Properties of Some Families of Meromorphic Multivalent Functions Associated with Generalized Hypergeometric Functions

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Abstract We introduce and study two subclasses $\Omega_{[\alpha_1]}(A, B, \lambda)$ and $\Omega^+_{[\alpha_1]}(A, B, \lambda)$ of meromorphic p-valent functions defined by certain linear operator involving the generalized hypergeometric function. The main object is to investigate the various important properties and characteristics of these subclasses of meromorphically multivalent functions. We extend the familiar concept of neighborhoods of analytic functions to these subclasses. We also derive many interesting results for the Hadamard products of functions belonging to the class $\Omega^+_{[\alpha_1]}(\alpha, \beta, \gamma, \lambda)$.

Keywords Generalized hypergeometric function; Hadamard product; meromorphic functions; neighborhoods

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1. Introduction

Let Σ_p denote the subclass of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p}, \quad p \in N = 1, 2, \ldots, \tag{1.1}$$

which are analytic and $p-$valent in the punctured unit $U^\star = \{z : z \in C \text{ and } 0 < |z| < 1\} = U \setminus \{0\}$. For a function $f(z) \in \Sigma_p$ given by (1.1) and $g(z) \in \Sigma_p$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_k z^{k-p}, \quad p \in N,$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f * g)(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k b_k z^{k-p} = (g * f)(z).$$

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ $(\beta_j \notin Z_0^- = \{0, -1, -2, \ldots\}, j = 1, 2, \ldots, s)$,

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we now define the generalized hypergeometric function $_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by

$$_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)k}{(\beta_1)_k \cdots (\beta_s)k} \frac{z^k}{k!}$$

$(q \leqslant s+1; q, s \in N_0 \cup \{0\})$, where $(\theta)_\nu$ is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$(\theta)_\nu = \frac{\Gamma(\theta + \nu)}{\Gamma(\theta)} \begin{cases} 1, & \nu = 0; \ \theta \in C \setminus \{0\}, \\ \theta(\theta + 1) \cdots (\theta + \nu - 1), & \nu \in N, \ \theta \in c. \end{cases}$$

Corresponding to the function $h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by

$$h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-p} \ _qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

we consider a linear operator $H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \Sigma_p \longrightarrow \Sigma_p$, which is defined by the following Hadamard product (or convolution):

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) = h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z).$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) = z^{-p} + \sum_{k=1}^{\infty} \Gamma_k(\alpha_1)a_k z^{k-p}, \tag{1.2}$$

where

$$\Gamma_m(\alpha_1) = \frac{(\alpha_1)_m \cdots (\alpha_q)_m}{(\beta_1)_m \cdots (\beta_s)_m \ m!}, \quad m \in N. \tag{1.3}$$

If, for convenience, we write $H_{p,q,s}(\alpha_1) = H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)$, then one can easily verify from (1.2) that

$$z(H_{p,q,s}(\alpha_1))' = \alpha_1 H_{p,q,s}(\alpha_1 + 1)f(z) - (\alpha_1 + p)H_{p,q,s}(\alpha_1)f(z).$$

The linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [1] , Aouf [2] and Aouf and Yassen [3]. In particular, for $q = 2, s = 1$ and $\alpha_2 = 1$, we obtain the linear operator $H_{p,q,s}(\alpha_1, 1, \beta_1)f(z) = \ell_p(\alpha_1, \beta_1)f(z)$ which was introduced and studied by Liu and Srivastava [4]. We also note, for any integer $n > -p$ and for $f(z) \in \Sigma_p$ that

$$H_p(n + p, 1; 1)f(z) = I^{n+p-1}f(z) = \frac{(1 + z)^{n+p}}{z^p} * f(z), \quad n > -p; f(z) \in \Sigma_p,$$

where $I^{n+p-1}f(z)$ is the differential operator studied earlier by (among others) Aouf [5] and Aouf and Srivastava [6]. Note also, similar approach in getting the function $G_{[\alpha_1],\lambda}(z)$ was studied extensively by Aouf and Mostafa [7].

Now, for $f \in \Sigma_p; p \in N; 0 \leqslant \lambda < \frac{1}{2}$, let

$$G_{[\alpha_1],\lambda}(z) = G_{p,q,s,(\alpha_1),\lambda} = (1 - \lambda)H_{p,q,s}(\alpha_1)f(z) + \frac{\lambda}{p+1}z^2[\frac{1}{z}(H_{p,q,s}(\alpha_1)f(z))]'$$

so that,

$$G_{[\alpha_1],\lambda}(z) = \frac{1 - 2\lambda}{z^p} + \sum_{n=1}^{\infty}[1 - \lambda + \lambda(\frac{k - (p+1)}{p+1})]\Gamma_k(\alpha_1)a_k z^{k-p}, \quad p \in N; \ 0 \leqslant \lambda < \frac{1}{2}. \tag{1.4}$$

From (1.4), it is easily verified that

$$zG'_{[\alpha_1],\lambda}(z) = \alpha_1 G_{[\alpha_1+1],\lambda}(z) - (\alpha_1 + p)G_{[\alpha_1],\lambda}(z). \tag{1.5}$$

For fixed parameters A, B, p and λ with $-1 \leqslant B < A \leqslant 1, p \in N$ and $0 \leqslant \lambda < \frac{1}{2}$, we say that a function $f(z) \in \Sigma_p$ is in the class $\Omega_{[\alpha_1](A,B,\lambda)}$ of meromorphically p-valent functions in U^*, if the function $G_{[\alpha_1],\lambda}(z)$ defined by (1.4) satisfies the following inequality:

$$|\frac{z^{p+1}G'_{[\alpha_1],\lambda}(z) + p(1-2\lambda)}{Bz^{p+1}G'_{[\alpha_1],\lambda}(z) + Ap(1-2\lambda)}| < 1, \quad z \in U^*. \tag{1.6}$$

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} |a_k| z^k, \quad p \in N, \tag{1.7}$$

which are analytic and p- valent in U^*. Furthermore, we say that a function $f(z) \in \Omega_{[\alpha_1]}^+(A, B, \lambda)$ whenever $f(z)$ is of the form (1.1) and satisfies (1.6).

We have the following interesting relationships with some of the special function classes which were investigated recently:

(i) For $q = 2$ $(\alpha_1, \alpha_2 = 1)$, $s = 1$ $(\beta = 1)$ and $\lambda = 0$, we have $\Omega_{[p,2,1,\alpha_1,\beta_1]}(\alpha A, \alpha B, 0) = S_{\alpha_1,\beta_1}(A, B, \alpha)$, and $\Omega_{[p,2,1,\alpha_1,\beta_1]}^+(\alpha A, \alpha B, 0) = S_{\alpha_1,\beta_1}^*(A, B, \alpha)$ $(\alpha > 0, -1 \leqslant B < A \leqslant 1, -1 \leqslant B \leqslant 0$ and $|B\alpha| \leqslant 1)$ (see [8]);

(ii) For $q = 2$ $(\alpha_1, \alpha_2 = 1)$, $s = 1$ $(\beta_1 = 1)$ and $\lambda = 0$, where have $\Omega_{[p,2,1,\alpha_1]}^+(A, B, 0) = H^*(p; A, B)$ $(0 \leqslant B \leqslant 1; -B \leqslant A < B)$ (see [9]);

(iii) For $p = 1, q = 2$ $(\alpha_1, \alpha_2 = 1)$, $s = 1$ $(\beta_1 = 1)$, $A = (1-2\gamma\alpha)\beta$, $B = (1-2\gamma)\beta$ and $\lambda = 0$, we have $\Omega_{[1,2,1,\alpha_1]}^+((1 - 2\gamma\alpha)\beta, (1 - 2\gamma)\beta, 0) = \Sigma_d(\alpha, \beta, \gamma)$ $(0 \leqslant \alpha < 1; \frac{1}{2} \leqslant \gamma \leqslant 1, 0 < \beta \leqslant 1)$ (see [10]);

(iv) For $p = 1, q = 2$ $(\alpha_1, \alpha_2 = 1)$, $s = 1$ $(\beta_1 = 1)$ and $\lambda = 0$, where have $\Omega_{[1,2,1,\alpha_1]}^+(A, B, 0) = \Sigma_d(A, B)$ $(-1 \leqslant B < A \leqslant 1, -1 \leqslant B < 0)$ (see [11]).

Also we note that:

$$\Omega_{[\alpha_1]}^+((1 - 2\gamma\frac{\alpha}{p})\beta, (1 - 2\gamma)\beta, \lambda) = \Omega_{[\alpha_1]}^+(\alpha, \beta, \gamma)$$

$$= \{f(z) \in \Sigma_p^* : |\frac{z^{p+1}G'_{[\alpha_1],\lambda}(z) + p(1-2\lambda)}{(2\gamma - 1)z^{p+1}G'_{[\alpha_1],\lambda}(z) + (2\lambda\alpha - p)(1-2\lambda)}| < \beta\}$$

for $(z \in U^*; 0 \leqslant \alpha < p, p \in N; \frac{1}{2} \leqslant \gamma \leqslant 1; 0 < \beta \leqslant 1)$.

Meromorphically multivalent functions have been extensively studied by (for example) Mogra [9,12], Uralegaddi and Ganigi [13], Uralegaddi and Somanatha [14], Aouf [5,15,16], Srivastava et al. [17], Owa et al. [18], Joshi and Aouf [19], Joshi and Srivastava [20], Aouf et al. [21], Raina and Srivastava [22] and Yang [23,24].

In this paper we investigate the various important properties and characteristics of the classes $\Omega_{[\alpha_1]}(A, B, \lambda)$ and $\Omega_{[\alpha_1]}^+(A, B, \lambda)$. Following the recent investigations by Altintas et al. [25, p.1668], we extend the concept of neighborhoods of analytic functions, which was considered earlier by (for example) Goodman [26] and Ruscheweyh [27], to meromorphically multivalent

functions belonging to the classes $\Omega_{[\alpha_1]}(A, B, \lambda)$ and $\Omega^+_{[\alpha_1]}(A, B, \lambda)$. We also derive many results for the Hadamard products of functions belonging to the class $\Omega^+_{[\alpha_1]}(\alpha, \beta, \gamma, \lambda)$.

2. Inclusion properties of the class $\Omega_{[\alpha_1]}(A, B, \lambda)$

We begin by recalling the following result (Jack's lemma), which we shall apply in proving our first inclusion theorem (Theorem 2.2 below).

Lemma 2.1 ([28]) *Let the (nonconstant) function $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then $z_0 w'(z_0 = \xi w(z_0)$, where ξ is a real number and $\xi \geq 1$.*

Theorem 2.2 *The following inclusion property holds true for the $\Omega_{[\alpha+1]}(A, B, \lambda), \alpha_1 > 0$:*

$$\Omega_{[\alpha+1]}(A, B, \lambda) \subset \Omega_{[\alpha_1]}(A, B, \lambda).$$

Proof Let $f(z) \in \Omega_{[\alpha+1]}(A, B, \lambda)$ and suppose that

$$z^{p+1} G'_{[\alpha_1],\lambda}(z) = -p(1 - 2\lambda)\frac{1 + Aw(z)}{1 + Bw(z)}, \tag{2.1}$$

where the function $w(z)$ is analytic in U and $w(0) = 0$. Then by using (1.5) and (2.1), we have

$$z^{p+1} G'_{[\alpha_1],\lambda}(z) = -p(1 - 2\lambda)\frac{1 + Aw(z)}{1 + Bw(z)} - \frac{p(1 - 2\lambda)}{\alpha_1} \cdot \frac{(A - B)zw'(z)}{(1 + Bw(z))^2}. \tag{2.2}$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwise there exist a point $z_0 \in U$ such that $\max_{|z| \leqslant |z_0|} |w(z)| = |w(z_0)| = 1$. Applying Jack's lemma, we have $z_0 w'(z_0) = \xi w(z_0)$ ($\xi \geqslant 1$). Writing $w(z_0) = e^{i\theta}$ ($0 \leqslant \theta \leqslant 2\pi$) and putting $z = z_0$ in (2.2), we get

$$|\frac{z_0^{p+1} G'_{[\alpha_1],\lambda}(z_0) + p(1 - 2\lambda)}{Bz_0^{p+1} G'_{[\alpha_1],\lambda}(z_0) + Ap(1 - 2\lambda)}|^2 - 1$$

$$= \frac{|\alpha_1 + \xi + \alpha_1 Be^{i\theta}|^2 - |\alpha_1 + B(\alpha_1 - \xi)e^{i\theta}|^2}{|\alpha_1 + B(\alpha_1 - \xi)e^{i\theta}|^2}$$

$$= \frac{\xi^2(1 - B^2) + 2\alpha_1 \xi(1 + B^2 + 2B\cos\theta)}{|\alpha_1 + B(\alpha_1 - \xi)e^{i\theta}|^2}, \tag{2.3}$$

which obviously contradicts our hypothesis that $f(z) \in \Omega_{[\alpha_1+1]}(A, B, \lambda)$. Thus we must have $|w(z)| < 1$ ($z \in U$), and so from (2.3), we conclude that $f(z) \in \Omega_{[\alpha_1]}(A, B, \lambda)$, which evidently completes the proof of Theorem 2.2. $\square$

Theorem 2.3 *Let μ be a complex number such that $\mathrm{Re}(\mu) > 0$. If $f(z) \in \Omega_{[\alpha_1]}(A, B, \lambda)$, then the function*

$$F_{[\alpha_1],\lambda}(z) = \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} G_{[\alpha_1],\lambda}(t)\mathrm{d}t, \tag{2.4}$$

is also in the same class $\Omega_{[\alpha_1]}(A, B, \lambda)$.

Proof From (2.4), we have

$$zF'_{[\alpha_1],\lambda}(z) = \mu G_{[\alpha_1],\lambda}(z) + (\mu + p)F_{[\alpha_1,\lambda]}(z). \tag{2.5}$$

Put

$$z^{p+1}F'_{[\alpha_1],\lambda}(z) = -p(1-2\lambda)\frac{1+Aw(z)}{1+Bw(z)}, \tag{2.6}$$

where $w(z)$ is analytic in U and $w(0) = 0$. Then, by using (2.5) and (2.6), we have

$$z^{p+1}G'_{[\alpha_1],\lambda}(z) = -p(1-2\lambda)\frac{1+Aw(z)}{1+Bw(z)} - \frac{p(1-2\lambda)(A-B)}{\mu}\cdot\frac{zw'(z)}{(1+Bw(z))^2}.$$

The remaining part of the proof is similar to that of Theorem 2.2 and so is omitted. $\square$

Theorem 2.4 $f(z) \in \Omega_{[\alpha_1]}(A, B, \lambda)$, if and only if

$$F_{[\alpha_1],\lambda}(z) = \frac{\alpha_1}{z^{\alpha_1+p}}\int_0^z t^{\alpha_1+p-1}f(t)\mathrm{d}t \in \Omega_{[\alpha_1+1]}(A, B, \lambda). \tag{2.7}$$

Proof In view of the definition of $F_{\alpha_1,\lambda}(z)$, we have

$$\alpha_1 f(z) = (\alpha_1 + p)F_{[\alpha_1],\lambda}(z) + zF'_{[\alpha_1],\lambda}(z). \tag{2.8}$$

By using (1.5) and (2.7), we have

$$\alpha_1 G_{\alpha_1,\lambda}f(z) = (\alpha_1 + p)G_{\alpha_1,\lambda}F_{\alpha_1,\lambda}(z) + z(G_{\alpha_1,\lambda}F_{\alpha_1,\lambda}(z))'.$$

The desired result follows immediately. $\square$

3. Properties of the class $\Omega^+_{[\alpha_1]}(A, B, \lambda)$

In the rest of the paper we assume further that $\alpha_j > 0$ $(j = 1,\ldots,q)$, $\beta_j > 0$ $(j = 1, l\ldots,s)$, $-1 \leqslant A < B \leqslant 1$, $-1 \leqslant B \leqslant 0$, $0 \leqslant \lambda < \frac{1}{2}$ and $p \in N$.

Theorem 3.1 Let $f(z) \in \Sigma^*_p$ be given by (1.7). Then $f(z) \in \Omega^+_{[\alpha_1]}(A, B, \lambda)$ if and only if

$$\sum_{k=p}^{\infty} k[1 + \lambda(\frac{k-(p+1)}{p+1})](1-B)\Gamma_{k+p}(\alpha_1)\,|a_k| \leqslant (A-B)p(1-2\lambda), \tag{3.1}$$

where Γ_m is given by (1.3).

Proof Let $f(z) \in \Omega^+_{[\alpha_1]}(A, B, \lambda)$ be given by (1.7). Then, from (1.6) and (1.7), we have

$$\left|\frac{z^{p+1}G'_{[\alpha_1+1],\lambda}(z) + p(1-2\lambda)}{Bz^{p+1}G'_{[\alpha_1+1],\lambda}(z) + Ap(1-2\lambda)}\right|$$

$$= \left|\frac{\sum_{k=p}^{\infty}k[1+\lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1)\,|a_k|\,z^{k+p}}{(A-B)p(1-2\lambda) + \sum_{k=p}^{\infty}Bk[1+\lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1)\,|a_k|\,z^{k+p}}\right| < 1, \quad z \in U.$$

Since $\mathrm{Re}(z) \leqslant |z|$ $(z \in C)$, we have

$$\mathrm{Re}\Big\{\frac{\sum_{k=p}^{\infty}k[1+\lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1)\,|a_k|}{(A-B)p(1-2\lambda) + \sum_{k=p}^{\infty}Bk[1+\lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1)\,|a_k|}\Big\} < 1. \tag{3.2}$$

Choose values of z on the real axis so that $z^{p+1}G'_{[\alpha_1+1],\lambda}(z)$ is real. Upon clearing the denominator in (3.1) and letting $z \to 1^-$ through real values we obtain (2.8). In order to prove the converse,

we assume that the inequality (2.8) holds true. Then, if we let $z \in \partial U$, we find from (1.7) and (2.8) that

$$\left| \frac{z^{p+1}G'_{[\alpha_1+1],\lambda}(z) + p(1-2\lambda)}{Bz^{p+1}G'_{[\alpha_1+1],\lambda}(z) + Ap(1-2\lambda)} \right|$$

$$\leqslant \frac{\sum_{k=p}^{\infty} k[1 + \lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1)\,|a_k|}{(A-B)p(1-2\lambda) + \sum_{k=p}^{\infty} Bk[1 + \lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1)\,|a_k|}, \quad z \in U$$

$$< 1, \quad z \in \partial U = \{z : z \in C \text{ and } |z| = 1\}. \tag{3.3}$$

Hence, by the maximum modulus theorem, we have $f(z) \in \Omega^+_{[\alpha_1]}$. This completes the proof of Theorem 3.1. □

Corollary 3.2 *If the function $f(z)$ defined by (1.7) is in the class $\Omega^+_{[\alpha_1]}(A, B, \lambda)$, then*

$$a_n \leq \frac{(A-B)p(1-2\lambda)}{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1-B)\Gamma_{k+p}(\alpha_1)}, \quad k \geqslant p;\ p \in N$$

with equality for the functions

$$f(z) = z^{-p} + \frac{(A-B)p(1-2\lambda)}{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1-B)\Gamma_{k+p}(\alpha_1)}, \quad k \geqslant p;\ p \in N. \tag{3.4}$$

Putting $A = (1 - 2\gamma\frac{\alpha}{p})$ and $B = (1 - 2\gamma)\beta$ $(0 \leqslant \alpha < p, 0 < \beta \leqslant 1, \frac{1}{2} \leqslant \gamma \leqslant 1$ and $p \in N)$ in Theorem 3.1.

Corollary 3.3 *A function $f(z)$ defined by (1.7) is in the class $\Omega^+_{[\alpha_1]}(\alpha, \beta, \gamma, \lambda)$ if and only*

$$\sum_{k=p}^{\infty} k[1 + \lambda(\frac{k-(p+1)}{p+1})](1 + 2\beta\gamma - \beta)\Gamma_{k+p}(\alpha_1)\,|a_k| \leqslant 2\beta\gamma(p-\alpha)(1-2\lambda).$$

The following property is an easy consequence of Theorem 3.1.

Theorem 3.4 *Let each of the function $f_j(z)$ defined by*

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,j}|\, z^k, \quad j = 1, 2, \ldots, m \tag{3.5}$$

be in the class $\Omega^+_{[\alpha_1]}(A, B, \lambda)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^{m} \zeta_j f_j(z), \quad \zeta_j \geqslant 0 \text{ and } \sum_{j=1}^{m} \zeta_j = 1$$

is also in the class $\Omega^+_{[\alpha_1]}(A, B, \lambda)$.

Next we prove the following growth and distortion properties for the class $\Omega^+_{[\alpha_1]}(A, B, \lambda)$.

Theorem 3.5 *If a function $f(z)$ defined by (1.7) is in the class $f(z) \in \Omega^+_{[\alpha_1]}(A, B, \lambda)$ and the sequence $\{C_k\} = \{k[1 + \lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1)\}$ $(k \geqslant p; p \in N; 0 \leqslant \lambda < \frac{1}{2})$ is nondecreasing, then*

$$\left\{ \frac{(p+m-1)!}{(p-1)!} - \frac{(A-B)(1-2\lambda)}{(1-B)C_p} \cdot \frac{p!}{(p-m)!} r^{2p} \right\} r^{-(p+m)}$$

$$\leqslant |f^{(m)(z)}| \leqslant \left\{ \frac{(p+m-1)!}{(p-1)!} + \frac{(A-B)(1-2\lambda)}{(1-B)C_p} \cdot \frac{p!}{(p-m)!} r^{2p} \right\} \tag{3.6}$$

$(0 < |z| = r < 1; 0 \leqslant \lambda < p; p \in N; m \in N_0 = N \bigcup \{0\}; p > m)$, where $\Gamma_m(\alpha_1)$ is given by (1.3). The result is sharp for the functions $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(1-2\lambda)}{(1-B)\Gamma_{2p}(\alpha_1)} z^p, \quad p \in N. \tag{3.7}$$

Proof In view of Theorem 3.1, we have

$$\Gamma_{2p}(\alpha_1) \frac{p}{p!} \sum_{k=p}^{\infty} k! |a_k| \leqslant \sum_{k=p}^{\infty} k[1 + \lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1) |a_k| \leqslant \frac{(A-B)p(1-2\lambda)}{(1-B)},$$

which tields

$$\sum_{k=p}^{\infty} k! |a_k| \leqslant \frac{(A-B)p(1-2\lambda)}{(1-B)\Gamma_{2p}(\alpha_1)}, \quad p \in N. \tag{3.8}$$

Now, by differentiating both sides of (1.7) m times with respect to z, we have

$$f^{(m)}(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-(p+m)} + \sum_{k=p}^{\infty} \frac{k!}{(k-m)!} |a_k| z^{k-m}, \tag{3.9}$$

$m \in N_0; p \in N; p > m$ and Theorem 3.5 follows easily from (3.6) and (3.7). Finally, it is easy to see that the bounds in (3.4) are attained for the function $f(z)$ given by (3.5). $\square$

Next we determine the radii of meromorphically p-valent starlikeness of order δ and meromorphically p-valent convexity of order δ ($0 \leqslant \delta < p$) for functions in the class $\Omega^+_{[\alpha_1]}(A, B, \lambda)$.

Theorem 3.6 *Let the function $f(z)$ defined by (1.7) is in the class $\Omega^+_{[\alpha_1]}(A, B, \lambda)$. Then we have:*

(i) $f(z)$ is meromorphically p-valent starlike of order δ ($0 \leqslant \delta < p$), $p \in N$ in the disk $|z| < r_1$, that is,

$$\mathrm{Re}\{-\frac{zf'(z)}{f(z)}\} > \delta, \quad |z| < r_1,$$

where

$$r_1 = \inf_{k \geqslant p} \left\{ \frac{(p-\delta)k[1 + \lambda(\frac{k-(p+1)}{p+1})](1-B)\Gamma_{k+p}(\alpha_1)}{(k+\delta)(A-B)p(1-2\lambda)} \right\}^{\frac{1}{k+p}}. \tag{3.10}$$

(ii) $f(z)$ is meromorphically p-valent convex of order δ ($0 \leqslant \delta < p$), $p \in N$ in the disk $|z| < r_2$, that is,

$$\mathrm{Re}\{-(1 + \frac{zf''(z)}{f'(z)})\} > \delta, \quad |z| < r_2,$$

where

$$r_2 = \inf_{k \geqslant p} \left\{ \frac{(p-\delta)[1 + \lambda(\frac{k-(p+1)}{p+1})](1-B)\Gamma_{k+p}(\alpha_1)}{(k+\delta)(A-B)p(1-2\lambda)} \right\}^{\frac{1}{k+p}}. \tag{3.11}$$

Each of these results is sharp for the function $f(z)$ given by (3.2).

Proof (i) From Eq. (1.7), we easily get

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{\frac{zf'(z)}{f(z)} - p + 2\delta} \right| \leqslant \frac{\sum_{k=p}^{\infty}(k+p)|a_k||z|^{k+p}}{2(p-\delta) - \sum_{k=p}^{\infty}(k-p+2\delta)|a_k||z|^{k+p}}.$$

Thus, we have the desired inequality

$$\Big|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\Big| \leqslant 1, \quad 0\leqslant \delta < p; \ p\in N.$$

If

$$\sum_{k=p}^{\infty}(\frac{k+\delta}{p-\delta})|a_k||z|^{k+p} \leqslant 1, \tag{3.12}$$

by Theorem 3.1, (3.10) will be true if

$$(\frac{k+\delta}{p-\delta})|z|^{k+p} \leqslant \Big\{\frac{k[1+\lambda(\frac{k-(p+1)}{p+1})](1-B)\Gamma_{k+p}(\alpha_1)}{(A-B)p(1-2\lambda)}\Big\}, \quad k\geqslant p; \ p\in N. \tag{3.13}$$

The last inequality (3.11) leads us immediately to the disk $|z| < r_1$, where r_1 is given by (3.8).

(ii) In order to prove the second assertion of Theorem 3.6, we find from the definition (1.7) that

$$\Big|\frac{1+\frac{zf''(z)}{f'(z)}+p}{\frac{1+zf''(z)}{f'(z)}-p+2\delta}\Big| \leqslant \frac{\sum_{k=p}^{\infty}k(k+p)|a_k||z|^{k+p}}{2p(p-\delta)-\sum_{k=p}^{\infty}k(k-p+2\delta)|a_k||z|^{k+p}}.$$

Thus, we have the desired inequality

$$\Big|\frac{1+\frac{zf''(z)}{f'(z)}+p}{\frac{1+zf''(z)}{f'(z)}-p+2\delta}\Big| \leqslant 1, \quad 0\leqslant \delta < p; \ p\in N.$$

If

$$\sum_{k=p}^{\infty}(\frac{k(k+\delta)}{p(p-\delta)})|a_k||z|^{k+p} \leqslant 1, \tag{3.14}$$

by Theorem 3.1, (3.10) will be true if

$$(\frac{k(k+\delta)}{p(p-\delta)})|z|^{k+p} \leqslant \Big\{\frac{k[1+\lambda(\frac{k-(p+1)}{p+1})](1-B)\Gamma_{k+p}(\alpha_1)}{(A-B)p(1-2\lambda)}\Big\}, \quad k\geqslant p; \ p\in N. \tag{3.15}$$

The last inequality (3.13) readily yields the disk $|z| < r_2$, where r_2 is given by (3.15). $\square$

4. Neighborhoods

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [26] and Ruscheweyh [27], and (more recently) by Altintas et al. [25,29,30], Liu [8], and Liu and Srivastava [1], we begin by introducing here the δ-neighborhood of a function $f(z)\in\Sigma_p$ of the form (1.1) by means of the definition given below:

$$N_\delta(f) = \Big\{g\in\Sigma_p: g(z) = z^{-p}+\sum_{k=1}^{\infty}b_kz^{k-p} \ \text{and}$$

$$\sum_{k=1}^{\infty}\frac{(1+|B|)(k+p)[1-\lambda+\lambda(\frac{k-(p+1)}{p+1})]}{(A-B)p(1-2\lambda)}|b_k-a_k| \leqslant \delta\Big\} \tag{4.1}$$

$$-1\leqslant B < A\leqslant 1; \ p\in N; \ 0\leqslant \lambda < \frac{1}{2}; \ \delta > 0.$$

Making use of the definition (3.15), we now prove Theorem 4.1 below

Theorem 4.1 *Let the function $f(z)$ defined by (1.1) be in the class $\Omega_{[\alpha_1]}(A, B, \lambda)$. If $f(z)$ satisfies the following condition:*

$$\frac{f(z) + \epsilon z^{-p}}{1 + \epsilon} \in \Omega_{[\alpha_1]}(A, B, \lambda), \quad \epsilon \in C; \ |\epsilon| < \delta; \ \delta > 0, \tag{4.2}$$

then

$$N_\delta(f) \subset \Omega_{[\alpha_1]}(A, B, \lambda). \tag{4.3}$$

Proof It is easily seen from (1.6) that $g(z) \in \Omega_{[\alpha_1]}(A, B, \lambda)$ if and only if for any complex number σ with $|\sigma| = 1$,

$$\frac{z^{p+1}G'_{[\alpha_1],\lambda}(z) + p(1 - 2\lambda)}{Bz^{p+1}G'_{[\alpha_1],\lambda}(z) + Ap(1 - 2\lambda)} \neq \sigma, \quad z \in U, \tag{4.4}$$

which is equivalent to

$$\frac{(g * h)(z)}{z^{-p}} \neq 0, \quad z \in U, \tag{4.5}$$

where, for convenience,

$$h(z) = z^{-p} + \sum_{k=1}^{\infty} c_k z^{k-p} = z^{-p} + \sum_{k=1}^{\infty} \frac{(1 - \sigma B)(k - p)[1 - \lambda + \lambda(\frac{k-(p+1)}{p+1})]\Gamma_k(\alpha_1)}{(B - A)p(1 - 2\lambda)\sigma} z^{k-p}. \tag{4.6}$$

From (4.5), we have

$$|c_k| = \left| \frac{(1 - \sigma B)(k - p)[1 - \lambda + \lambda(\frac{k-(p+1)}{p+1})]\Gamma_k(\alpha_1)}{(B - A)p(1 - 2\lambda)\sigma} \right|$$

$$\leqslant \frac{(1 + |B|)(k + p)[1 - \lambda + \lambda(\frac{k-(p+1)}{p+1})]\Gamma_k(\alpha_1)}{(A - B)p(1 - 2\lambda)}, \quad k, p \in N; \ 0 \leqslant \lambda < \frac{1}{2}.$$

Now, if $f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \in \Sigma_p$ satisfies the condition (4.1), then (4.4) yields

$$\left| \frac{(f * h)(z)}{z^{-p}} \right| \geqslant \delta, \quad z \in U; \ \delta > 0.$$

By letting $g(z) = z^{-p} + \sum_{k=1}^{\infty} b_k z^{k-p} \in N_\delta(f)$, we get that

$$\left| \frac{[g(z) - f(z)] * h(z)}{z^{-p}} \right| = \left| \sum_{k=1}^{\infty} (b_k - a_k)c_k z^k \right|$$

$$\leqslant |z| \sum_{k=1}^{\infty} \frac{(1 + |B|)(k + p)[1 - \lambda + \lambda(\frac{k-(p+1)}{p+1})]\Gamma_k(\alpha_1)}{(A - B)p(1 - 2\lambda)} |b_k - a_k| < \delta, \quad z \in U; \ \delta > 0.$$

Then we have (4.4), and hence also (4.3) for any $\sigma \in C$ such that $|\sigma| = 1$, which implies that $g(z) \in \Omega_{[\alpha_1]}(A, B, \lambda)$. This evidently proves the assertion (4.3) of Theorem 4.1. $\square$

We now define the δ-neighborhood of a function $f(z) \in \Sigma_p$ of the form (1.7) as follows

$$N_\delta^+(f) = \left\{ g \in \Sigma_p^* : g(z) = z^{-p} + \sum_{k=p}^{\infty} |b_k| z^k \text{ and} \right.$$

$$\sum_{k=1}^{\infty} \frac{(1+|B|)k[1-\lambda+\lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1)}{(A-B)p(1-2\lambda)}\|b_k| - |a_k\| \leqslant \delta\Big\},$$

$$-1 \leqslant B < A \leqslant 1; \ p \in N; \ 0 \leqslant \lambda < \frac{1}{2}; \ \delta > 0.$$

Theorem 4.2 *Let the function $f(z)$ defined by (1.7) be in the class $\Omega^+_{[\alpha_1+1]}(A,B,\lambda)$ ($-1 \leqslant B < A \leqslant 1, -1 \leqslant B \leqslant 0, P \in N$ and) $0 \leqslant \lambda < \frac{1}{2}$. Then*

$$N^+_\delta(f) \subset \Omega^+_{[\alpha_1]}(A,B,\lambda), \quad \delta = \frac{2p}{\alpha_1+2p}.$$

The result is sharp in the sense that δ cannot be increased.

Proof Making use the same method as in the proof of Theorem 4.1, we can show that (4.6)

$$h(z) = z^{-p} + \sum_{k=p}^{\infty} c_k z^k = z^{-p} + \sum_{k=p}^{\infty} \frac{(1-\sigma B)k[1-\lambda+\lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1)}{(B-A)p(1-2\lambda)\sigma} z^k.$$

Thus, under the hyppothesis $-1 \leqslant B < A \leqslant 1, -1 \leqslant B \leqslant 0, P \in N$ and $0 \leqslant \lambda < \frac{1}{2}$, $f(z) \in \Omega^+_{[\alpha_1+1]}(A,B,\lambda)$ is given by (1.7), we obtain

$$\Big|\frac{(f*h)(z)}{z^{-p}}\Big| = \Big|1 + \sum_{k=p}^{\infty} c_k|a_k|z^{k+p}\Big|$$

$$\geqslant 1 - \frac{\alpha_1}{\alpha_1+2p}\sum_{k=p}^{\infty} \frac{(1-B)k[1-\lambda+\lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1+1)}{(A-B)p(1-2\lambda)}|a_k|.$$

Also, from Theorem 3.1, we obtain

$$\Big|\frac{(f*h)(z)}{z^{-p}}\Big| \geqslant 1 - \frac{\alpha_1}{\alpha_1+2p} = \frac{2p}{\alpha_1+2p} = \delta.$$

The remaining part of thr proof of Theorem 4.2 is similar to that of Theorem 4.1, and we skip the details involved. To show the sharpness, we consider the function $f(z)$ anf $g(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(1-2\lambda)}{(1-B)\Gamma_{2p}(\alpha_1+1)}z^p \in \Omega^+_{[\alpha_1+1]}(A,B,\lambda)$$

and

$$g(z) = z^{-p} + \Big[\frac{(A-B)(1-2\lambda)}{(1-B)\Gamma_{2p}(\alpha_1+1)} + \frac{(A-B)(1-2\lambda)\delta'}{(1-B)\Gamma_{2p}(\alpha_1+1)}\Big]z^p,$$

where $\delta' > \delta = \frac{2p}{\alpha_1+2p}$. Clearly, the function $g(z)$ belongs to $N^+_{\delta'}(A,B,\lambda)$. Thus the proof of Theorem 4.2 is completed. □

Theorem 4.3 *Let $f(z) \in \Sigma_p$ be given by (1.1) and define the partial sums $s_1(z)$ and $s_n(z)$ as*

$$s_1(z) = z^{-p} \text{ and } s_n(z) = z^{-p} + \sum_{k=1}^{n-1} a_k z^{k-p}, \quad n \in N\{1\}.$$

Suppose also that

$$\sum_{k=1}^{\infty} d_k|a_k| \leqslant 1, \quad d_k = \frac{(1+|B|)(k+p)[1-\lambda+\lambda(\frac{k-(p+1)}{p+1})]\Gamma_k(\alpha_1)}{(A-B)p(1-2\lambda)}. \tag{4.7}$$

Then:

    (i) $f(z) \in \Omega_{[\alpha_1]}(A, B, \lambda)$.

    (ii) *If* $\{\Gamma_k(\alpha_1)\}$ $(k \in N)$ *is nondecreasing and*

$$\Gamma_1(\alpha_1) > \frac{(A - B)p(1 - 2\lambda)}{(1 + |B|)(1 + p)[1 - \lambda + \lambda(\frac{k-(p+1)}{p+1})]},$$

then

$$\text{Re}\{\frac{f(z)}{s_n(z)}\} > 1 - \frac{1}{d_n}, \quad z \in U; \ n \in N, \tag{4.8}$$

and

$$\text{Re}\{\frac{s_n(z)}{f(z)}\} > \frac{d_n}{1 + d_n}, \quad z \in U; \ n \in N. \tag{4.9}$$

Each of the bounds in (4.7) and (4.8) is the best possible for each $n \in N$.

Proof (i) It is not difficult to see that $z^{-p} \in \Omega_{[\alpha_1]}(A, B, \lambda)$ $(p \in N)$. Thus, from Theorem 4.1 and the hypothesis (4.6), we have $N_1(z^{-p}) \subset \Omega_{[\alpha_1]}(A, B, \lambda)$ as asserted by Theorem 4.1.

    (ii) Under the hypothesis in Part (ii) of Theorem 4.3, we can see from (4.6) that $d_{k+1} > d_k > 1$, $k \in N$. Therefore, we have

$$\sum_{k=1}^{n-1} |a_k| + d_n \sum_{k=n}^{\infty} |a_k| \leqslant \sum_{k=1}^{\infty} d_k |a_k| \leqslant 1, \tag{4.10}$$

by using hypothesis (4.6) again. By setting

$$g_1(z) = d_n[\frac{f(z)}{s_n(z)} - (1 - \frac{1}{d_n})] = 1 + \frac{d_n \sum_{k=n}^{\infty} a_k z^k}{1 + \sum_{k=1}^{n-1} a_k z^k},$$

and appling (4.9), we find that

$$|\frac{g_1(z) - 1}{g_1(z) + 1}| \leqslant \frac{d_n \sum_{k=n}^{\infty} |a_k|}{2 - 2\sum_{k=1}^{n-1} |a_k| - d_n \sum_{k=n}^{\infty} |a_k|} \leqslant 1, \quad z \in U,$$

which readily yields the assertion (4.7). If we take

$$f(z) = z^{-p} - \frac{z^{n-p}}{d_n}, \tag{4.11}$$

then

$$\frac{f(z)}{s_n(z)} = 1 - \frac{z^n}{d_n} \longrightarrow 1 - \frac{1}{d_n}, \quad z \longrightarrow 1^-,$$

which shows that bound in (4.7) is the best possible for each $n \in N$. Similary, if we put

$$g_2(z) = (1 + d_n)(\frac{s_n(z)}{f(z)} - \frac{d_n}{1 + d_n}) = 1 - \frac{(1 + d_n)\sum_{k=n}^{\infty} a_k z^k}{1 + \sum a_k^{z^k}},$$

and making use of (4.9), we can deduce that

$$|\frac{g_2(z) - 1}{g_2(z) + 1}| \leqslant \frac{(1 + d_n)\sum_{k=n}^{\infty} |a_k|}{2 - 2\sum_{k=1}^{n-1} |a_k| + (1 - d_n)\sum_{k=n}^{\infty} |a_k|} \leqslant 1, \quad z \in U,$$

which leads immediately to assertion (4.10). The bound in (4.8) is sharp for each $n \in N$, with the extremal function $f(z)$ given by (4.10). The proof of Theorem 4.3 is thus completed. $\square$

5. Convolution properties for the class $\Omega^+_{[\alpha_1]}(\alpha, \beta, \gamma, \lambda)$

For the function $f_j(z)$ $(j = 1, 2)$ defined by (3.3) we denote by $(f_1 * f_1)(z)$ the Hadmard product (or convolution) of the function $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_1)(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,1}| \, |a_{k,2}| \, z^k.$$

Throughout this section, we assume further that the sequence $\{k[1 + \lambda(\frac{k-(p+1)}{p+1})]\Gamma_{k+p}(\alpha_1)\}$ $(k \geqslant p; p \in N, 0 \leqslant \lambda < \frac{1}{2})$ is nondecreasing.

Theorem 5.1 *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (3.5) be in the class $\Omega^+_{[\alpha_1]}(\alpha, \beta, \gamma, \lambda)$. Then $(f_1 * f_2)(z) \in \Omega^+_{[\alpha_1]}(\zeta, \beta, \gamma, \lambda)$, where*

$$\zeta = p - \frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z^{-p} + \frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)} z^p, \quad j = 1, 2; \ p \in N. \tag{5.1}$$

Proof Employing the technique used earlier by Schild and Silverman [22], we need to find the largest ζ such that

$$\sum_{k=p}^{\infty} \frac{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1+2\beta\gamma-\beta)\Gamma_{k+p}(\alpha_1)}{2\beta\gamma(p-\zeta)(1-2\lambda)} |a_{k,1}| \, |a_{k,2}| \leqslant 1$$

for $f_j(z) \in \Omega^+_{[\alpha_1]}(\alpha, \beta, \gamma, \lambda)$ $(j = 1, 2)$. Since $f_j(z) \in \Omega^+_{[\alpha_1]}(\alpha, \beta, \gamma, \lambda)$ $(j = 1, 2)$, we readily see that

$$\sum_{k=p}^{\infty} \frac{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1+2\beta\gamma-\beta)\Gamma_{k+p}(\alpha_1)}{2\beta\gamma(p-\alpha)(1-2\lambda)} |a_{k,j}| \leqslant 1, \quad j = 1, 2.$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \frac{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1+2\beta\gamma-\beta)\Gamma_{k+p}(\alpha_1)}{2\beta\gamma(p-\alpha)(1-2\lambda)} \sqrt{|a_{k,1}| \, |a_{k,2}|} \leqslant 1. \tag{5.2}$$

This implies that we only need to show that

$$\frac{1}{(p-\zeta)} \cdot |a_{k,1}| \, |a_{k,2}| \leqslant \frac{1}{(p-\alpha)} \cdot \sqrt{|a_{k,1}| \, |a_{k,2}|}, \quad k \geqslant p,$$

or, equivalently, that $\sqrt{|a_{k,1}||a_{k,2}|} \leqslant \frac{p-\zeta}{p-\alpha}$ $(k \geqslant p)$. Hence, by the inequality (5.1), it is sufficient to prove that

$$\frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1+2\beta\gamma-\beta)\Gamma_{k+p}(\alpha_1)} \leqslant \frac{p-\zeta}{p-\alpha}, \quad k \geqslant p. \tag{5.3}$$

It follows from (5.2) that

$$\zeta \leqslant p - \frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1+2\beta\gamma-\beta)\Gamma_{k+p}(\alpha_1)}, \quad k \geqslant p.$$

Now, defining the function $\Phi(k)$ by

$$\Phi(z) = p - \frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{k[1+\lambda(\frac{k-(p+1)}{p+1})](1+2\beta\gamma-\beta)\Gamma_{k+p}(\alpha_1)}, \quad k \geqslant p,$$

we have

$$\Phi(k+1) - \Phi(z) = \frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{(1+2\beta\gamma-\beta)}\Gamma_{k+p}(\alpha_1) \times$$

$$\{\frac{(k+1)(a+k+p)[1+\lambda(\frac{k+1-(p+1)}{p+1})] - k(c+k+p)[1+\lambda](\frac{k-(p+1)}{p+1})}{k(k+1)(a+k+p)[1+\lambda(\frac{k-(p+1)}{p+1})][1+\lambda](\frac{k+1-(p+1)}{p+1})}\} > 0,$$

that is, $\Phi(k)$ is an increasing function of k $(k \geqslant p)$. Therefore, we conclude that

$$\zeta \leqslant \Phi(p) = p - \frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{p(1+2\beta\gamma-\beta)\Gamma_{k+p}(\alpha_1)},$$

which evidently completes the proof of Theorem 5.1. $\square$

Using arguments similar to these in the proof of Theorem 5.1, we obtain the following result.

Theorem 5.2 *Let the functions $f_1(z)$ defined by (3.3) be in the class $\Omega^+_{[\alpha_1]}(\alpha, \beta, \gamma, \lambda)$. Suppose also that the function $f_2(z)$ defined (3.3) is in the class $\in \Omega^+_{[\alpha_1]}(\theta, \beta, \gamma, \lambda)$. Then $(f_1 * f_2)(z) \in \Omega^+_{[\alpha_1]}(\tau, \beta, \gamma, \lambda)$ where*

$$\tau = p - \frac{2\beta\gamma(p-\alpha)(1-2\lambda)}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_1(z) = z^{-p} + \frac{2\beta\gamma(p-\alpha)(1-2\lambda)}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)}z^p, \quad p \in N,$$

$$f_1(z) = z^{-p} + \frac{2\beta\gamma(p-\theta)(1-2\lambda)}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)}z^p, \quad p \in N.$$

Theorem 5.3 *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (3.3) be in the class $\Omega^+_{[\alpha_1]}(\alpha, \beta, \gamma, \lambda)$. Then the function $h(z)$ defined by*

$$h(z) = z^{-p} + \sum_{k=p}^{\infty}(|a_{k,1}|^2 + |a_{k,2}|^2)z^k$$

belongs to the class $\Omega^+_{[\alpha_1]}(\varphi, \beta, \gamma, \lambda)$, where

$$\varphi = p - \frac{4\beta\gamma(p-\alpha)^2(1-2\lambda)}{p(1+2\beta\gamma-\beta)\Gamma_{2p}(\alpha_1)}.$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ defined by (4.11).

Proof Noting that

$$\Big(\sum_{k=p}^{\infty} \frac{k[1+\lambda(\frac{k-(p+1)}{p+1})](1+2\beta\gamma-\beta)\Gamma_{k+p}(\alpha_1)}{2\beta\gamma(p-\alpha)(1-2\lambda)}\Big)^2 |a_{k,j}|^2$$

$$\leqslant \Big(\sum_{k=p}^{\infty} \frac{k[1+\lambda(\frac{k-(p+1)}{p+1})](1+2\beta\gamma-\beta)\Gamma_{k+p}(\alpha_1)}{2\beta\gamma(p-\alpha)(1-2\lambda)} |a_{k,j}|\Big)^2 \leqslant 1, \quad j = 1, 2,$$

for $f_j(z) \in \Omega^+_{[\alpha_1]}(p, \alpha, \beta, \gamma, \lambda)$ $(j = 1, 2)$, we have

$$\sum_{k=p}^{\infty} \frac{1}{2} \Big[\frac{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1 + 2\beta\gamma - \beta)\Gamma_{k+p}(\alpha_1)}{2\beta\gamma(p-\alpha)(1-2\lambda)} \Big]^2 (|a_{k,1}|^2 + |a_{k,2}|^2) \leqslant 1.$$

Therefore, we have to find the largest φ such that

$$\frac{1}{(p-\varphi)} \leqslant \frac{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1 + 2\beta\gamma - \beta)\Gamma_{k+p}(\alpha_1)}{4\beta\gamma(p-\alpha)^2(1-2\lambda)}, \quad k \geqslant p, \qquad (5.4)$$

that is,

$$\varphi \leqslant p - \frac{4\beta\gamma(p-\alpha)^2(1-2\lambda)}{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1 + 2\beta\gamma - \beta)\Gamma_{k+p}(\alpha_1)}, \quad k \geqslant p.$$

Now, defining the function $\Psi(k)$ by

$$\Psi(z) = p - \frac{4\beta\gamma(p-\alpha)^2(1-2\lambda)}{k[1 + \lambda(\frac{k-(p+1)}{p+1})](1 + 2\beta\gamma - \beta)\Gamma_{k+p}(\alpha_1)}, \quad k \geqslant p,$$

we observe that $\Psi(z)$ is an increasing function of k $(k \geqslant p)$. Thus, we conclude that

$$\varphi \leqslant \Psi(z) = p - \frac{4\beta\gamma(p-\alpha)^2(1-2\lambda)}{p(1 + 2\beta\gamma - \beta)\Gamma_{k+p}(\alpha_1)}, \quad k \geqslant p,$$

which completes the proof of Theorem 5.3. $\square$

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