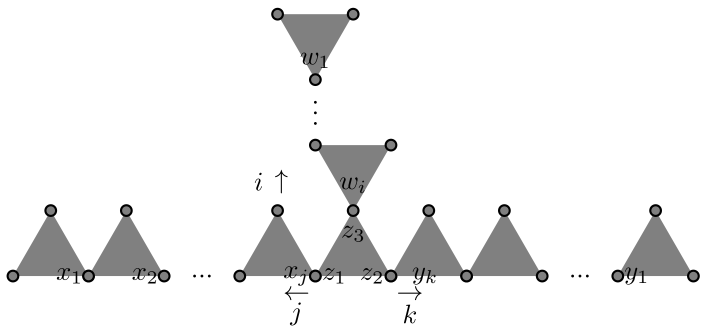


Figure 34 Hypergraphs $F_{i,j,k}^{(3)}$ and the labellings

Set $x_1 = y_1 = w_1 = \beta$, $x_j = f^{j-1}(\beta)$, $y_k = f^{k-1}(\beta)$, $w_i = f^{i-1}(\beta)$, $z_1 = 1 - x_j = 1 - f^{j-1}(\beta)$, $z_2 = 1 - y_k = 1 - f^{k-1}(\beta)$, $z_3 = 1 - w_i = 1 - f^{i-1}(\beta)$. Then, we consider the following cases.
(a) When $i = 1, j, k \to \infty$, by Lemma 2.12, we get $\rho(F_{1,\infty,\infty}) = \rho_3'$. Since $\rho(F_{1,5,6}) = \rho(F_{1,4,8}) = \rho(F_{1,3,14}) = \rho_3$, when $i = 1, j = 5, k > 6$, or $i = 1, j > 5, k \geq 6$,or $i = 1, j = 4, k > 8$, or $i = 1, j > 4, k \geq 8$, or $i = 1, j = 3, k > 14$, or $i = 1, j > 3, k \geq 14$, we have $\rho_3 < \rho(F_{i,j,k}) < \rho_3'$.

(b) When $i = j = 2, k \to \infty$, by Lemma 2.10, we set $y_k = \frac{1-\sqrt{1-4\beta}}{2} - \varepsilon$ and $z_2 = 1 - y_k = \frac{1+\sqrt{1-4\beta}}{2} + \varepsilon$. We can check that $z_1 \cdot z_2 \cdot z_3 > z_1 \cdot z_3 \cdot \frac{1+\sqrt{1-4\beta}}{2} \approx 0.2599 > \beta$, so, we get $\rho(F_{2,2,\infty}) < \rho_3'$. Thus, since $\rho(F_{2,2,7}) = \rho_3$, we get that $\rho_3 < \rho(F_{2,2,k}) < \rho_3'$ for any $k \geq 8$.

When $i = 2, j = 3$, we consider $\rho(F_{2,3,k})$. When $k \to \infty$, we set $z_2 = 1 - y_k == \frac{1+\sqrt{1-4\beta}}{2} + \varepsilon$, and we can check that $z_1 \cdot z_2 \cdot z_3 > z_1 \cdot z_3 \cdot \frac{1+\sqrt{1-4\beta}}{2} = \beta$. Since $\rho(F_{2,3,4}) = \rho_3$, we get $\rho_3 < \rho(F_{2,3,k}) < \rho_3'$ for any $k \geq 5$.

When $i = 2, j = 4$, if $k = 6$, we can check that $z_1 \cdot z_2 \cdot z_3 \approx 0.2462 > \beta$; if $k = 7$, we can check that $z_1 \cdot z_2 \cdot z_3 \approx 0.2436 < \beta$. Since $\rho(F_{2,3,4}) = \rho_3$, we get $\rho_3 < \rho(F_{2,4,k}) < \rho_3'$ for any $4 \leq k \leq 6$.

When $i = 2, j = 5$, if $k = 5$, we can check that $z_1 \cdot z_2 \cdot z_3 \approx 0.2444 < \beta$. So, we get $\rho(F_{2,5,5}) > \rho_3'$. Thus, when $i = 2, k \geq j \geq 5$, by Lemma 2.1, we have $\rho(F_{2,j,k}) > \rho_3'$.

(c) When $i = 3$, if $j = 3$, $k = 4$, we can check that $z_1 \cdot z_2 \cdot z_3 \approx 0.2496 > \beta$; if $j = 3$,$k = 5$, we can check that $z_1 \cdot z_2 \cdot z_3 \approx 0.2441 < \beta$. Since $\rho(F_{3,3,3}) > \rho_3$, we get $\rho_3 < \rho(F_{3,3,k}) < \rho_3'$ for any $k = 3, 4$, and $\rho(F_{3,3,k}) > \rho_3'$ for any $k \geq 5$.

When $i = 3$, if $j = 4$, $k = 4$, we can check that $z_1 \cdot z_2 \cdot z_3 \approx 0.2411 < \beta$. So, when $i = 3$, if $k \geq j \geq 4$, we have $\rho(F_{3,j,k}) > \rho_3'$.

(d) When $i = 4$, if $j = 4$, $k = 4$, we can check that $z_1 \cdot z_2 \cdot z_3 \approx 0.2328 < \beta$. So, when $k \geq j \geq i \geq 4$, we have $\rho(F_{i,j,k}) > \rho_3'$.

(3) If H has exactly two branching edges, then $H = G^{(3)}_{i,j:m:l,k}$ $(i \leq j, l \leq k)$. We label this graph as follows.

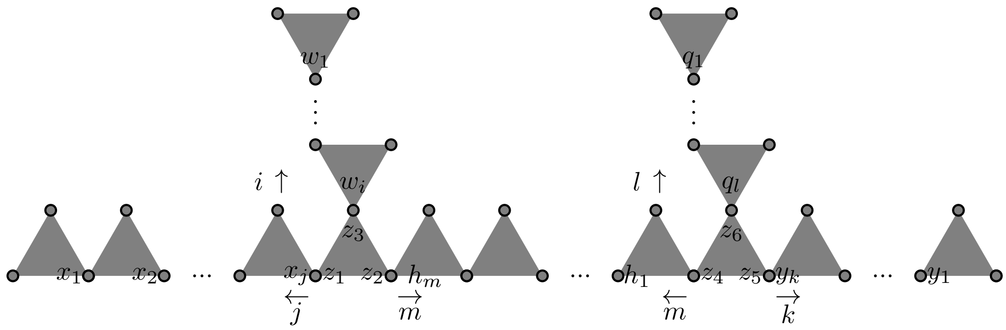


Figure 35 Hypergraphs $G^{(3)}_{i,j:m:l,k}$ and the labellings

We set $x_1 = y_1 = w_1 = q_1 = \beta$, $x_j = f^{j-1}(\beta)$, $y_k = f^{k-1}(\beta)$, $w_i = f^{i-1}(\beta)$, $q_l = f^{l-1}(\beta)$, $z_1 = 1 - x_j = 1 - f^{j-1}(\beta)$, $h_1 = \frac{\beta}{1-z_4}$, $h_s = \frac{\beta}{1-h_{s-1}}$ for any $2 \leq s \leq m$, $z_2 = 1 - h_m$, $z_3 = 1 - w_i = 1 - f^{i-1}(\beta)$, $z_5 = 1 - y_k = 1 - f^{k-1}(\beta)$, $z_6 = 1 - q_l = 1 - f^{l-1}(\beta)$, $z_4 = \frac{\beta}{z_5 \cdot z_6}$.

Firstly, we assume that $i \neq 1$ and $l \neq 1$ at the same time and consider the following cases.

(a) When $i+j=5$ and $l+k=3$, that is $i=2$, $j=3$, $l=1$, $k=2$. We can compute that $z_4=\frac{\beta}{1-2\beta}=f(2\beta)$. So, we set $h_1=f_2(2\beta)$, $h_m=f_{m+1}(2\beta)$. Since $\frac{1-\sqrt{1-4\beta}}{2}<2\beta<\frac{1+\sqrt{1-4\beta}}{2}$, by Lemma 2.10, we get that h_m is decreasing with respect to m, and $z_2=1-h_m$ is increasing with respect to m. So, when $m\to\infty$, we get $h_m=\frac{1-\sqrt{1-4\beta}}{2}+\varepsilon$ and $z_2=1-h_m=\frac{1+\sqrt{1-4\beta}}{2}-\varepsilon$. We can check that $z_1\cdot z_2\cdot z_3<z_1\cdot z_3\cdot\frac{1+\sqrt{1-4\beta}}{2}=\beta$. So, for any $m\geq 0$, we get $\rho(G_{2,3:m:1,2})>\rho_3'$. Thus, for any $m\geq 0, i+j\geq 5$, and $l+k\geq 3$, we have $\rho(G_{i,j:m:l,k})>\rho_3'$.

(b) When $i+j=5$ and $l+k=2$, that is $i=2$, $j=3$, $l=1$, $k=1$. We can compute that $z_4=\frac{\beta}{(1-\beta)^2}$. Since $\frac{\beta}{(1-\beta)^2}=\frac{1-\sqrt{1-4\beta}}{2}$, we get $h_m=\frac{1-\sqrt{1-4\beta}}{2}$. So, $z_2=\frac{1+\sqrt{1-4\beta}}{2}$. We can check that $z_1\cdot z_2\cdot z_3=\beta$. Thus, for any $m\geq 0$, we have $\rho(G_{2,3:m:1,1})=\rho_3'$.

(c) When $i+j=4$ and $l+k=4$, if $i=2$, $j=2$, $l=2$, $k=2$, we get $z_4=\frac{\beta(1-\beta)^2}{(1-2\beta)^2}$ and we can compute that $\frac{1-\sqrt{1-4\beta}}{2}\approx 0.4302<z_4\approx 0.5375<\frac{1+\sqrt{1-4\beta}}{2}\approx 0.5698$. By Lemma 2.10, we get h_m is decreasing with respect to m. We can check that when $m=8$, $z_1\cdot z_2\cdot z_3\approx 0.2433<\beta$, while when $m=9$, $z_1\cdot z_2\cdot z_3\approx 0.2465>\beta$. So, for any $m\geq 9$, we get $\rho(G_{2,2:m:2,2})<\rho_3'$, and for any $0\leq m<9$, $\rho(G_{2,2:m:2,2})>\rho_3'$.

If $i=2$, $j=2$, $l=1$, $k=3$, since $z_4=0.5098$, we can get the same results as $\rho(G_{2,2:m:2,2})$, that is, for any $m\geq 9$, we get $\rho(G_{2,2:m:1,3})<\rho_3'$, and for any $0\leq m<9$, $\rho(G_{2,2:m:1,3})>\rho_3'$.

(d) When $i+j=4$ and $l+k=3$, that is $i=2$, $j=2$, $l=1$, $k=2$, since $z_4=\frac{\beta}{1-2\beta}=f(2\beta)$, we get $h_m=f_{m+1}(2\beta)$. By Lemma 2.10, we get that h_m is decreasing with respect to m, and $z_2=1-h_m$ is increasing with respect to m. So, we can check that when $m=1$ $z_1\cdot z_2\cdot z_3\approx 0.2442<\beta$, while when $m=2$ $z_1\cdot z_2\cdot z_3\approx 0.2473>\beta$. So, for any $m\geq 2$, we get $\rho(G_{2,2:m:1,2})<\rho_3'$, and for any $0\leq m<2$, $\rho(G_{2,2:m:1,2})>\rho_3'$.

(e) When $i+j=4$ and $l+k=2$, that is $i=2$, $j=2$, $l=1$, $k=1$, since $\rho(G_{2,3:m:1,1})=\rho_3'$, by Lemma 2.1, we get $\rho(G_{2,2:m:1,1})<\rho_3'$.

When $i+j\leq 3$ and $l+k\leq 3$, that is a special case of the following cases.

Now, we consider $i=l=1$. When $j=1$, we set $x_1=y_1=w_1=q_1=\beta$,$z_1=z_3=z_6=1-\beta$, $z_2=\frac{\beta}{(1-\beta)^2}$. Since $\frac{\beta}{(1-\beta)^2}=\frac{1-\sqrt{1-4\beta}}{2}$, we get $z_4=\frac{1+\sqrt{1-4\beta}}{2}$. When $k\to\infty$, $z_5=1-y_k=1-f^{k-1}(\beta)=\frac{1+\sqrt{1-4\beta}}{2}+\varepsilon$. We can check that $z_4\cdot z_5\cdot z_6<(1-\beta)(\frac{1+\sqrt{1-4\beta}}{2})^2=\beta$. Therefore, for any $m\geq 0, k\geq 1$, we have $\rho(G_{1,1:m:1,k})<\rho_3'$. Since $\rho(G_{1,1:0:1,4})=\rho_3$ and $\rho(G_{1,1:6:1,3})=\rho_3$, so, if $\rho_3<\rho(G_{1,1:m:1,k})<\rho_3'$, we have $m\geq 0, k\geq 5$, or $m>0, k=4$, or $m\geq 7, k\geq 3$. When $j=2$, since $z_2=\frac{\beta}{1-2\beta}=f(2\beta)$, we have the same results as in the sixth item of Case 6 when there is one branching edge in the third branch.

	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
3	2	4	6	7	9	10	11	12	13	14	15	16	17	18	19	20	20	21	22	22	23	23	24
4		6	8	9	11	12	13	14	15	16	17	18	19	20	21	22	22	23	24	24	25	25	26
5			10	11	12	14	15	16	17	18	19	20	21	22	22	23	24	25	25	26	27	27	27
6				12	14	15	16	17	18	19	20	21	22	23	24	25	25	26	27	27	28	28	29

Table 1 The values of j,k and m

When $j\geq 3$, if $\rho(G_{1,j:m:1,k})<\rho_3'$, we have the following table. In the first column of the table we set $j = 3, 4, 5, 6$, and in the first row we set $k = 3, 4, \ldots, 25$. In the table, the left values denote the minimum that m can get corresponding to the values that j and k get.

But from Table 1, we cannot get obvious rules that j, k and m can follow.

(4) H contains at least three branching edges. Since all degrees of vertices are at most 2, any branching edges lie in a path. We consider the following graph $G_{1,1:m:1:n:1,1}$

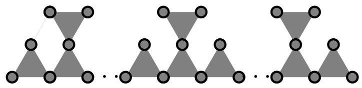


Figure 36 Hypergraphs $G_{1,1:m:1:n:1,1}$

We label this graph as follows.

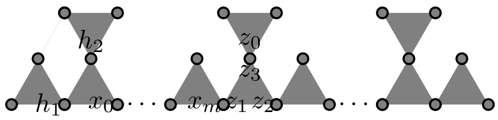


Figure 37 The labelling of $G_{1,1:m:1:n:1,1}$

Set $h_1 = h_2 = z_0 = \beta$, $x_0 = \frac{\beta}{(1-\beta)^2}$, $z_3 = 1 - \beta$. Since $\frac{\beta}{(1-\beta)^2} = \frac{1-\sqrt{1-4\beta}}{2}$, we get $x_m = \frac{1-\sqrt{1-4\beta}}{2}$. So, $z_1 = 1 - x_m = \frac{1+\sqrt{1-4\beta}}{2}$. By the symmetry, we set $z_2 = \frac{1+\sqrt{1-4\beta}}{2}$. We can check that $z_1 \cdot z_2 \cdot z_3 = \beta$. Thus, for any $m \geq 0, n \geq 0$, $\rho(G_{1,1:m:1:n:1,1}) = \rho'_3$. If H contains $G_{1,1:m:1:n:1,1}$ as a subgraph, then $\rho(H) > \rho(G_{1,1:m:1:n:1,1}) = \rho'_3$.

Therefore, all hypergraphs with spectral radius $\rho(H)$ satisfying $\rho_3 < \rho(H) \leq \rho'_3$ are in the list of Theorem 3.1.

4. General k-uniform hypergraphs

For any integer $r \geq 2$, let $\rho'_r = \beta^{-\frac{1}{r}}$. In this section, we will classify all r-uniform connected hypergraphs with spectral radius at most ρ'_r for all $r \geq 4$.

A hypergraph $H = (V, E)$ is called *reducible* if every edge e contains at least one leaf vertex v_e. In this case, we can define an $(r-1)$-uniform multi-hypergraph $H' = (V', E')$ by removing v_e from each edge e, i.e., $V' = V \setminus \{v_e \colon e \in E\}$ and $E' = \{e - v_e \colon e \in E\}$. We say that H' is reduced from H while H extends H'.

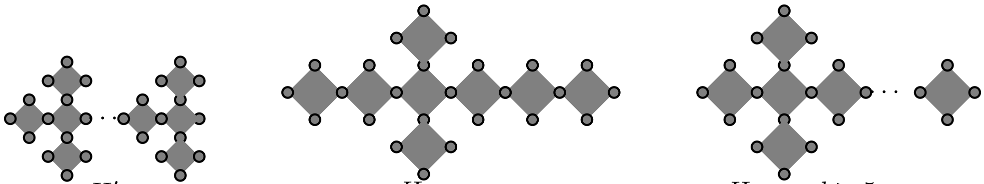
From [2] we have the following lemma and corollary.

Lemma 4.1 *If H extends H', then H is consistently β-normal if and only of H' is consistently β-normal for the same value of β.*

Corollary 4.2 *If H extends H', then $\rho(H) = \rho'_r$ if and only if $\rho(H') = \rho'_{r-1}$, and $\rho(H') < \rho'_{r-1}$ if and only if $\rho(H) < \rho'_r$.*

We will use a similar notion for those special r-uniform hypergraphs with spectral radius at most ρ'_r. We can extend the graphs in Theorem 3.1 by Corollary 4.2. Are there any new hypergraphs not extended from the list of Theorem 3.1

Theorem 4.3 *For $r \geq 5$, every r-uniform hypergraph with spectral radius at most ρ'_r is reducible. For $r = 4$, irreducible hypergraphs with spectral radius at most ρ'_r are the following hypergraphs.*



H'_2 $\quad H_{1,1,2,3}$ $\quad H_{1,1,1,l}, l \geq 5$

Figure 38 Irreducible 4-uniform hypergraphs

Proof Let H be an r-uniform hypergraph with $\rho_r < \rho(H) \leq \rho'_r$.

(1) If H is not simple, then H contains a subgraph that consists of two edges intersecting on $s \geq 2$ vertices. Call this subgraph $G_s^{(r)}$. Define a weighted incident matrix B of $G_s^{(r)}$ as follows: for any vertex v and edge e (called the other edge e'),

$$B(v, e) = \begin{cases} \dfrac{1}{2}, & \text{if } v \in e \cap e', \\ 1, & \text{if } v \in e \setminus e', \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that when $s \geq 3$ we have $(\frac{1}{2})^s < \beta$, so $G_s^{(r)}$ is consistently β-supernormal and $\rho(H) > \rho'_r$. When $s = 2$, $G_s^{(r)}$ is reducible to $C_n^{(3)+}$. So, if H contains $G_s^{(r)}$, $\rho(H) > \rho'_r$.

(2) Now we assume that H is simple. If H is not a simple hypertree, then H contains a cycle. Let $C_l = v_0 e_1 v_1 e_1 \cdots v_{l-1} e_l v_0$ be a cycle of the minimum length in H. Observe that any vertex in e_i other than v_{i-1} and v_i must be a leaf vertex in C_l. This cycle must be equal to $C_l^{(r)+}$, which is β-supernormal. We have $\rho(H) > \rho(C_l^{(r)+}) > \rho'_r$.

(3) Finally, we assume that H is a simple hypertree. Now assume that H is irreducible. Following the proof in [2], we take an edge, saying $F_0 = \{v_1, v_2, \ldots, v_r\}$ so that each vertex v_i is in another edge F_i, for $i = 1, 2, \ldots, r$. The subgraph consisting of edges $F_0, F_1, \ldots, F_r$ is called an edge-star, denoted by $S_r^{(r)}$. Now we define $B(v_i, F_i) = \frac{1}{4}$, $B(v_i, F_0) = \frac{3}{4}$, and $B(v, F_i) = 1$ for each vertex $v \neq v_i$ in F_i. Note $\prod_{i=1}^r B(e_i, F_0) = (\frac{3}{4})^r \leq 0.2373 < \beta$ if $r \geq 5$. Thus $S_r^{(r)}$ is β-supernormal for $r \geq 5$. We have $\rho(H) \geq \rho(S_r^{(r)}) > \rho'_r$. Contradiction. Thus, every r-uniform hypergraph for $r \geq 5$ with spectral radius at most ρ'_r is reducible.

H'_1 $\quad H'_2$

Figure 39 Hypergraphs H'_1 and H'_2

(4) It remains to consider the case $r = 4$. Firstly, if there is a branch containing either a branching vertex or a branching edge, then H contains one of the following subgraphs H_1' and H_2'.

We label H_1' and H_2' as follows.

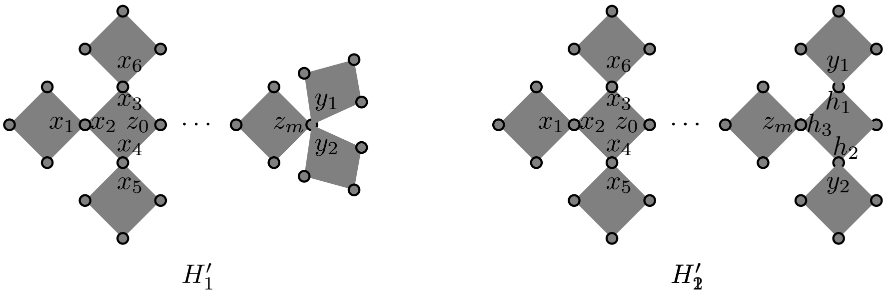


Figure 40 The labellings of H_1' and H_2'

In both H_1' and H_2', setting $x_1 = x_5 = x_6 = y_1 = y_2 = \beta$, $x_2 = x_3 = x_4 = h_1 = h_2 = 1-\beta$, $h_3 = \frac{\beta}{(1-\beta)^2} = \frac{1-\sqrt{1-4\beta}}{2}$ $z_0 = \frac{\beta}{(1-\beta)^3}$, we can check $\frac{\beta}{(1-\beta)^3} = \frac{1+\sqrt{1-4\beta}}{2}$. By Lemma 2.10, we get $z_m = \frac{1+\sqrt{1-4\beta}}{2}$. In H_1', we can check that $y_1 + y_2 + z_m \approx 1.0601 > 1$, while in H_2', $h_3 + z_m = \frac{1-\sqrt{1-4\beta}}{2} + \frac{1+\sqrt{1-4\beta}}{2} = 1$. Thus, for any $m \geq 0$, $\rho(H_1') > \rho_r'$, $\rho(H_2') = \rho_r'$. If H contains H_1' or H_2' as a proper subgraph, then $\rho(H) > \rho_r'$.

Now, we consider that all four branches of F_0 are paths. We denote H by $H^{(4)}_{i,j,k,l}$, where i, j, k, and l ($i \leq j \leq k \leq l$) are the length of the four paths. We will first show that $H_{1,2,2,2}$ is strictly β-supernormal. We label this graph as follows.

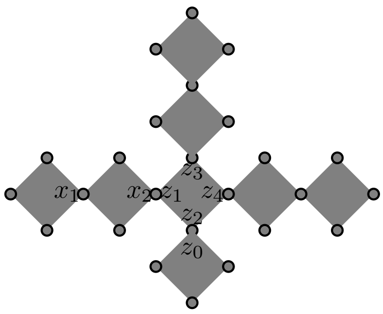


Figure 41 Hypergraphs $H_{1,2,2,2}$ and the labellings

Set $x_1 = z_0 = \beta$, $x_2 = f(\beta)$, $z_1 = \frac{1-2\beta}{1-\beta}$, by the symmetry, $z_2 = z_3 = z_4 = \frac{1-2\beta}{1-\beta}$. We can check that $z_1 \cdot z_2 \cdot z_3 \cdot z_4 =\approx 0.2325 < \beta$. So, $\rho(H_{1,2,2,2}) > \rho_r'$.

When $i = j = 1$, $k = 2$, let the following graph denote $H_{1,1,2,l}$.

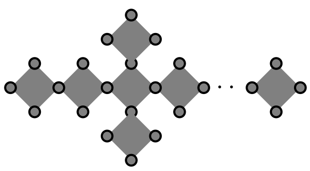


Figure 42 Hypergraphs $H_{1,1,2,l}$

We label this graph as follows.

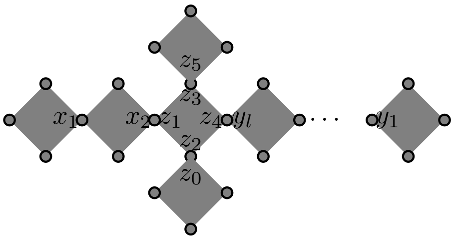


Figure 43 The labelling of $H_{1,1,2,l}$

Setting $x_1 = y_1 = z_0 = z_5 = \beta$, $x_2 = f(\beta)$, $z_1 = \frac{1-2\beta}{1-\beta}$, $z_2 = z_3 = 1-\beta$, $y_l = f^{l-1}(\beta)$, $z_4 = 1 - y_l$. When $l = 3$, we can check that $z_1 \cdot z_2 \cdot z_3 \cdot z_4 = \beta$. So, $\rho(H_{1,1,2,3}) = \rho'_3$. When $l = 4$, we can check that $z_1 \cdot z_2 \cdot z_3 \cdot z_4 \approx 0.2367 < \beta$. So, $\rho(H_{1,1,2,4}) > \rho'_r$. Since $\rho(H_{1,1,2,2}) = \rho_r$, we have that only when $l = 3$, $\rho_r < \rho(H_{1,1,2,3}) = \rho'_r$.

When $i = j = k = 1$, let the following graph denote $H_{1,1,1,l}$,

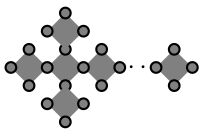


Figure 44 Hypergraphs $H_{1,1,2,l}$

We label this graph as follows.

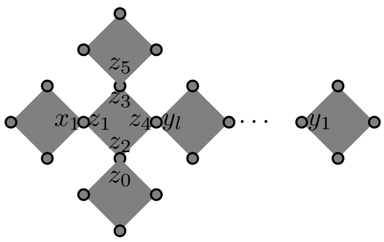


Figure 45 The labelling of $H_{1,1,2,l}$

Set $x_1 = y_1 = z_0 = z_5 = \beta$, $z_1 = z_2 = z_3 = 1 - \beta$, $y_l = f^{l-1}(\beta)$,$z_4 = 1 - y_l$. When $l \to \infty$, by Lemma 2.12, we set $y_l = \frac{1-\sqrt{1-4\beta}}{2} - \varepsilon$ and $z_4 = 1 - y_l = \frac{1+\sqrt{1-4\beta}}{2} + \varepsilon$. We can check that $z_1 \cdot z_2 \cdot z_3 \cdot z_4 > z_1 \cdot z_2 \cdot z_3 \cdot \frac{1+\sqrt{1-4\beta}}{2} = \beta$. Since $\rho(H_{1,1,1,4}) < \rho_4$ and $\rho(H_{1,1,1,5}) > \rho_4$, we have that for any $l \geq 5$, $\rho_r < \rho(H_{1,1,1,l}) < \rho'_r$.

Therefore, all irreducible hypergraphs with spectral radius between at most ρ'_r are classified in the list of Theorem 4.3. □

From Corollary 4.2, Theorems 4.3 and 3.1, we have the following theorem.

Theorem 4.4 *Let $r \geq 4$, $\rho_r = \sqrt[r]{4}$ and $\rho'_r = \beta^{-1/r}$. If the spectral radius of a connected r-uniform hypergraph H is in (ρ_r, ρ'_r), then H must be one of the following hypergraphs:*

(1) r-uniform hypergraphs obtained by extending the hypergraphs on the list of Theorem 3.1 by $r-3$ times.

(2) r-uniform hypergraphs obtained by extending the hypergraphs on the list of Theorem 4.3 by $r-4$ times.

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