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Nullity of Hermitian-Adjacency Matrices of Mixed Graphs

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Abstract A mixed graph means a graph containing both oriented edges and undirected edges. The nullity of the Hermitian-adjacency matrix of a mixed graph G, denoted by $\eta_H(G)$, is referred to as the multiplicity of the eigenvalue zero. In this paper, for a mixed unicyclic graph G with given order and matching number, we give a formula on $\eta_H(G)$, which combines the cases of undirected and oriented unicyclic graphs and also corrects an error in Theorem 4.2 of [Xueliang LI, Guihai YU. The skew-rank of oriented graphs. Sci. Sin. Math., 2015, 45: 93-104 (in Chinese)]. In addition, we characterize all the n-vertex mixed graphs with nullity $n-3$, which are determined by the spectrum of their Hermitian-adjacency matrices.

Keywords nullity; mixed graph; unicyclic graph; Hermitian-adjacency matrix

MR(2010) Subject Classification 05C50

1. Introduction

The nullity of an undirected simple graph is defined as the multiplicity of the eigenvalue zero of the adjacency matrix, which has received intensive study for recent years and originates from the problem of characterizing all the singular or nonsingular graphs, proposed by Collatz and Sinogowitz [1]. In particular, if G is bipartite and the nullity of G is greater than zero, then as described in [2] the alternant hydrocarbon, corresponding to G, is unstable. Moreover, this topic is also attracting in mathematics for it is in close relation to the problem of minimum rank of symmetric matrices with patterns described by graphs [3].

Due to the important application in chemistry and independent interest in mathematics, many papers focusing on this theme have been published. The n-vertex graphs with nullity $n-2$ or $n-3$ were characterized by Cheng and Liu [4]. The nullity of some special graph classes, such as trees, bipartite graphs, unicyclic graphs, bicyclic graphs, graphs with pendant trees, were investigated in [5–13]. The problem of determining the graphs with given rank is equivalent to that of the nullity, since for the rank $r(G)$ and the nullity $\eta(G)$ of an n-vertex undirected graph G, $r(G)+\eta(G)=n$. In this case, Cheng et al. obtained the graphs with rank 4 and 5 in [14] and [15], respectively. Furthermore, Fan et al. [16] generalized the nullity of undirected graphs to signed graphs.

The adjacency matrix and Laplacian matrix of a mixed graph have been studied intensively [17–21]. Let G be a mixed graph of order n with vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$. Denote

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by $A(G) = (a_{st})$ the adjacency matrix of G, where $a_{st} = 1$ if the vertices v_s and v_t are joined by an undirected edge, $a_{st} = a_{ts} = -1$ if there is an oriented edge between v_s and v_t and $a_{st} = 0$ otherwise. Bapat, Grossman and Kulkarni [17] introduced the Laplacian matrix $L(G) = D(G) + A(G)$ for a mixed graph G and gave a generalization of the Matrix Tree Theorem for mixed graphs. In [22], the authors introduced the adjacency matrices of the (edge) weighted directed graphs, which generalize the adjacency (resp., Laplacian) matrices of mixed graphs, and so do the adjacency matrices of 3-colored digraphs [22–24]. Let X be a weighted directed graph with simple underlying graph and $V(X) = \{v_1, v_2, \ldots, v_n\}$. Then the adjacency matrix $A(X) = (a_{ij})$ of X is defined by $a_{ij} = w_{ij}$ if there is an oriented edge from v_i to v_j, $a_{ij} = \overline{w_{ji}}$ if there is an oriented edge from v_j to v_i and $a_{ij} = 0$ otherwise, where w_{ij}, a complex number of unit modulus with nonnegative imaginary part, is the weight of the corresponding edge and $\overline{w_{ji}}$ is the complex conjugate of w_{ji}.

For a weighted directed graph X, if we let the weights of edges belong to $\{1,\ i\}$, then the adjacency matrix $A(X)$ may be identified with the Hermitian-adjacency matrix for a mixed graph, named by Liu and Li [25]. Notice that the Hermitian-adjacency matrices for mixed graphs may be considered as a kind of the adjacency matrices for weighted directed graphs (3-colored digraphs), which can be seen more clearly from the following definition. Let G be a mixed graph of order n with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The Hermitian-adjacency matrix of G is written as $H(G) = (h_{st})$ and

$$h_{st} = \begin{cases} i, & \text{if } v_s \to v_t; \\ -i, & \text{if } v_t \to v_s; \\ 1, & \text{if } v_s \sim v_t; \\ 0, & \text{otherwise,} \end{cases}$$

where i is the imaginary number unit and $v_s \to v_t$ means there is an oriented edge from v_s to v_t and $v_s \sim v_t$ means there is an undirected edge between v_s and v_t. Motivated by the conclusion of [16] and the importance of the nullity, we here extend the nullity of the adjacency matrix of undirected graphs to the Hermitian-adjacency matrix of mixed graphs. For convenience, the nullity of a mixed graph is always referred to as the nullity of the Hermitian-adjacency matrix for the mixed graph.

The paper is arranged as follows. Prior to showing our main results, in Section 2 we list some elementary notations and useful lemmas. Besides, we characterize the nullity of the mixed cycles. In Section 3, the nullity of the mixed unicyclic graphs is formulated, which corrects an error in a result of the oriented unicyclic graphs [26, Theorem 4.2]. In addition, the mixed graphs with nullity $n-3$ are characterized definitely, from which we can see that they are determined by the spectrum of their Hermitian adjacency matrices under the switching equivalence.

2. Preliminaries

Throughout this paper, only simple graphs are considered. A graph G is called mixed, if part of the edges in G are given some orientations, i.e., G contains both oriented edges and undirected edges. Clearly, the oriented graphs and the undirected graphs are two extreme cases of the mixed graphs. The undirected simple graph corresponding to a mixed graph G is called the underlying graph of G and denoted by G_U. A mixed graph is said to be a mixed unicyclic graph if its underlying graph is a unicyclic graph. For a mixed graph G, if the edge e_{uv} is oriented from u to v, then we write $u \to v$; if e_{uv} is undirected, then we write $u \sim v$.

From the definition of the Hermitian-adjacency matrix for a mixed graph G, it is obvious that $H(G)$ is a Hermitian matrix, so its eigenvalues are real. Suppose C is a mixed cycle with $V(C) = \{v_1, v_2, \dots, v_k\}$ labelled clockwise in C. If the value of $h_{12}h_{23}\dots h_{(k-1)k}h_{k1}$ is positive (resp., negative), then we say the mixed cycle C is positive (resp., negative). Otherwise, the value of $h_{12}h_{23}\dots h_{(k-1)k}h_{k1}$ is not real and we call C non-real cycle. In fact, the sign of a mixed cycle (positive, negative, or non-real) is independent of the direction choice in calculation [25]. A mixed graph is positive if each mixed cycle of it is positive. A mixed graph is called an elementary graph if each component is a mixed cycle or an edge, and every edge-component is viewed as undirected. A real spanning elementary subgraph of a mixed graph G is an elementary graph such that its order is the same as G and all the mixed cycles of it are real (positive or negative). By $\eta_H(G)$ we denote the nullity of a mixed graph G for $H(G)$ to distinguish from the nullity $\eta(X)$ of an undirected simple graph X for its adjacency matrix.

The following notations and definitions of mixed graphs are only based on their underlying graphs. A vertex u is a pendant vertex if its degree is one, and the vertex adjacent to u is said quasi-pendant. A matching of G is a subset of edges such that any two edges in it share no common vertex. The maximum matching of G is a matching with the largest cardinality and if it covers all the vertices of G, then we call it perfect matching. The matching number of G, denoted by $\mu(G)$, is defined as the cardinality of a maximum matching.

Let U be a vertex subset of G and $G - U$ be the subgraph obtained from G by deleting all the vertices of U together with the edges incident to them. Sometimes, for a subgraph G_1 of G we use $G - G_1$ instead of $G - V(G_1)$. By $G_1 \cup G_2$ we denote the disjoint union of G_1 and G_2. For the terminologies and notations not mentioned here, we refer the readers to [7] and [25].

Lemma 2.1 ([25]) *Let G be a positive mixed graph. Then $Sp_H(G) = Sp_H(G_U)$, where $Sp_H(G)$ and $Sp_H(G_U)$ mean the spectrum of the Hermitian-adjacency matrices of G and its underlying graph G_U, respectively.*

An edge is called cut-edge in a mixed graph G if its deletion leads to the increase of the number of components of G. Then from [25, Theorem 2.16], the following corollary holds immediately.

Corollary 2.2 *Let e_{uv} be an oriented cut-edge of a mixed graph G. If it is changed to be an undirected cut-edge, then the spectrum of $H(G)$ remains invariant.*

From the Lemma 2.1 and Corollary 2.2, we see that the following two results for the nullity of undirected simple graphs also hold for mixed graphs.

Lemma 2.3 ([27]) *Let T be a mixed tree (or mixed forest) of order n. Then for the nullity of*

T, $\eta_H(T) = n - 2\mu(T)$.

Lemma 2.4 ([27]) *Let C be a positive mixed cycle with order n. Then $\eta_H(C) = 2$ if $n \equiv 0 \pmod 4$, and $\eta_H(C) = 0$ otherwise.*

Suppose G is a mixed graph of order n and the characteristic polynomial of $H(G)$ is written as

$$\Phi(G;\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_n.$$

Lemma 2.5 ([25]) *For an n-vertex mixed graph G, let $R(G) = n-c$ and $S(G) = m-n+c$, where m and c denote the number of edges and components of G, respectively. Then the coefficients of $\Phi(G;\lambda)$ are given by*

$$(-1)^k c_k = \sum_{G'} (-1)^{R(G')+L(G')} \cdot 2^{S(G')},$$

where the summation takes over all the real elementary subgraphs G' of G with k vertices and $L(G')$ is the number of negative mixed cycles in G'.

Lemma 2.6 *Let C be a mixed cycle with order n.*

(i) Suppose C is negative. If n is odd or $n \equiv 0 \pmod 4$, then $\eta_H(C) = 0$. If $n \equiv 2 \pmod 4$, then $\eta_H(C) = 2$.

(ii) Suppose C is non-real. Then $\eta_H(C) = 0$ if n is even, and $\eta_H(C) = 1$ if n is odd.

Proof (i) Note that if the coefficient $c_n \neq 0$ in the characteristic polynomial $\Phi(C;\lambda)$, then $\lambda = 0$ is not the root of $\Phi(C;\lambda) = 0$, i.e., zero is not an eigenvalue of $H(C)$.

Let C be a negative mixed cycle. If n is odd, then there is exactly one real elementary subgraph, C itself. So by Lemma 2.5,

$$(-1)^n c_n = (-1)^{n-1+1} \cdot 2 = (-1)^n \cdot 2 \neq 0$$

and thus $\eta_H(C) = 0$ holds.

If n is even, it is easy to see that there are three real elementary subgraphs, namely C and two real elementary subgraphs constructed by the two perfect matchings of C. Hence applying Lemma 2.5 we have

$$(-1)^n c_n = (-1)^{n-1+1} \cdot 2 + 2((-1)^{n-\frac{n}{2}} \cdot 2^{\frac{n}{2}-n+\frac{n}{2}}) = (-1)^n \cdot 2 + (-1)^{\frac{n}{2}} \cdot 2.$$

Furthermore, if $n \equiv 0 \pmod 4$, then from the above equation $(-1)^n c_n = (-1)^n \cdot 2 + (-1)^{\frac{n}{2}} \cdot 2 = 4$, which implies that $\eta_H(C) = 0$. If $n \equiv 2 \pmod 4$, then $(-1)^n c_n = (-1)^n \cdot 2 + (-1)^{\frac{n}{2}} \cdot 2 = 0$ and we proceed to consider c_{n-1} and c_{n-2}. Clearly, in this case C contains no $(n-1)$-vertex real elementary subgraph, and thus $c_{n-1} = 0$. Moreover, one can easily obtain that n matchings containing $\frac{n}{2}-1$ edges construct all the $(n-2)$-vertex real elementary subgraphs of C. Therefore,

$$(-1)^{n-2} c_{n-2} = n((-1)^{n-2-(\frac{n}{2}-1)} \cdot 2^{\frac{n}{2}-1-(n-2)+\frac{n}{2}-1}) = n(-1)^{\frac{n}{2}-1} \neq 0.$$

So the characteristic polynomial can be written as $\Phi(C;\lambda) = \lambda^2(\lambda^{n-2} + c_1\lambda^{n-3} + \cdots + c_{n-2})$, together with $c_{n-2} \neq 0$ which implies that $\eta_H(C) = 2$.

(ii) Let C be a non-real mixed cycle. If n is even, there are exactly two real elementary subgraphs induced by the two perfect matchings of C. Then

$$(-1)^n c_n = 2((-1)^{n-\frac{n}{2}} \cdot 2^{\frac{n}{2}-n+\frac{n}{2}}) = (-1)^{\frac{n}{2}} \cdot 2 \neq 0,$$

which proves $\eta_H(C) = 0$.

Now assume n is odd, and it is obvious that C contains no real elementary subgraph. As a result, $c_n = 0$ follows. We further consider c_{n-1} and obtain that n matchings containing $\frac{n-1}{2}$ edges build all the $(n-1)$-vertex real elementary subgraphs of C. Hence

$$(-1)^{n-1} c_{n-1} = n((-1)^{n-1-(\frac{n-1}{2})} \cdot 2^{\frac{n-1}{2}-(n-1)+\frac{n-1}{2}}) = n(-1)^{\frac{n-1}{2}} \neq 0,$$

and finally we have $\eta_H(C) = 1$. The proof is completed. □

The following result, similar to that in undirected simple graphs, is of importance for the research of nullity. Denote by $r_H(G)$ the rank of the Hermitian-adjacency matrix of mixed graph G.

Lemma 2.7 *Let G be a mixed graph and v_1 be a pendant vertex with its quasi-pendant vertex v_2. Let $G' = G - \{v_1, v_2\}$. Then $\eta_H(G) = \eta_H(G')$.*

Proof Denote by $r_H(G)$ the rank of the Hermitian-adjacency matrix of G. Then we have

$$r_H(G) = r\left(\begin{bmatrix} 0 & h_{12} & \mathbf{0} \\ h_{21} & 0 & \alpha \\ \mathbf{0^T} & \overline{\alpha} & H(G') \end{bmatrix}\right) = r\left(\begin{bmatrix} 0 & h_{12} & \mathbf{0} \\ h_{21} & 0 & \mathbf{0} \\ \mathbf{0^T} & \mathbf{0^T} & H(G') \end{bmatrix}\right),$$

where $\overline{\alpha}$ is the conjugation and transposition of the row vector α and $h_{12} \in \{\pm i, 1\}$. Thus $r_H(G) = r_H(G') + 2$, which points out $\eta_H(G) = \eta_H(G')$ from $\eta_H(G) + r_H(G) = n$. □

In undirected simple graphs, Gong et al. [7] extended a result as Lemma 2.7 to graphs with pendant trees, and Fan et al. [16] obtained an analogous generalized conclusion in signed graphs. Here we say that the similar results also hold in mixed graphs. See Theorems 2.9 and 2.10.

Definition 2.8 ([7]) *Let T be a mixed tree with a vertex u and G be an n-vertex mixed graph, disjoint with T. For $1 \leq k \leq n$, denote by $T(u) \odot^k G$ the k-joining graph of T and G with respect to u, obtained from $T \cup G$ by joining u and any k vertices of G with (oriented or undirected) edges. Then we call $T(u) \odot^k G$ a mixed graph with the pendant tree T.*

Let T be a mixed tree with at least two vertices and $v \in V(T)$. Suppose there is a maximum matching not cover v in T, then v is said mismatched in T, and v is called matched, otherwise. The following two theorems are similar to Theorems 3.1 and 3.3 of [7] and the proofs are absolutely parallel. The readers can refer to [7].

Theorem 2.9 *Let T be a mixed tree of order p with u a matched vertex and G be a mixed graph with n vertices. Then for $1 \leq k \leq n$,*

$$\eta_H(T(u) \odot^k G) = \eta_H(T) + \eta_H(G).$$

Theorem 2.10 *Let T be a mixed tree of order p with u a mismatched vertex. Let G be a mixed* *graph with n vertices. Then for $1 \le k \le n$,*

$$\eta_H(T(u) \odot^k G) = \eta_H(T-u) + \eta_H(G+u) = \eta_H(T) - 1 + \eta_H(G+u),$$

where $G+u$ denotes the subgraph of $T(u) \odot^k G$ induced by u and the vertices in G.

3. Nullity of the mixed graphs

In this section, all mixed graphs are connected. We first introduce some definitions. Let G be a mixed unicyclic graph with the unique cycle C. For $v \in V(C)$, by $G\{v\}$ we denote the induced connected component of G with largest possible number of vertices, containing v and no other vertices of C. Clearly, $G\{v\}$ is a mixed tree. A unicyclic graph G is called Type I if there is a vertex $v \in V(C)$ such that v is matched in $G\{v\}$; otherwise G is of Type II. We see that if G is of Type I, then $G\{v\}$ and $G - G\{v\}$ are mixed trees.

Lemma 3.1 *Let G be a mixed unicyclic graph with matching number $\mu(G)$. If G is of Type I and $v \in V(C)$ is matched in $G\{v\}$, then $\mu(G) = \mu(G\{v\}) + \mu(G - G\{v\})$; otherwise, $\mu(G) = \mu(C) + \mu(G - C)$.*

Proof First we suppose G is of Type I and $v \in V(C)$ is matched in $G\{v\}$. Note that a maximum matching of $G\{v\}$, together with one of $G - G\{v\}$, forms a matching of G. Thus we have $\mu(G) \ge \mu(G\{v\}) + \mu(G - G\{v\})$. Next we assume that any maximum matchings of $G\{v\}$ and $G - G\{v\}$ cannot form a maximum matching of G. Then in this case one can observe that a maximum matching of G consists of a maximum matching of $G\{v\} - v$ and one of $G - \{G\{v\} - v\}$. Moreover, we obtain $\mu(G\{v\} - v) = \mu(G\{v\}) - 1$ and $\mu(G - \{G\{v\} - v\}) \le \mu(G - G\{v\}) + 1$, which implies that

$$\mu(G) = \mu(G\{v\} - v) + \mu(G - \{G\{v\} - v\}) \le \mu(G\{v\}) + \mu(G - G\{v\}).$$

Hence the first conclusion holds from the above.

Analogous with the above proof, the second result is clear. □

Let $\mathcal{U}$ be the graph set consisting of all the mixed unicyclic graphs with order n and cycle length l. Denote by $\mathcal{U}_1$ a subset of $\mathcal{U}$, consisting of all the following graphs: Type I mixed graphs, Type II mixed graphs with non-real cycle, Type II mixed graphs with negative cycle and $l \equiv 0 \pmod 4$, Type II mixed graphs with positive cycle and $l \equiv 2 \pmod 4$. Let $\mathcal{U}_2 \subset \mathcal{U}$ be composed of Type II mixed graphs with odd length positive or negative cycle, and $\mathcal{U}_3 = \mathcal{U} \setminus \{\mathcal{U}_1, \mathcal{U}_2\}$.

Now we formulate the nullity of a mixed unicyclic graph in terms of its matching number.

Theorem 3.2 *Let G be a mixed unicyclic graph with order n and matching number $\mu(G)$. Denote by C the unique cycle of G with length l. Then it follows that*

$$\eta_H(G) = \begin{cases} n - 2\mu(G), & \text{if } G \in \mathcal{U}_1; \\ n - 2\mu(G) - 1, & \text{if } G \in \mathcal{U}_2; \\ n - 2\mu(G) + 2, & \text{if } G \in \mathcal{U}_3, \end{cases}$$

where $\mathcal{U}_i$ $(1 \le i \le 3)$ are the graph sets shown above.

Proof Assume that G is of Type I with $v \in V(C)$ matched in $G\{v\}$. Then from Theorem 2.9 and Lemmas 2.3 and 3.1, we obtain

$$\begin{aligned}\eta_H(G) &= \eta_H(G\{v\}) + \eta_H(G - G\{v\}) = n - 2(\mu(G\{v\}) + \mu(G - G\{v\})) \\ &= n - 2\mu(G).\end{aligned} \tag{1}$$

Next suppose that G is always of Type II. Then by Theorem 2.10, we have

$$\eta_H(G) = \eta_H(G - C) + \eta_H(C). \tag{2}$$

The remaining proof can be divided into the following cases with respect to the sign of C.

Case 1 Suppose the mixed cycle C of G is non-real.

If the length l of C is even, then $\eta_H(C) = 0$ from Lemma 2.6 (ii). Therefore, by (2) and Lemmas 2.3 and 3.1,

$$\eta_H(G) = \eta_H(G - C) = n - l - 2\mu(G - C) = n - l - 2(\mu(G) - \frac{l}{2}) = n - 2\mu(G).$$

If l is odd, then $\eta_H(C) = 1$ from Lemma 2.6 (ii), and thus

$$\begin{aligned}\eta_H(G) &= \eta_H(G - C) + 1 = n - l - 2\mu(G - C) + 1 \\ &= n - l - 2(\mu(G) - \frac{l-1}{2}) + 1 = n - 2\mu(G).\end{aligned}$$

Case 2 Assume the mixed cycle C of G is negative.

If l is odd or $l \equiv 0 \pmod 4$, then $\eta_H(C) = 0$ from Lemma 2.6 (i). Applying (2) and Lemmas 2.3 and 3.1, we derive

$$\eta_H(G) = \eta_H(G - C) = n - l - 2\mu(G - C) = n - l - 2(\mu(G) - \lfloor \frac{l}{2} \rfloor),$$

where $\lfloor a \rfloor$ denotes the largest integer not more than a. Thus $\eta_H(G) = n - 2\mu(G)$ if $l \equiv 0 \pmod 4$, and $\eta_H(G) = n - 2\mu(G) - 1$ if l is odd.

If $l \equiv 2 \pmod 4$, then $\eta_H(C) = 2$ from Lemma 2.6 (i). Similarly, we have

$$\begin{aligned}\eta_H(G) &= \eta_H(G - C) + 2 = n - l - 2\mu(G - C) + 2 \\ &= n - l - 2(\mu(G) - \frac{l}{2}) + 2 = n - 2\mu(G) + 2.\end{aligned}$$

Case 3 Let the mixed cycle C of G be positive. Applying Lemma 2.4, one can easily prove the result with the analogous method in Case 2, omitted. □

Remark 3.3 From Theorem 3.2, we see that it combines the similar results for undirected unicyclic graphs [8, Theorem 2.1] and for oriented unicyclic graphs [26, Theorem 4.2], and besides it also contains the case of non-real cycle in G. Furthermore, we find there exists an error in the condition $\mu(G) = 2\mu(G - C)$ for the result of [26, Theorem 4.2]. The oriented graph in Figure 1 may be a counter-example, which is evenly-oriented and not satisfy the condition $\mu(\Gamma) = 2\mu(\Gamma - C)$, but the rank of its Hermitian-adjacency or skew-adjacency matrix is also $2\mu(\Gamma) - 2$, not $2\mu(\Gamma)$. So, here we show our statement and proof in Theorem 3.2.

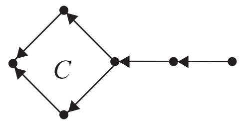

Figure 1 A counter-example for [26, Theorem 4.2]

Moveover, from Theorem 3.2 we easily derive the mixed unicyclic graphs with nonsingular Hermitian-adjacency matrix (i.e., $\eta_H(G) = 0$). See the following corollary.

Corollary 3.4 *Let G be a mixed unicyclic graph with order n. Then $\eta_H(G) = 0$ if and only if $G \in \mathcal{U}_1$ with perfect matching or $G \in \mathcal{U}_2$ with $G - C$ having perfect matching.*

In [28], the author determined all the n-vertex mixed graphs with $\eta_H = n-2$ (i.e., $r_H = 2$). Then in the following we will characterize all the mixed graphs with nullity $\eta_H = n-3$ (i.e., $r_H = 3$). Prior to demonstrating the result, we give some concepts used in [25] and [28].

Let D be a diagonal matrix with diagonal entries belonging to $\{1, \pm i\}$. For two mixed graphs G_1 and G_2, we say G_1 and G_2 are switching equivalent, denoted by $G_1 \sim G_2$, if there exists D such that $H(G_1) = D^{-1}H(G_2)D$. Then we see the switching equivalent graphs have the same spectrum. Two vertices u, v of a mixed graph G are called twin points, if G is switching equivalent to a mixed graph G' in which u and v have the same neighborhood and same nature (undirected or oriented) edges. Note that the deletion of twin points does not change the rank of Hermitian-adjacency matrix. If a mixed graph contains no twin points, then we call it reduced. By $K^1_{a,b,c}$ and $K^2_{a,b,c}$ we denote two mixed complete tripartite graphs, whose reduced forms are switching equivalent to positive mixed C_3 and negative mixed C_3 (see Figure 2 for example), respectively.

Figure 2 A positive mixed C_3 and a negative mixed C_3

The following Lemmas are necessary for us.

Lemma 3.5 ([25]) *Let G be a mixed bipartite graph. Then the spectrum of $H(G)$ is symmetry about zero.*

Denote by S_n^+ the mixed graph obtained from the mixed star S_n by adding an (undirected or oriented) edge joining two pendant vertices. The mixed complete graph of order n is written as K_n.

Lemma 3.6 *Let G be a connected mixed graph of order n with $r_H(G) = 3$. Then G contains no S_4^+ and K_4 as induced subgraphs.*

Proof Lemma 2.7 implies that $r_H(S_4^+) = 4$, thus the first result holds. Next we will discuss the rank of K_4. If we also have $r_H(K_4) = 4$, then the result follows immediately.

Let the characteristic polynomial of $H(K_4)$ be

$$\Phi(K_4;\lambda) = \lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda^1 + c_4.$$

It suffices to prove $c_4 \neq 0$. We can observe that there are three real elementary subgraphs of K_4 with 4 vertices, which only consist of edges (i.e., three maximum matchings). In addition, there are also three spanning cycles C_4 in K_4. If each spanning C_4 is non-real, then there are only real elementary subgraphs composed of edges. Thus from Lemma 2.5, $c_4 = 3\cdot(-1)^{4-2}\cdot 2^{2-4+2} = 3 \neq 0$. Suppose there exists positive or negative spanning C_4, then by Lemma 2.5, $S(C_4) = 4-4+1 = 1$. Therefore, we see that each real spanning C_4 contributes an even number to c_4, but all the real elementary subgraphs consisting of edges contribute an odd number 3 to c_4. So, in this case we also derive $c_4 \neq 0$. Finally, $r_H(K_4) = 4$ and the result holds. □

Lemma 3.7 *Let G be a connected mixed graph with order n. If $r_H(G) = 3$, then G is a mixed complete tripartite graph.*

Proof If G is a mixed bipartite graph, then Lemma 3.5 tells us $r_H(G)$ is an even number, a contradiction. Thus G is non-bipartite. Suppose C_l is the longest induced cycle of G. Then Lemmas 2.4 and 2.6 imply that $l \leq 4$, otherwise, if $l \geq 5$ then the rank of $H(C_l)$, as a principal submatrix of $H(G)$, is not less than 4. So we see $r_H(G) \geq 4$, a contradiction. Now from the above proof we claim that G must contain C_3.

Next we proceed to prove that G is a mixed complete tripartite graph. Let the vertex set of C_3 be $V(C_3) = \{x, y, z\}$. Let Q be the largest induced mixed complete tripartite subgraph of G, containing C_3. By X, Y, Z we denote the three color classes of Q and $x \in X, y \in Y, z \in Z$. Suppose $Q \neq G$, then there is a vertex $u \in V(G) \setminus V(Q)$ adjacent to the vertices of Q. Without loss of generality, let u be adjacent to the vertices of $V(C_3)$. Then from Lemma 3.6, u is exactly adjacent to two vertices, say x and y, of $V(C_3)$. Applying Lemma 3.6 again, we have u is not adjacent to any vertex of Z. Otherwise, G contains K_4 as an induced subgraph, a contradiction by Lemma 3.6. Furthermore, u must be adjacent to any vertex of X and Y. If not, G has S_4^+ as an induced subgraph which contradicts $r_H(G) = 3$ from Lemma 3.6. Now we obtain a larger induced complete tripartite subgraph than Q by adding u to Z, which contradicts the assumption for Q. Consequently, the result holds. □

Theorem 3.8 *Let G be a connected mixed graph of order n. Then $\eta_H(G) = n - 3$ if and only if G is switching equivalent to $K^1_{a,b,c}$ or $K^2_{a,b,c}$, where $1 \leq a \leq b \leq c$ and $a + b + c = n$.*

Proof For the sufficiency part, since the deletion of twin points does not change the rank, from Lemmas 2.4 and 2.6 we easily get $r_H(K^1_{a,b,c}) = r_H(K^2_{a,b,c}) = 3$, and thus $\eta_H(K^1_{a,b,c}) = \eta_H(K^2_{a,b,c}) = n - 3$.

Next we verify the other direction. By Lemma 3.7, we shall suppose $G = K_{a,b,c}$ is a mixed complete tripartite graph. Denote by X, Y, Z the three color classes of G. Then we claim that the vertices of X, Y, Z are twin points, respectively. Otherwise, suppose two vertices u, v of X are not twin points without loss of generality. Then the two rows (columns) indexed by u, v

in $H(G)$ will be linear independent, which implies that $r_H(G) \geq 4$, a contradiction. Hence, we obtain the reduced form of G is a mixed triangle, and furthermore it is a positive or a negative mixed triangle from Lemmas 2.4 and 2.6. Finally, we see G is switching equivalent to $K^1_{a,b,c}$ or $K^2_{a,b,c}$. This completes the proof. $\Box$

A mixed graph G is said determined by the spectrum of $H(G)$, if any mixed graph sharing the same spectrum with G is switching equivalent to G. The problem on cospectral (undirected) graphs (or on graph determined by its spectrum) has long history, due to G$\ddot{u}$nthard and Primas [29]. In [28], the author showed that the mixed graphs with nullity $n-2$ are not always determined by their spectrum of Hermitian-adjacency matrices. In the following we give a conclusion for the mixed graphs with nullity $n-3$.

Theorem 3.9 *Let G be an n-vertex mixed graph with nullity $n-3$. Then G is determined by its spectrum of $H(G)$.*

Proof Suppose G is a mixed graph of order n with nullity $n-3$. Then from Theorem 3.8 G is switching equivalent to either $K^1_{a,b,c}$ or $K^2_{a,b,c}$. Since $K^1_{a,b,c}$ and $K^2_{a,b,c}$ are not switching equivalent, it suffices to prove that the spectrum of $H(K^1_{a,b,c})$ is different from that of $H(K^2_{a,b,c})$. Let c^1_3 and c^2_3 be the characteristic polynomial coefficients corresponding to λ^{n-3} for $H(K^1_{a,b,c})$ and $H(K^2_{a,b,c})$, respectively. If we can prove $c^1_3 \neq c^2_3$, then this completes the proof. Note that each subgraph C_3 of $K^1_{a,b,c}$ (resp., $K^2_{a,b,c}$) is positive (resp., negative). Thus from Lemma 2.5 we obtain $c^1_3 = -c^2_3$. Consequently, the result follows. $\Box$

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