Journal of Mathematical Research with Applications
Jan., 2018, Vol. 38, No. 1, pp. 34–42
DOI:10.3770/j.issn:2095-2651.2018.01.003
Http://jmre.dlut.edu.cn

Flag-Transitive Point-Primitive $(v, k, 4)$-Symmetric Designs with $\mathrm{PSL}_n(q)$ as Socle

Yan ZHU[1], **Shenglin ZHOU**[2,*]

1. *School of Basic Education Sciences, Xuzhou Medical University, Jiangsu* 221004*, P. R. China;*
2. *School of Mathematics, South China University of Technology, Guangdong* 510640*, P. R. China*

Abstract Let $\mathcal{D}$ be a nontrivial symmetric $(v, k, 4)$ design, and $G \le \mathrm{Aut}(\mathcal{D})$ be flag-transitive and point-primitive with $\mathrm{PSL}_n(q)$ as socle. Then $\mathcal{D}$ is a 2-$(15, 8, 4)$ symmetric design and $\mathrm{Soc}(G) = \mathrm{PSL}_2(9)$.

Keywords symmetric design; flag-transitive; point-primitive; classical group

MR(2010) Subject Classification 05B05; 05B25; 20B25

1. Introduction

In recent years, there has been tremendous interest in the inner relationship between the theory of combinatorial designs with some kinds of symmetric properties and the theory of finite groups. Studying the symmetric properties of designs such as point-transitivity and flag-transitivity can make us better understand the structure of some groups. We are concerned in this paper with the flag-transitive (v, k, λ)-symmetric designs when $\lambda = 4$.

A 2-(v, k, λ) symmetric design is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a non-empty set of points and $\mathcal{B}$ is the set of blocks with an incidence relation such that: (i) $|\mathcal{P}| = |\mathcal{B}| = \sqsubseteq$, (ii) every block is incident with exactly k points, and (iii) every 2-element subset of points is incident with exactly λ blocks. The design $\mathcal{D}$ is called nontrivial if $2 < k < v-1$, and denoted by (v, k, λ)-symmetric design for simplicity. For a nontrivial (v, k, λ)-symmetric design, let r denote the number of the blocks through a given point, it is well-known that $r = k$ and $k(k-1) = \lambda(v-1)$.

A flag of $\mathcal{D}$ is an incident pair of point and block. An automorphism of a design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is a permutation of $\mathcal{P}$ which leaves the set $\mathcal{B}$ invariant. The group consisting of all automorphisms of $\mathcal{D}$ is the full automorphism group of $\mathcal{D}$, denoted by $\mathrm{Aut}(\mathcal{D})$. For $G \le \mathrm{Aut}(\mathcal{D})$, the design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is called flag-transitive if G is transitive on the set of flags, and point-primitive if G is primitive on $\mathcal{P}$.

In fact, 2-$(v, k, 1)$ design is a linear space. Saxl [1] classified the finite linear spaces with an automorphism group which is an almost simple group of Lie type acting flag-transitively. In 2007, Regueiro [2] gave the classification of biplanes which admits a flag-transitive automorphism

Received March 28, 2017; Accepted September 15, 2017
Supported by the National Natural Science Foundation of China (Grant Nos. 11471123; 11626206) and the Scientific Research Foundation for Excellent Talents of Xuzhou Medical University (Grant No. D2016002).
* Corresponding author
E-mail address: zhuyan@xzhmu.edu.cn (Yan ZHU); slzhou@scut.edu.cn (Shenglin ZHOU)

group of almost simple type with classical socle. Zhou, Dong and Fang [3] proved that if $\mathcal{D}$ is a nontrivial triplane with a flag-transitive point-primitive automorphism group G which is almost simple and classical socle, then $\mathcal{D}$ has parameters $(11,6,3)$ and $(45,12,3)$. Zhou and Tian proved in [4] that if a 2-$(v,k,4)$ symmetric design admits a flag-transitive point-primitive automorphism group G with $\mathrm{PSL}_2(q)$ as socle, then $\mathcal{D}$ is a 2-$(15,8,4)$ symmetric design.

The main purpose of this paper is to generalize above result to the case of $X := \mathrm{Soc}(G) = \mathrm{PSL}_n(q)$, with $n \geq 3$, and $(n,q) \neq (3,2)$.

Now, we give the result as follows.

Theorem 1.1 *If $\mathcal{D}$ is a nontrivial $(v,k,4)$-symmetric design admitting a flag-transitive, point-primitive automorphism group G of almost simple type, then the socle of G cannot be $\mathrm{PSL}_n(q)$, with $n \geq 3$, and $(n,q) \neq (3,2)$.*

This, together with [4, Theorem 1.1], yields the following:

Corollary 1.2 *Let $\mathcal{D}$ be a $(v,k,4)$-symmetric design, which admits a flag-transitive, point-primitive automorphism group G of almost simple type and $X = \mathrm{PSL}_n(q)$ for $n \geq 2$ and $(n,q) \neq (3,2)$. Then $\mathcal{D}$ is a 2-$(15,8,4)$ symmetric design with $X = \mathrm{PSL}_2(9)$ and $X_x = \mathrm{PGL}_2(3)$, where x is a point of $\mathcal{D}$.*

2. Preliminary results

Lemma 2.1 ([5]) *If $\mathcal{D}$ is a $(v,k,4)$-symmetric design and G is a flag-transitive point-primitive automorphism group of $\mathcal{D}$, then*

(i) $k(k-1) = 4(v-1)$;

(ii) $16v - 15$ *is a square;*

(iii) $k^2 > 4v$, *and* $|G_x|^3 > 4|G|$, *where* $x \in \mathcal{P}$;

(iv) $k \mid 4d_i$, *where d_i is any subdegree of G;*

(v) $k \mid 4 \cdot \gcd(v-1, |G_x|)$.

Lemma 2.2 ([6, 1.6]) *If X is a simple group of Lie type in characteristic p, then any proper subgroup of index prime to p is contained in a parabolic subgroup of X.*

Lemma 2.3 *Suppose $\mathcal{D}$ is a $(v,k,4)$-symmetric design with a primitive, flag-transitive almost simple automorphism group G with simple socle X of Lie type in characteristic p, and the stabilizer G_x is not a parabolic subgroup of G. If p is odd, then p does not divide k; and if $p = 2$, then 8 does not divide k. Hence $|G| < 4|G_x|_{p'}^2 \cdot |G_x|$.*

Proof By Lemma 2.1, p divides $v = [G : G_x]$. Since k divides $4(v-1)$, if p is odd, then $(k,p) = 1$, and if $p = 2$, then $(k,p) \leq 2$. Hence k divides $4|G_x|_{p'}$, and since $k^2 > 4v$, we have $|G| < 4|G_x|_{p'}^2 \cdot |G_x|$.

Lemma 2.4 ([2, Lemma 9]) *Suppose p divides v, and G_x contains a normal subgroup H of Lie type in characteristic p which is quasisimple and $p \nmid |Z(H)|$; then k is divisible by $[H : P]$, for* *some parabolic subgroup P of H.*

Lemma 2.5 ([7, Lemma 3.9]) *If X is a group of Lie type in characteristic p, acting on the set of cosets of a maximal parabolic subgroup, and X is not $\mathrm{PSL}_d(q)$, $P\Omega^+_{2m}(q)$ (with m odd), nor $E_6(q)$, then there is a unique subdegree which is a power of p.*

3. Proof of Theorem 1.1

In this section, let $\mathcal{D}$ be a nontrivial $(v,k,4)$-symmetric designs, and $G \leq \mathrm{Aut}(\mathcal{D})$ be a flag-transitive, point-primitive with $X = \mathrm{PSL}_n(q)$, where $n \geq 3$ and $(n,q) \neq (3,2)$.

Let $q = p^m$, and take $\{v_1, \ldots, v_n\}$ to be a basis for the natural n-dimensional vector space V for X. Since the stabilizer G_x is a maximal subgroup of G, then by Aschbacher's Theorem in [8], G_x lies in one of the families $\mathcal{C}_i$ of subgroups of $\Gamma L_n(q)$, or in the set $\mathcal{S}$ of almost simple subgroups which is not contained in any of these families. We will analyze each of these cases separately. In order to describe the Aschbacher subgroups, we denote the pre-image of the group H by $\hat{\ }H$ in the corresponding linear group.

$\mathcal{C}_1$) In this case, G_x is reducible. That is to say, $G_x \cong P_i$ stabilizes an i-dimension subspace of V. Suppose $G_x \cong P_1$. Then G is 2-transitive, and this case has already been discussed by Kantor [9].

Now suppose $G_x \cong P_i$ $(1 < i < n)$ fixes W which is an i-subspace of V. Since our arguments are arithmetic and for i and $n-i$ we have the same calculations, we will assume $i \leq n/2$. Considering the G_x-orbits of the i-spaces intersecting W in $(i-1)$-dimensional spaces, we know k divides

$$\frac{4q(q^i-1)(q^{n-i}-1)}{(q-1)^2}.$$

Also,

$$v = \frac{(q^n-1)\cdots(q^{n-i+1}-1)}{(q^i-1)\cdots(q-1)} > q^{i(n-i)},$$

but $k^2 > 4v$, we have

$$\frac{4q^2(q^i-1)^2(q^{n-i}-1)^2}{(q-1)^4} > q^{i(n-i)},$$

so $i = 2; i = 3, n = 6,7,8,9,10$ or $i = 4, n = 8$.

Case 1 $i = 4$ and $n = 8$. Then

$$k \mid \frac{4q(q^4-1)^2}{(q-1)^2},\ v = \frac{(q^8-1)(q^7-1)(q^6-1)(q^5-1)}{(q^4-1)(q^3-1)(q^2-1)(q-1)} > q^{16}.$$

The inequality $k^2 > 4v$ implies $q = 2$ or 3. But for every q, $16v - 15$ is not a square.

Case 2 $i = 3$ and $n = 6,7,8,9$ or 10.

Subcase 2.1 $q = 2$.

For $n = 6,7,8,9$ or 10, we have $v = 1395, 11811, 97155, 788035$ or 6347715, respectively, and it is easily known that $16v - 15$ is not a square for every case.

Subcase 2.2 $q > 2$.

If $n = 6$, then k divides

$$4(\frac{q(q^3-1)(q^3-1)}{(q-1)^2}, \frac{(q^6-1)(q^5-1)(q^4-1)}{(q^3-1)(q^2-1)(q-1)} - 1)$$

and $k^2 > 4v$, and we have $4 > q^5(q^2 - q + 1)$, which is a contradiction.

The other cases can be ruled out in the same way.

Case 3 $i = 2$.

Here, $v = \frac{(q^n-1)(q^{n-1}-1)}{(q^2-1)(q-1)}$, and G has suborbits $|\{$2-subspaces H: $\dim(H \cap W)$=1$\}|$ and $|\{$2-subspaces H: $H \cap W = \overline{0}\}|$ with sizes:

$$\frac{q(q+1)(q^{n-2}-1)}{q-1} \quad \text{and} \quad \frac{q^4(q^{n-2}-1)(q^{n-3}-1)}{(q^2-1)(q-1)}.$$

If n is even, then k divides $\frac{4q(q^{n-2}-1)}{q^2-1}$, since $q+1$ is prime to $\frac{q^{n-3}-1}{q-1}$. Then $k \leq \frac{4q(q^{n-2}-1)}{q^2-1}$, $k^2 \leq \frac{16q^2(q^{n-2}-1)^2}{(q^2-1)^2} = (\frac{q^{n-2}-1}{q-1})^2 \cdot (q+1)^2 \cdot \frac{16q^2}{(q+1)^4} < 4 \cdot \frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)} \cdot \frac{16q^2}{(q+1)^4} < 4 \cdot \frac{(q^n-1)(q^{n-1}-1)}{(q-1)(q^2-1)} = 4v$, which contradicts the condition $k^2 > 4v$.

Hence n is odd, and k divides $\frac{4q(q^{n-2}-1)}{q-1}(q+1, \frac{n-3}{2})$. First assume $n = 5$. Then $v = (q^2+1)(q^4+q^3+q^2+q+1)$, and $k \mid 4q(q^2+q+1)$. Suppose $k = \frac{4q(q^2+q+1)}{u}$, by $k^2 > 4v$, we have

$$\frac{4q^2(q^2+q+1)^2}{u^2} > (q^2+1)(q^4+q^3+q^2+q+1),$$

which implies $u^2 < \frac{4q^2(q^2+q+1)^2}{(q^2+1)(q^4+q^3+q^2+q+1)} < \frac{4(q^4+2q^3+3q^2+2q+1)}{q^4} = 4(1 + \frac{2}{q} + \frac{3}{q^2} + \frac{2}{q^3} + \frac{1}{q^4}) < 4(1 + \frac{2}{2} + \frac{3}{2^2} + \frac{2}{2^3} + \frac{1}{2^4}) = 12 + \frac{1}{4}$, then $u = 1, 2, 3$.

If $u = 1$ or 2, then $k = 4q(q^2+q+1)$ or $2q(q^2+q+1)$ which contradicts the basic equation $k(k-1) = 4(v-1)$.

If $u = 3$, then $k = \frac{4}{3}q(q^2+q+1)$. By $k^2 > 4v$, we have

$$\frac{16}{9}q^2(q^2+q+1)^2 > 4(q^2+1)(q^4+q^3+q^2+q+1).$$

It follows that

$$5q^6 + q^5 + 6q^4 + 10q^3 + 14q^2 + 9q + 9 < 0,$$

a contradiction.

Therefore $n \geq 7$. Here

$$v = (q^{n-1} + q^{n-2} + \cdots + q^2 + q + 1)(q^{n-3} + q^{n-5} + \cdots + q^4 + q^2 + 1),$$

and $k \mid 4dc$, where $d = q(q^{n-3} + q^{n-4} + \cdots + q + 1) = \frac{q(q^{n-2}-1)}{q-1}$ and $c = (q+1, \frac{n-3}{2})$. Say $k = \frac{4dc}{e}$, then $k^2 = \frac{16d^2c^2}{e^2} > 4v$ implies $e^2 < \frac{4d^2c^2}{v} < 4 \cdot 4q^2 = 16q^2$. Thus, we have $e < 4q$.

Then we have the following equality

$$\frac{v-1}{d} = q^{n-2} + q^{n-4} + \cdots + q^3 + q + 1,$$

and also, since $k(k-1) = 4(v-1)$, we have

$$k = \frac{4(v-1)}{k} + 1 = \frac{eq^{n-2} + eq^{n-4} + \cdots + eq^3 + eq + e + c}{c},$$

then

$$kc = eq^{n-2} + eq^{n-4} + \cdots + eq^3 + eq + e + c.$$

Thus we have $(kc, q) = (e + c, q)$ and $(kc, q^{n-3} + q^{n-4} + \cdots + q^2 + q + 1)$ divides $(kc, e(q^{n-3} + q^{n-4} + \cdots + q^2 + q + 1))$ which equals to $(eq^{n-4} + eq^{n-6} + \cdots + eq^3 + eq + 2e + c, (2e + c)q + e + c)$.

From $k \mid 4dc$, we have $kc \mid 4c^2 d$. Then, $kc = (kc, 4c^2 d)$ divides $(kc, 4c^2) \cdot (kc, d)$, which equals to $(kc, 4c^2) \cdot (kc, q(q^{n-3} + q^{n-4} + \cdots + q + 1))$. But $(kc, q(q^{n-3} + q^{n-4} + \cdots + q + 1))$ divides $(kc, q) \cdot (kc, q^{n-3} + q^{n-4} + \cdots + q + 1)$ which implies kc divides $4c^2 \cdot (e + c, q) \cdot [(2e + c)q + e + c]$.

Thus $kc \mid 4c^2 q[(2e + c)q + e + c]$. Then $kc \le 4c^2 q[(2e + c)q + e + c]$. It follows that

$$\begin{aligned} q^{n-2} + q^{n-4} + \cdots + q^3 + q + 1 + 1 &\le eq^{n-2} + eq^{n-4} + \cdots + eq^3 + eq + e + c = kc \\ &\le 4c^2 q[(2e + c)q + e + c] \\ &< 4(\frac{n-3}{2})^2 q[(8q + \frac{n-3}{2})q + 4q + \frac{n-3}{2}] \\ &= (n-3)^2(8q^3 + \frac{n+5}{2}q^2 + \frac{n-3}{2}q). \end{aligned}$$

If $n = 7$, then we get $q^5 + q^3 + q + 2 < 128q^3 + 96q^2 + 32q$. It follows $q \le 11$, then $q = 2, 3, 4, 5, 7, 8, 9$ or 11.

If $n = 9$, then we have $q^7 + q^5 + q^3 + q + 2 < 288q^3 + 252q^2 + 108q$. It follows that $q = 2, 3$ or 4.

If $n = 11$, then $q^9 + q^7 + q^5 + q^3 + q + 2 < 512q^3 + 512q^2 + 256q$ forces $q = 2$.

If $n = 13$, then $q^{11} + q^9 + q^7 + q^5 + q^3 + q + 2 < 800q^3 + 900q^2 + 500q$ implies $q = 2$.

If $n \ge 15$, we get

$$\begin{aligned} q^{n-2} + q^{n-4} + \cdots + q^3 + q + 2 &\le eq^{n-2} + eq^{n-4} + \cdots + eq^3 + eq + e + c = kc \\ &\le 4c^2 q[(2e + c)q + e + c] \\ &< 4(q+1)^2 q[(8q + q + 1)q + 4q + q + 1] \\ &= 36q^5 + 96q^4 + 88q^3 + 32q^2 + 4q. \end{aligned}$$

But $q^{n-2} + q^{n-4} + \cdots + q^3 + q + 2 > 36q^5 + 96q^4 + 88q^3 + 32q^2 + 4q$ for all $n \ge 15$, a contradiction.

Now, we consider 4 possible cases: (1) $n = 7, q = 2, 3, 4, 5, 7, 8, 9$ or 11; (2) $n = 9, q = 2, 3$ or 4; (3) $n = 11, q = 2$; (4) $n = 13, q = 2$.

Subcase (1) $n = 7, q = 2, 3, 4, 5, 7, 8, 9$ or 11.

If $q = 2$, then $c = 1$ and $1 \le e < 8$. But $kc = 43e + 1 \nmid 40e + 24$ for every e.

If $q = 3$, then $c = 2$ and $1 \le e < 12$. For every e, $kc = 2(137e + 1) \nmid 48(7e + 8)$, which is impossible.

If $q = 4$, then $c = 1$ and $1 \le e < 16$. For every e, $kc = (4^5 + 4^3 + 5)e + 1 > 144e + 80$, a contradiction.

The other cases can be ruled out similarly.

Subcase (2) $n = 9, q = 2, 3$ or 4.

If $q=2$, then $c=3$ and $1\le e<8$. For every e, $kc=(2^7+2^5+2^3+3)e+3\nmid 360e+648$.

If $q=3$, then $c=1$ and $1\le e<12$. For every e, $kc=(3^7+3^5+3^3+4)e+1>84e+48$.

If $q=4$, then $c=1$ and $1\le e<16$. For every e, $kc=(4^7+4^5+4^3+5)e+1>144e+80$.

Subcase (3) $n=11, q=2$.

If $q=2$, then $c=1$ and $1\le e<8$. But then, $kc=(2^9+2^7+2^5+2^3+3)e+1>40e+24$.

Subcase (4) $n=13, q=2$.

If $q=2$, then $c=1$ and $1\le e<8$. For every e, $kc=(2^{11}+2^9+2^7+2^5+2^3+3)e+1>40e+24$.

$\mathcal{C}_1'$) Now G contains a graph automorphism and the stabilizer G_x stabilizes a pair $\{U,W\}$ of subspaces of dimension i and $n-i$ respectively, with $i<n/2$. Write G^0 for the subgroup $G\cap P\Gamma L_n(q)$ of index 2 in G.

First suppose $U\subset W$. By Lemma 2.5, there is a subdegree with a power of p. On the other hand, if p is odd, then the highest power of p dividing $v-1$ is q, it is $2q$ if $q>2$ is even, and is at most 2^{n-1} if $q=2$. Hence $k^2<v$, a contradiction.

Now assume $V=U\oplus W$. Here p divides v, so $(k,p)\le 4$. First suppose $i=1$. If $x=\{\langle v_1\rangle,\langle v_2,\dots,v_n\rangle\}$, then consider $y=\{\langle v_1,\dots,v_{n-1}\rangle,\langle v_n\rangle\}$, so $[G_x:G_{xy}]=\frac{q^{n-2}(q^{n-1}-1)}{q-1}$ and k divides $\frac{4(q^{n-1}-1)}{q-1}$. However, $v=\frac{q^{n-1}(q^n-1)}{q-1}=q^{n-1}(q^{n-1}+q^{n-2}+\cdots+q^2+q+1)>q^{n-1}\cdot q^{n-1}=q^{2(n-1)}$ implies $k^2\le\frac{16(q^{n-1}-1)^2}{(q-1)^2}\le 4q^{2(n-1)}<4v$, a contradiction.

Next suppose $i>1$. Consider $x=\{\langle v_1,\dots,v_i\rangle,\langle v_{i+1},\dots,v_n\rangle\}$ and $y=\{\langle v_1,\dots,v_{i-1},v_i+v_n\rangle,\langle v_{i+1},\dots,v_n\rangle\}$. Then $[G_x^0:G_{xy}^0]_{p'}$ divides $4(q^i-1)(q^{n-i}-1)$, so $k<4q^n$. On the other hand, $v>q^{2i(n-i)}$, so again $k^2<16q^{2n}\le 4q^{2i(n-i)}<4v$, a contradiction.

$\mathcal{C}_2$) In this case, G_x preserves a partition $V=V_1\oplus\cdots\oplus V_a$, with each V_i of the same dimension b, where $n=ab$.

Now let $b=1$ and $n=a$, and consider $x=\{\langle v_1\rangle,\dots,\langle v_n\rangle\}$ and $y=\{\langle v_1+v_2\rangle,\langle v_2\rangle,\dots,\langle v_n\rangle\}$. Since $n>2$, we see k divides $8n(n-1)(q-1)$. Now $v>\frac{q^{n(n-1)}}{n!}$ and $k^2>4v$, so $n=3, q=2,3,4,5,7$ or $n=4, q=2$. But there is no such value of k satisfying the equation $k(k-1)=4(v-1)$.

Then consider $b>1$, and let $x=\{\langle v_1,\dots,v_b\rangle,\langle v_{b+1},\dots,v_{2b}\rangle,\dots\}$ and $y=\{\langle v_1,\dots,v_{b-1},v_{b+1}\rangle,\langle v_b,v_{b+2},\dots,v_{2b}\rangle,\dots,\langle v_{n-b+1},\dots,v_n\rangle\}$. Then, $k\mid\frac{4a(a-1)(q^b-1)^2}{q-1}$, so $v>\frac{q^{n(n-b)}}{a!}$, forcing $n=4, q=2,3,4,5$ and $a=2=b$.

Since $v=|G:G_x|$, we have $v\mid |G|$. By $X=\mathrm{PSL}_n(q)$ and $X\trianglelefteq G\le \mathrm{Aut}(X)=X\cdot\mathrm{Out}(X)$, we get $|G|\mid |X|\cdot|\mathrm{Out}(X)|$. Then $v\mid |X|\cdot|\mathrm{Out}(X)|$.

For every (n,q) pair ($n=4,\ 2\le q\le 5$), by $k\mid\frac{4a(a-1)(q^b-1)^2}{q-1}$, $v>\frac{q^{n(n-b)}}{a!}$ and the equation $k(k-1)=4(v-1)$, we can get the possible (v,k) parameters. Since $X=\mathrm{PSL}_n(q)$, $|\mathrm{Out}(X)|$ and $|X|$ are known. The contradiction can be obtained by $v\nmid |X|\cdot|\mathrm{Out}(X)|$.

For instance, if $q=2$ and $n=4$, since $k\mid 72$, then $v>2^7=128$. By $k(k-1)=4(v-1)$, the possible (v,k) parameters are $(139,24),(316,36)$ and $(1279,72)$. Since $X=\mathrm{PSL}_4(2)$, then $|\mathrm{Out}(X)|=2$ and $|X|=20160=2^6\cdot3^2\cdot5\cdot7$. But $v\nmid |X|\cdot|\mathrm{Out}(X)|$ for every v, a contradiction.

$\mathcal{C}_3$) Here G_x is an extension field subgroup. Since $4|G_x|\cdot|G_x|_{p'}^2>|G|$, by Lemma 2.3, either:

(1) $n=3$ and $X\cap G_x = \hat{}(q^2+q+1)\cdot 3 < \mathrm{PSL}_3(q)=X$, or

(2) n is even and $G_x = N_G(\hat{}\mathrm{PSL}_{\frac{n}{2}}(q^2))$.

First consider Subcase (1). Here $v=\frac{q^3(q^2-1)(q-1)}{3}$, where $q=p^m$, so k divides $12m(q^2+q+1)$, and $k^2>4v$ implies $m=1, q=2,3,5,7$ or 11; $m=2, q=2^2$ or 3^2; $m=3, q=2^3$ or 3^3; $m=4, q=2^4$ and $m=5, q=2^5$. But for every case, $16v-15$ is not a square.

Now consider Subcase (2). Write $n=2m_0$. Since p divides v, we have $(k,p)\le 4$. First suppose $n\ge 8$ and let W be a 2-subspace of V considered as a vector space over the field of q^2 elements, then W is a 4-subspace over a field of q elements. If we consider the stabilizer of W in G_x and in G, we see that in $G_W\backslash G_{xW}$ there is an element g such that $G_x\cap G_x^g$ contains the point-wise stabilizer of W in G_x. Therefore, $k\mid 4(q^n-1)(q^{n-2}-1)$, contrary to $k^2>4v$.

Now let $n=6$. Then since $(k,p)\le 4$, Lemma 2.4 implies k is divisible by the index of a parabolic subgroup of G_x, so it is divisible by the primitive prime divisor of q^3-1, but this divides the index of G_x in G, which is v, a contradiction.

Therefore $n=4$. Then $v=\frac{q^4(q^3-1)(q-1)}{2}$, and so k is odd and prime to $q-1$. The fact $(v-1,q+1)=1$ implies k is also prime to $q+1$, and hence $k\mid m(q^2+1)$, contrary to $k^2>4v$, another contradiction.

$\mathcal{C}_4$) In this case, G_x stabilizes a tensor product of spaces of different dimensions, and $n\ge 6$. In all these cases $v>k^2$, which Contradicts the fact $k^2>4v$.

$\mathcal{C}_5$) Here G_x is the stabilizer in G of a subfield space. So $G_x=N_G(\mathrm{PSL}_n(q_0))$, where $q=q_0^t$ and t prime.

If $t>2$, then $4|G_x|\cdot|G_x|_{p'}^2>|G|$ forces $n=2$, a contradiction. Hence $t=2$.

If $n=3$, then $v=\frac{(q_0^3+1)(q_0^2+1)q_0^3}{(q_0+1,3)}$. Since p divides v, we have $(k,p)\le 4$, so Lemma 2.4 implies G_{xB} is contained in a parabolic subgroup of G_x, where B is a block incident with x. Therefore, $q_0^2+q_0+1$ divides k, and $(v-1,q_0^2+q_0+1)$ divides $2q_0+(q_0+1,3)$, forcing $q_0=2$ and $v=120$, but then $16v-15=3\cdot 5\cdot 127$ is not a square.

If $n=4$, then by Lemma 2.4 we see q_0^2+1 divides k, but q_0^2+1 also divides v, a contradiction.

Hence $n\ge 5$. Considering the stabilizers of a 2-dimensional subspace of V, we see that k divides $4(q_0^n-1)(q_0^{n-1}-1)$, but then $k^2<v$, which is also a contradiction.

$\mathcal{C}_6$) In this case, G_x is an extraspecial normalizer.

Since $4|G_x|\cdot|G_x|_{p'}^2>|G|$, we have $n\le 4$. Now, $n>2$ implies that $G_x\cap X$ is either 2^4A_6 or 3^2Q_8, with X either $\mathrm{PSL}_4(5)$ or $\mathrm{PSL}_3(7)$, respectively. Since k divides $4(v-1,|G_x|)$, we obtain that $k\le 6$, contrary to the fact $k^2>4v$.

Let $n=2$. Then $G_x\cap X=A_4.a<L_2(p)=X$, with $a=2$ precisely when $p\equiv\pm1 \pmod 8$, and $a=1$ otherwise (and there are a conjugacy classes in X). From $|G|<|G_x|^3$, we obtain $p\le 13$. That is to say, $a=2$ and $p=7$ or $a=1$ and $p=5,11,13$.

For every pair (a,p), it is easy to know that $16v-15$ is not a square.

$\mathcal{C}_7$) In this case, G_x stabilizes the tensor product of a space of the same dimension, say b, and $n=b^a$. Since $|G_x|^3>|G|$, we have $n=4$ and $G_x\cap X=(\mathrm{PSL}_2(q)\times\mathrm{PSL}_2(q))2^d<X=\mathrm{PSL}_4(q)$, where $d=(2,q-1)$. Then $v=\frac{q^4(q^2+1)(q^3-1)}{z}>\frac{q^9}{z}$, with $z=2$ unless $q\equiv 1 \pmod 4$, in which case $z=4$. So k divides $4m(q^2-1)^2$, and if q is odd, then k divides $\frac{m(q^2-1)^2}{16}$.

If q is odd, then $k^2 < \frac{q^9}{16} < \frac{q^9}{z} < v$, a contradiction. Hence q is even, and so $k = \frac{4m(q^2-1)^2}{y}$, where y is an integer, and since $k^2 > 4v$, we get $q = 2, 2^2, 2^3, 2^4, 2^5$. In this case, $v = \frac{q^4(q^2+1)(q^3-1)}{4}$.

All these possibilities can be ruled out by the following strategy. For every $q = 2^i$ $(1 \le i \le 5)$, first computing the value of v, then by $k^2 > 4v$, we get the lower bound of k. Then combining it with $k = \frac{4m(q^2-1)^2}{y}$ obtains the values of k. But $k \nmid 4(v-1)$, we get contradiction.

For example, if $q = 2$, $v = \frac{q^4(q^2+1)(q^3-1)}{4} = 140$. By $k^2 > 4v$, we have $k \ge 24$. However, $k = \frac{4m(q^2-1)^2}{y} = \frac{36}{y}$, we have $k = 36$. But $k \nmid 4(v-1)$, since $4(v-1) = 2^2 \cdot 139$.

$\mathcal{C}_8$) Now suppose G_x to be a classical group.

Case 1 First assume G_x is a symplectic group, so n is even. By Lemma 2.2, k is divisible by a parabolic index in G_x.

If $n = 4$, then $v = \frac{q^2(q^3-1)}{(2,q-1)}$, and $\frac{q^4-1}{q-1}$ divides k, however $(v-1, q^2+1)$ divides 2, which is a contradiction.

If $n = 6$, then $v = \frac{q^6(q^5-1)(q^3-1)}{(3,q-1)}$ and q^3+1 divides k, but q^3+1 divides $4(v-1)$ only if $q = 2$, so $k = 9$, too small.

Then let $n \ge 8$. If we consider the stabilizers of a 4-dimensional subspace of G_x and G, we see that k divides twice the odd part of $(q^n-1)(q^{n-2}-1)$. Also, $(k, q-1) \le 2$, so k divides $\frac{4(q^n-1)(q^{n-2}-1)}{(q-1)^2}$, and therefore $k \le 16q^{2n-4}$. The inequality $k^2 > 4v$ forces $n = 8$. In this case, $v = \frac{q^{12}(q^7-1)(q^5-1)(q^3-1)}{(q-1,4)}$ which implies $q \le 3$, and in neither of these two cases is $16v - 15$ a square.

Case 2 Next let G_x be orthogonal. Then q is odd, since that is the case with odd dimension, and with even dimension it is a consequence of the maximality of G_x in G. The case in which $n = 4$ and G_x is of type O_4^+ will be investigated later. In all other cases Tits Lemma (Lemma 2.2) implies that k is divisible by a parabolic index in G_x and is therefore even, but it is not divisible by 4 since v is also even and $(k, v) \le 4$. This and the fact that q does not divide k implies $k > v$. This is impossible.

Case 3 Finally consider G_x to be a unitary group over the field of q_0 elements, with $q = q_0^2$. If $n \ge 4$, then considering the stabilizers of a nonsingular 2-subspace of V in G and G_x, we see k divides $4(q_0^n - (-1)^n)(q_0^{n-1} - (-1)^{n-1})$. The inequality $k^2 > 4v$ implies $n = 4$, and in this case $v = \frac{q_0^6(q_0^4+1)(q_0^3+1)(q_0^2+1)}{(q_0-1,4)}$. Since k divides $4(q_0^4-1)(q_0^3+1)$ and $(k, (q_0^2+1)(q_0-1)) \le 2$, we see that k divides $4(q_0^3+1)(q_0+1)$, so $k^2 \le 4v$. It is impossible. Thus $n = 3$, and by Lemma 2.2, $q_0^2 - q_0 + 1$ divides k, and k divides $4(v-1)$ with $v = \frac{q_0^3(q_0^3-1)(q_0^2+1)}{z}$ with z either 1 or 3. This implies $q_0 = 2$, but then $v = 280$, and $16v - 15$ is not a square.

$\mathcal{S}$) In the end, we consider the case where G_x is an almost simple group (modulo the scalars), not contained in the Aschbacher subgroups of G. From [10, Theorem 4.2], we have the possibilities $|G_x| < q^{2n+4}$, $G_x' = A_{n-1}$ or A_{n-2}, or $G_x \cap X$ and X are as in [10, Table 4].

From Lemma 2.1, we have $|G| < |G_x|^3$ with $|G| \le q^{n^2-n-1}$ forcing $n \le 7$, and by the bound $4|G_x||G_x|_{p'}^2 > |G|$ we need only to consider the following possibilities [11, Chapter 5]:

(1) $n=2$, and $G_x \cap X = A_5$, with $q = 11, 19, 29, 31, 41, 49, 59, 61$ or 121;

(2) $n=3$, and $G_x \cap X = A_6 < \mathrm{PSL}_3(4) = X$;

(3) $n=4$, and $G_x \cap X = U_4(2) < \mathrm{PSL}_4(7) = X$.

Case 1 $n=2$, $q = 11, 19, 29, 31, 41, 49, 59, 61$ or 121.

For every case, it is easy to know that $16v - 15$ is not a square.

Case 2 $n=3$, $q=4$. Then $|X| = 20160$ and $v = 56$. But $16v - 15 = 881$ is a prime.

Case 3 $n=4$, $q=7$. Then $|X| = 2317591180800$ and $v = 89413240$. But $16v - 15 = 5^2 \cdot 57224473$ is not a square.

This completes the proof of Theorem 1.1. $\Box$

Acknowledgements We thank the referees for their time and comments.

References

[1] J. SAXL. *On finite linear spaces with almost simple flag-transitive automorphism groups*. J. Combin. Theory Ser. A, 2002, **100**(2): 322–348.
[2] E. O'REILLY-REGUEIRO. *Biplanes with flag-transitive automorphism groups of almost simple type, with classical socle*. J. Algebraic Combin., 2007, **26**(4): 529–552.
[3] Shenglin ZHOU, Huili DONG, Weidong FANG. *Finite classical groups and flag-transitive triplanes*. Discrete Math., 2009, **309**(16): 5183–5195.
[4] Shenglin ZHOU, Delu TIAN. *Flag-transitive point-primitive 2-$(v,k,4)$ symmetric designs and two dimensional classical groups*. Appl. Math. J. Chinese Univ., 2011, **26**(3): 334–341.
[5] Weidong FANG, Huili DONG, Shenglin ZHOU. *Flag-transitive 2-$(v,k,4)$ symmetric designs*. Ars Combin., 2010, **95**: 333–342.
[6] G. M. SEITZ. *Flag-transitive subgroups of Chevalley groups*. Ann. Math., 1973, **97**(1): 27–56.
[7] M. W. LIEBECK, J. SAXL, G. M. SEITZ. *On the overgroups of irreducible subgroups of the finite classical groups*. Proc. London Math. Soc. (3), 1987, **55**(3): 507–537.
[8] M. ASCHBACHER. *On the maximal subgroups of the finite classical groups*. Invent. Math., 1984, **76**(3): 469–514.
[9] W. KANTOR. *Classification of 2-transitive symmetric designs*. Graphs Combin., 1985, **1**(2): 165–166.
[10] M. W. LIEBECK. *On the orders of maximal subgroups of the finite classical groups*. Proc. London Math. Soc. (3), 1985, **50**(3): 426–446.
[11] P. B. KLEIDMAN. *The subgroup structure of some finite simple groups*. PH. D. Thesis, University of Cambridge, 1987.