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The Principle of Numerical Calculations for the Eigenvalue Comparison on Parameterized Surfaces

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Abstract In this note, we show the principle of doing numerical calculations for the eigenvalue comparison on a given complete parameterized surface.

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1. Introduction

Mathematicians care about the eigenvalue problem (or more precisely, eigenvalue inequalities) since it has the close relation with the isoperimetric problem, and this fact can be easily seen from some classical results, for instance, the Rayleigh-Faber-Krahn inequality (see, e.g., Theorem 2 on page 87 of [1]), the Szegö-Weinberger inequality (see, e.g., Remark 6 on page 94 of [1]), the Payne-Polya-Weinberger inequality [2], and so on. A classical result in the eigenvalue problem on manifolds, Cheng's eigenvalue comparison theorem (Cheng's ECT for short), tells us that for an n-dimensional ($n \geq 2$) complete Riemannian manifold M and a point $q \in M$, if its Ricci curvature is greater than and equal to $(n-1)K$, then the first Dirichlet eigenvalue $\lambda_1(B(q,r))$ of the Laplace operator on the geodesic ball $B(q,r)$ satisfies

$$\lambda_1(B(q,r)) \leq \lambda_1(V_n(r)), \tag{1.1}$$

where $V_n(r)$ is the geodesic ball of radius r in the n-dimensional space form of constant sectional curvature K, with $\lambda_1(V_n(r))$ the first Dirichlet eigenvalue of the Laplacian on $V_n(r)$, and the equality in (1.1) holds if and only if $B(q,r)$ is isometric to $V_n(r)$; conversely, if its sectional curvature is less than and equal to K, then for the geodesic ball $B(q,r)$ within the cut-locus of q, we have

$$\lambda_1(B(q,r)) \geq \lambda_1(V_n(r)), \tag{1.2}$$

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where $V_n(r)$, $B(q,r)$ and $\lambda_1(\cdot)$ have the same meanings as those in (1.1), and, as above, the equality in (1.2) hods if and only if $B(q,r)$ and $V_n(r)$ are isometric. We know that, in general, it is difficult to calculate the first Dirichlet eigenvalue of geodesic balls on a general manifold, but in the case of space forms, $\lambda_1(V_n(r))$ is actually the lowest positive real number such that there exists at least a nontrivial function $u(t)$ satisfying

$$\begin{cases} u''(t) + \frac{(n-1)S_K'(t)}{S_K(t)}u'(t) + \lambda_1 \cdot u(t) = 0, \\ u'(0) = u(r) = 0, \end{cases}$$

with

$$S_K(t) = \begin{cases} \sin(\sqrt{K}t)/\sqrt{K}, & K > 0, \\ t, & K = 0, \\ \sinh(\sqrt{-K}t)/\sqrt{-K}, & K < 0. \end{cases}$$

We refer readers to [1, Section 5, Chapter II] for this fact. The above ordinary differential equation (ODE for short) can be solved by suitable change of variables, and once K, n and r are confirmed, then $\lambda_1(V_n(r))$ can be accurately computed. This implies that we can give bounds, which can be given by accurate numbers, to $\lambda_1(B(q,r))$ provided we can find bounds for curvatures of the given complete manifold. Cheng's ECT is a useful tool in the estimation of eigenvalues of the Laplacian, and one can easily find examples for this fact [3, Theorem 2.1, Corollaries 2.2, 2.3, 2.4 and Theorem 4.2].

By considering more general curvature conditions, Freitas, Mao and Salavessa [4] have extended Cheng's ECT to more generalized versions [4, Theorems 3.6 and 4.4]. More precisely, for an n-dimensional ($n \geq 2$) complete Riemannian manifold M and a point $q \in M$, let $t := d(q, \cdot)$ be the Riemannian distance on M to the point q, if the radial Ricci curvature of M is bounded from below by $(n-1)k(t)$ with respect to q (see [4, Definition 2.2] for the precise statement of "how to define a lower bound, which is given by a continuous function w.r.t. t, for the radial Ricci curvature"), then the first Dirichlet eigenvalue $\lambda_1(B(q,r))$ of the Laplace operator on the geodesic ball $B(q,r)$ satisfies

$$\lambda_1(B(q,r)) \leq \lambda_1\left(V_n(q^-,r)\right), \tag{1.3}$$

where $V_n(q^-,r)$ is the geodesic ball with center q^- and radius r on the spherically symmetric manifold $M^- := [0,l) \times_{f(t)} \mathbb{S}^{n-1}$ with q^- as its base point, and $f(t)$ determined by

$$\begin{cases} f''(t) + k(t)f(t) = 0, \quad t \geq 0, \\ f'(0) = 1, \\ f(0) = 0, \\ f(t) > 0 \ \text{ on } (0,l), \end{cases} \tag{1.4}$$

and $\lambda_1(V_n(q^-,r))$ denotes the first Dirichlet eigenvalue of the Laplacian on $V_n(q^-,r)$. If $l < \infty$ and $f(l) = 0$, then M^- "closes" at $t = l$; if $l = +\infty$, this implies that the initial value problem (1.4) has a positive solution on $(0,\infty)$, and in this case, without considering the confusion between our parameter t here and the usual time-parameter, we say that M^- exists "for all the time", or equivalently, M^- has the "long-time existence". Here the radius-parameter r in (1.3) satisfies

$$r < \min\{l, \max_{\xi} \mathrm{d}_\xi(q)\}, \tag{1.5}$$

with $\xi \in S_q^{n-1} \subset T_qM$ and

$$\mathrm{d}_\xi(q) := \sup\{t > 0 |\ \gamma_\xi(s) = \gamma_{(q,\xi)}(s) := \exp_q(s\xi) \text{ is the unique}$$
$$\text{minimal geodesic joining } q \text{ and } \gamma_\xi(t)\},$$

and moreover, the equality in (1.3) holds if and only if $B(q,r)$ is isometric to $V_n(q^-, r)$. Conversely, if the radial sectional curvature of M is bounded from above by $k(t)$ with respect to q (see [4, Definition 2.3]) for the precise statement of "how to define an upper bound, which is given by a continuous function w.r.t. t, for the radial sectional curvature"), then $\lambda_1(B(q,r))$ satisfies

$$\lambda_1(B(q,r)) \geq \lambda_1\left(V_n(q^+, r)\right), \tag{1.6}$$

where $B(q,r)$ and $\lambda_1(\cdot)$ have the same meanings as those in (1.3), and $V_n(q^+, r)$ is the geodesic ball with center q^+ and radius r on the spherically symmetric manifold $M^+ := [0,l) \times_{f(t)} \mathbb{S}^{n-1}$ with q^+ as its base point, and $f(t)$ determined by (1.4). Here the radius-parameter r in (1.6) satisfies

$$r < \min\{\mathrm{inj}(q), l\}, \tag{1.7}$$

with $\mathrm{inj}(q)$ the injective radius of q, and the equality in (1.6) holds if and only if $B(q,r)$ and $V_n(q^+, r)$ are isometric. Similarly, the long-time existence of M^+ can also be defined as before. We call M^- and M^+ the "model spaces" of the original manifold M. The requirements (1.5) and (1.7) are necessary, since they are the preconditions for the validity of volume comparisons [4, Corollary 3.5, Theorem 4.2] which are the main tool to prove (1.3) and (1.6) in [4]. Especially, if $k(t)$ is a constant function, then the eigenvalue comparisons (1.3) and (1.6) degenerate into Cheng's ECT (1.1) and (1.2) directly. We also would like to point out one thing here, that is, in the case of spherically symmetric manifolds, $\lambda_1(V_n(q^-, r))$ or $\lambda_1(V_n(q^+, r))$ is actually the lowest positive real number such that there exists at least a nontrivial function $u(t)$ satisfying

$$\begin{cases} u''(t) + \frac{(n-1)f'(t)}{f(t)} u'(t) + \lambda_1 \cdot u(t) = 0, \\ u'(0) = u(r) = 0, \end{cases} \tag{1.8}$$

with $f(t)$ determined by (1.4). Clearly, this initial value problem can also be solved no matter $f(t)$ is given explicitly or numerically. Therefore, the bound $\lambda_1(V_n(q^-, r))$ or $\lambda_1(V_n(q^+, r))$ for the first Dirichlet eigenvalue $\lambda_1(B(q,r))$ can also be given (explicitly or numerically) by an accurate number once $k(t)$, n and r are confirmed.

The eigenvalue comparison (1.3) has been extended to the case of p-Laplcian $(1 < p < \infty)$ by Mao [5] (see [5, Theorem 3.2] for the precise statement), but the eigenvalue comparison (1.6) cannot be similarly generalized to the case of p-Laplcian, since in this case, Barta's lemma [4, Lemma 4.3] for the precise statement cannot be directly used any more.

2. The principle of numerical calculations

When the dimension of the original manifold M is 2 (i.e., $n = 2$), the radial Ricci curvature and the radial sectional curvature coincide with each other and are equal to the Gaussian curvature of the surface M. For the case of surfaces, we have the following result.

Theorem 2.1 *Let Σ be a complete surface, which can be parameterized, and a point $q \in \Sigma$. For the geodesic disk $B(q,r)$ with center q and radius r, where r satisfies (1.5) and (1.7), an optimal upper bound and an optimal lower bound can always be given simultaneously for the first Dirichlet eigenvalue $\lambda_1(B(q,r))$ of the Laplacian on $B(q,r)$. Moreover, these two optimal bounds can be computed numerically.*

Proof Let $(u, v, h(u,v))$ be a parametrization of Σ, and let $t := d(q,\cdot)$ be the Riemannian distance to q. If one wants to apply (1.3) and (1.6) to give an upper bound and a lower bound simultaneously for the first Dirichlet eigenvalue on $B(q,r)$, one needs to find upper and lower bounds for the Gaussian curvature with respect to q, and these bounds should be given as continuous functions of the variable t.

In general, one can compute the Gaussian curvature of Σ directly by the parameterization $(u, v, h(u,v))$, and the Gaussian curvature should be a continuous function of u, v. Without loss of generality, let $K(u,v)$ be the Gaussian curvature of Σ. Now, it is necessary to set up the relation between parameters u, v and the distance-parameter t. Hence, it cannot be avoided to compute geodesics on Σ starting from q, which generally do not have explicit expressions. But, fortunately, the software Mathematica can supply us the expressions of these geodesics given by interpolating functions. Actually, if one wants to get the expressions for geodesics starting from q, it is equivalent to solve the following system of ordinary differential equations (ODEs for short)

$$\begin{cases} u' = \ell, \\ v' = s, \\ \ell' = -\Gamma^1_{11}\ell^2 - 2\Gamma^1_{12}\ell s - \Gamma^1_{22}s^2, \\ s' = -\Gamma^2_{11}\ell^2 - 2\Gamma^2_{12}\ell s - \Gamma^2_{22}s^2, \end{cases} \tag{2.1}$$

with initial conditions

$$\begin{cases} u(0) = 0, \\ v(0) = 0, \\ u'(0) = \ell(0) = \cos\theta, \\ v'(0) = \ell(0) = \sin\theta, \end{cases}$$

where $\theta \in [0, 2\pi)$ and Γ^i_{jk}, $1 \leq i,j,k \leq 2$, denote the Christoffel symbols of Σ. Solving (2.1) by Mathematica, the Gaussian curvature can be rewritten as $K\left(u(t,\theta), v(t,\theta)\right)$, which now is a function of variables t, θ. Let

$$k_-(q,t) := \min_{\theta\in[0,2\pi)} K\left(u(t,\theta), v(t,\theta)\right), \quad k_+(q,t) := \max_{\theta\in[0,2\pi)} K\left(u(t,\theta), v(t,\theta)\right), \tag{2.2}$$

which, for a fixed t, are actually the minimal and the maximal values of the Gaussian curvature on the geodesic circle $c(q,t)$ with center q and radius t. Clearly, $k_-(q,t)$ and $k_+(q,t)$ are optimal bounds of the Gaussian curvature with respect to q. Besides, by applying the uniform continuity of continuous functions on compact sets, one can easily get that $k_-(q,t)$ and $k_+(q,t)$ are continuous functions w.r.t. t. Then the first claim of this theorem follows by applying the eigenvalue comparisons (1.3) and (1.6) directly. Once the Gaussian curvature bounds are obtained by (2.1) and (2.2), the model surfaces M^- and M^+ can be determined by solving the initial value problem (1.4), and then the values of bounds for the first Dirichlet eigenvalue $\lambda_1(B(q,r))$ can be obtained numerically by solving the boundary value problem (1.8). This completes the proof of Theorem 2.1. $\square$

Remark 2.2 The system of ODEs given by (2.1) for finding relations between the parameters u, v and the distance-parameter t can also be found in [6, Example 2.5.2]. Three interesting examples about torus, elliptic paraboloid and saddle [4, Examples 6.1, 6.2 and 6.3] and related numerical calculations have been dealt with in [4, Section 6]. However, the general way of finding $k_-(q,t)$ and $k_+(q,t)$ defined by (2.2) for the Gaussian curvature was not mentioned therein. Therefore, for the purpose that readers, who are interested in those examples, know the details of numerical calculations done in [4, Section 6] clearly, we give the above main theorem and its proof. From the proof of the above theorem, we know that this principle (which, roughly speaking, means the system of ODEs (2.1), with its initial conditions, and the optimal choice of bounds for the Gaussian curvature given by (2.2)) is the core of doing numerical calculations for the eigenvalue comparison on a given complete parameterized surface.

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