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The Eigenvalue Problem for $p(x)$-Laplacian Equations Involving Robin Boundary Condition

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Abstract This paper studies the eigenvalue problem for $p(x)$-Laplacian equations involving Robin boundary condition. We obtain the Euler-Lagrange equation for the minimization of the Rayleigh quotient involving Luxemburg norms in the framework of variable exponent Sobolev space. Using the Ljusternik-Schnirelman principle, for the Robin boundary value problem, we prove the existence of infinitely many eigenvalue sequences and also show that, the smallest eigenvalue exists and is strictly positive, and all eigenfunctions associated with the smallest eigenvalue do not change sign.

Keywords variable exponents; eigenvalue; Robin boundary condition; $p(x)$-Laplacian equations

MR(2010) Subject Classification 35J60; 35P30; 58E05

1. Introduction

In the last decade, the study of partial differential equations and variational problems with variable exponent growth conditions has been a very attractive field. These investigations are stimulated mainly by the development of the study of electrorheological fluids [1], image restoration [2] and the theory of nonlinear elasticity [3–5]. The eigenvalue problem involving variable exponent is one of the important research field as well.

Nonlinear eigenvalue problem for the p-Laplacian equations with Dirichlet, Neumann or Robin boundary conditions has been extensively studied by many authors, and it has many interesting results as explained in the works of [6–10] and references therein. The nonlinear eigenvalue problem for $p(x)$-Laplacian equations which possesses more complicated nonlinearity than the p-Laplacian equations is also considered, but there is still a gape existing in the literature. We point out that Fan [11,12] has made contributions to the study of $p(x)$-Laplacian eigenvalue problem. Moreover, the Steklov and Robin eigenvalue problems involving $p(x)$-Laplacian equations have been respectively studied by Deng [13] and Deng et al. [14]. These investigations mainly have relied on variational methods and deduce there exist infinitely many eigenvalue sequences, and also give some suitable conditions for which the infimum of all eigenvalues is either zero or positive. Mihăilescu and Rădulescu have also studied the variable exponent eigenvalue problem in different cases [15–17]. We notice that all results obtained depend on Rayleigh quotient

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of two modulars, which are similarly defined in the case of p-Laplacian equations. The main shortcoming of this method is lack of homogeneity of Rayleigh quotient with variable exponent. To overcome the above mentioned shortcoming scholars [11–14] imposed some constrained conditions. In their view, the difficulty is to compare the minimizers obtained from different normalization constants.

Furthermore, the definition of eigenvalues using the Rayleigh quotient of two modulars is not a proper generalization of the constant exponent case. Recently, Franzina and Lindqvist studied the Dirichlet eigenvalue problem, which is based on replacing the modulars by Luxemburg norms of the variable exponent Lebesgue space in the Rayleigh quotient [18]. This homogeneous definition of first eigenvalue of the $p(x)$-Laplacian has been firstly introduced in [18] as an appropriate replacement for the previous nonhomogeneous notions in the literature. Franzina and Lindqvist proved that the first eigenvalue is positive, and all eigenfunctions associated with the first eigenvalue are continuous and strictly positive. According to the definition of the Rayleigh quotient introduced by Franzina and Lindqvist, the asymptotic behavior and stability of eigenvalue for variable exponent problem were studied in [19–21]. Motivated by [18], we discuss the eigenvalue problem for $p(x)$-Laplacian equations with Robin boundary condition in this paper.

This paper is organized as follows. In Section 2, we recall some important notions concerning the variable exponent Lebesgue and Sobolev spaces. In Section 3, we establish the Euler-Lagrange equation for the minimization of a Rayleigh quotient of two Luxemburg norms. In Section 4, we prove the existence of infinitely many eigenvalue sequences for Robin boundary value problem.

2. Preliminaries

In order to deal with the variable exponent eigenvalue problem, we recall the theory of Lebesgue and Sobolev spaces with variable exponents [22–26].

Let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. For any Lipschitz continuous function $p : \overline{\Omega} \to (1, +\infty)$, we denote

$$1 < p^- := \inf_{x\in\Omega} p(x) \leq p(x) \leq \sup_{x\in\Omega} p(x) := p^+ < N, \quad \text{for all } \ \mathrm{x} \in \Omega. \tag{2.1}$$

We consider the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ defined as follows

$$L^{p(x)}(\Omega) = \Big\{u \ |u \, \text{is a measurable function on} \, \Omega \text{ such that} \int_\Omega |u(x)|^{p(x)} \mathrm{d}x \ < +\infty\Big\}.$$

This space is equipped with Luxemburg norm

$$|u|_{p(x)} = \inf\Big\{\lambda > 0; \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} \frac{\mathrm{d}x}{p(x)} \ \leq 1\Big\}.$$

Let variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with its norm

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},$$

which is equivalent to

$$\|u\|_{p(x)} = \inf\Big\{\lambda > 0; \int_{\Omega} [|\frac{u}{\lambda}|^{p(x)} + |\frac{\nabla u}{\lambda}|^{p(x)}] \frac{\mathrm{d}x}{p(x)} \le 1\Big\}.$$

Proposition 2.1 ([22,23,25,26]) *Both $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ and $(W^{1,p(x)}(\Omega), \|\cdot\|_{p(x)})$ are separable and reflexive Banach spaces.*

Proposition 2.2 ([22,25]) *Assume that $u \in L^{p(x)}(\Omega)$, $v \in L^{q(x)}(\Omega)$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ for all $x \in \Omega$. Then*

$$\int_{\Omega} |uv| \mathrm{d}x \le 2|u|_{p(x)}|v|_{q(x)}.$$

Proposition 2.3 ([27]) *Let $\varrho_{p(x)}(u) = \int_{\Omega}[|u|^{p(x)} + |\nabla u|^{p(x)}]\frac{\mathrm{d}x}{p(x)}$. For any $u, u_k \in W^{1,p(x)}(\Omega)$ $(k = 1, 2, \ldots)$, then we have*

(1) $\|u\|_{p(x)} < 1 \ (= 1; > 1) \ \Leftrightarrow \varrho_{p(x)}(u) < 1 \ (= 1; > 1)$.

(2) $\|u\|_{p(x)} \le 1 \Rightarrow \|u\|^{p^+}_{p(x)} \le \varrho_{p(x)}(u) \le \|u\|^{p^-}_{p(x)}$.

(3) $\|u\|_{p(x)} \ge 1 \Rightarrow \|u\|^{p^-}_{p(x)} \le \varrho_{p(x)}(u) \le \|u\|^{p^+}_{p(x)}$.

(4) $\|u_k - u\|_{p(x)} \to 0 \Leftrightarrow \varrho_{p(x)}(u_k - u) \to 0$.

Proposition 2.4 ([23–26]) *If $q(x) \in C(\bar{\Omega})$ and $1 \le q(x) < p^*(x)$, $\forall x \in C(\bar{\Omega})$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = +\infty$ if $p(x) \ge N$.*

Proposition 2.5 ([22,28,29]) *Let f_n be a sequence of measurable function in $L^{p(x)}(\Omega)$, if $f_n \to f$ a.e. in Ω, and $|f_n(x)| \le g(x)$ a.e. $x \in \Omega$ with $f, g \in L^{p(x)}(\Omega)$, then $f_n \to f$ in $L^{p(x)}(\Omega)$.*

Let $a : \partial\Omega \to \mathbb{R}$ be a real function with $a \in L^{\infty}(\partial\Omega)$ and $a^- := \inf_{x\in\partial\Omega} a(x) > 0$, and we define the weighted variable exponent Lebesgue space as follows

$$L^{p(x)}_{a(x)}(\partial\Omega) = \Big\{u | u : \partial\Omega \to \mathbb{R} \text{ is a measurable and} \int_{\partial\Omega} a(x)|u(x)|^{p(x)} \mathrm{d}\sigma < +\infty\Big\},$$

equipped with the norm

$$|u|_{(p(x),\ a(x))} = \inf\Big\{\lambda > 0; \int_{\partial\Omega} a(x)|\frac{u(x)}{\lambda}|^{p(x)} \frac{\mathrm{d}\sigma}{p(x)} \le 1\Big\}$$

where $d\sigma$ is the measure on the boundary $\partial\Omega$. Notice that it is easy to prove that $L^{p(x)}_{a(x)}(\partial\Omega)$ is a Banach space.

Proposition 2.6 ([13,22]) *Let $\rho_{p(x)}(u) = \int_{\partial\Omega}[a(x)|u|^{p(x)}]\frac{\mathrm{d}\sigma}{p(x)}$. For any $u, u_k \in L^{p(x)}_{a(x)}(\partial\Omega)$ $(k = 1, 2, \ldots)$, then we have*

(1) $|u|_{(p(x),\ a(x))} < 1 \ (= 1; > 1) \ \Leftrightarrow \rho_{p(x)}(u) < 1 \ (= 1; > 1)$.

(2) $|u|_{(p(x),\ a(x))} \le 1 \Rightarrow |u|^{p^+}_{(p(x),\ a(x))} \le \rho_{p(x)}(u) \le |u|^{p^-}_{(p(x),\ a(x))}$.

(3) $|u|_{(p(x),\ a(x))} \ge 1 \Rightarrow |u|^{p^-}_{(p(x),\ a(x))} \le \rho_{p(x)}(u) \le |u|^{p^+}_{(p(x),\ a(x))}$.

(4) $|u_k - u|_{(p(x),\ a(x))} \to 0 \Leftrightarrow \rho_{p(x)}(u_k - u) \to 0$.

Proposition 2.7 ([13]) *Suppose that $a(x) \in L^{r(x)}(\partial\Omega)$, $r(x) \in C(\partial\Omega)$ with $r(x) > \frac{p^1(x)}{p^1(x)-1}$ for all $x \in \partial\Omega$. If $q(x) \in C(\partial\Omega)$ and $1 \le q(x) < p^1_{r(x)}(x)$ for all $x \in C(\partial\Omega)$, then there is a compact* *embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{a(x)}(\partial\Omega)$, *where* $p^1(x) = \frac{(N-1)p(x)}{N-p(x)}$ *if* $p(x) < N$ *and* $p^1(x) = +\infty$ *if* $p(x) \geq N$, $p^1_{r(x)}(x) := \frac{r(x)-1}{r(x)} p^1(x)$.

3. The Euler-Lagrange equation

In this section we prove the existence of a non-trivial minimizer and the Euler-Lagrange equation corresponding to the minimizer. Moreover, we show λ_1 is the smallest eigenvalue and the existence of the strictly positive first eigenfunction.

Define

$$\beta : \partial\Omega \to \mathbb{R} \text{ to be a real function with } \beta \in \mathrm{L}^\infty(\partial\Omega) \text{ and } \beta^- := \inf_{\mathrm{x}\in\partial\Omega} \beta(\mathrm{x}) > 0, \tag{3.1}$$

and

$$\lambda_1 = \inf_{u\in W^{1,p(x)}(\Omega),\ u\not\equiv 0} \frac{|\nabla u|_{p(x)} + |u|_{(p(x),\ \beta(x))}}{|u|_{p(x)}}. \tag{3.2}$$

In the following theorem we establish the existence of a nonnegative minimizer.

Theorem 3.1 *There is a non-negative minimizer* $u \in W^{1,p(x)}(\Omega)$ *and* $u \not\equiv 0$ *for (3.2) and* $\lambda_1 > 0$.

Proof (1) In view of (3.2), we state that $\lambda_1 \geq 0$.

(2) We can choose a minimizing sequence $\{u_n\}$ such that

$$\lambda_1 = \lim_{n\to\infty} (|\nabla u_n|_{p(x)} + |u_n|_{(p(x),\beta(x))})$$

with $|u_n|_{p(x)} = 1$. Note that the sequence $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$. Then there exists a subsequence which still denotes $\{u_n\}$ and a measurable function $u \in W^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$. Combining Proposition 2.4 with Proposition 2.7, we get $u_n \to u$ in $L^{p(x)}(\Omega)$, $\nabla u_n \rightharpoonup \nabla u$ in $L^{p(x)}(\Omega)$ and $u_n \to u$ in $L^{p(x)}_{\beta(x)}(\partial\Omega)$. Therefore, together with the weak lower semi-continuity of the norm, we infer that $|u|_{p(x)} = 1$ and

$$\lambda_1 \leq |\nabla u|_{p(x)} + |u|_{(p(x),\beta(x))} \leq \liminf_{n\to\infty} (|\nabla u_n|_{p(x)} + |u_n|_{(p(x),\beta(x))}) = \lambda_1.$$

(3) If $\lambda_1 = 0$, then there exists a minimizer u with $|u|_{p(x)} = 1$ such that

$$0 = \lambda_1 = |\nabla u|_{p(x)} + |u|_{(p(x),\ \beta(x))}.$$

It follows that

$$|\nabla u|_{p(x)} = 0, |u|_{(p(x),\beta(x))} = 0.$$

Hence $u \equiv 0$, it is a contradiction with $|u|_{p(x)} = 1$, consequently $\lambda_1 > 0$.

This shows that $u \in W^{1,p(x)}(\Omega)$ is a minimizer. If u is a minimizer, so is $|u|$, which can be obtained from the definition of λ_1.

The following theorem is giving the Euler-Lagrange equation associated to a minimizer.

Theorem 3.2 *The Euler-Lagrange equation corresponding to the minimization of the Rayleigh quotient (3.2) is given by*

$$\begin{cases} -\mathrm{div}(|\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\nabla u}{K(u)}) = \lambda_1(\Omega) S(u) |\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)} & \text{in } \Omega, \\ |\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x) T(u) K(u) |\frac{u}{H(u)}|^{p(x)-2}\frac{u}{H(u)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where

$$K(u) := |\nabla u|_{p(x)},\ k(u) := |u|_{p(x)},\ H(u) := |u|_{(p(x),\ \beta(x))}$$
$$S(u) := \frac{\int_\Omega |\frac{\nabla u}{K(u)}|^{p(x)}\mathrm{d}x}{\int_\Omega |\frac{u}{k(u)}|^{p(x)}\mathrm{d}x},\ T(u) := \frac{\int_\Omega |\frac{\nabla u}{K(u)}|^{p(x)}\mathrm{d}x}{\int_{\partial\Omega} |\frac{u}{H(u)}|^{p(x)}\mathrm{d}\sigma}. \tag{3.4}$$

Proof Assume that $v \in W^{1,p(x)}(\Omega)$ and $\varepsilon > 0$ is small enough, let $Q(x) = u(x) + \varepsilon v(x)$ and we write

$$f_1(\varepsilon) = |\nabla Q|_{p(x)},\ f_2(\varepsilon) = |Q|_{p(x)} \text{ and } f_3(\varepsilon) = |Q|_{(p(x),\ \beta(x))}.$$

If $u \in W^{1,p(x)}(\Omega)$ is a minimizer of (3.2), then

$$\lambda_1 = \frac{K(u) + H(u)}{k(u)} \le \frac{f_1(\varepsilon) + f_3(\varepsilon)}{f_2(\varepsilon)},\quad \forall \varepsilon > 0.$$

That is

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\frac{f_1(\varepsilon) + f_3(\varepsilon)}{f_2(\varepsilon)}) = \frac{(f_1'(\varepsilon) + f_3'(\varepsilon))f_2(\varepsilon) - (f_1(\varepsilon) + f_3(\varepsilon))f_2'(\varepsilon)}{f_2^2(\varepsilon)} = 0,\ \text{at } \varepsilon = 0.$$

Thus we get

$$f_1'(0) + f_3'(0) = \frac{(f_1(0) + f_3(0))f_2'(0)}{f_2(0)}. \tag{3.5}$$

Applying [18, Lemma A.1], then for any v belonging to $W^{1,p(x)}(\Omega)$, we obtain

$$f_1'(0) = \langle K'(u), v\rangle = \frac{\int_\Omega |\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\nabla u}{K(u)}\nabla v\mathrm{d}x}{\int_\Omega |\frac{\nabla u}{K(u)}|^{p(x)}\mathrm{d}x}, \tag{3.6}$$

$$f_2'(0) = \langle k'(u), v\rangle = \frac{\int_\Omega |\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)}v\mathrm{d}x}{\int_\Omega |\frac{u}{k(u)}|^{p(x)}\mathrm{d}x}, \tag{3.7}$$

$$f_3'(0) = \langle H'(u), v\rangle = \frac{\int_{\partial\Omega} \beta(x)|\frac{u}{H(u)}|^{p(x)-2}\frac{u}{H(u)}v\mathrm{d}\sigma}{\int_{\partial\Omega} \beta(x)|\frac{u}{H(u)}|^{p(x)}\mathrm{d}\sigma}. \tag{3.8}$$

Recalling equations (3.5)–(3.8), we have

$$\begin{aligned}&\frac{\int_\Omega |\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\nabla u}{K(u)}\nabla v\mathrm{d}x}{\int_\Omega |\frac{\nabla u}{K(u)}|^{p(x)}\mathrm{d}x} + \frac{\int_{\partial\Omega} \beta(x)|\frac{u}{H(u)}|^{p(x)-2}\frac{u}{H(u)}v\mathrm{d}\sigma}{\int_{\partial\Omega} \beta(x)|\frac{u}{H(u)}|^{p(x)}\mathrm{d}\sigma}\\ &= \frac{(K(u) + H(u))\int_\Omega |\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)}v\mathrm{d}x}{k(u)\int_\Omega |\frac{u}{k(u)}|^{p(x)}\mathrm{d}x},\end{aligned}$$

so that

$$\begin{aligned}&\int_\Omega |\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\nabla u\cdot\nabla v}{K(u)}\mathrm{d}x + T(u)\int_{\partial\Omega}\beta(x)|\frac{u}{H(u)}|^{p(x)-2}\frac{uv}{H(u)}\mathrm{d}\sigma\\ &= \lambda_1 S(u)\int_\Omega |\frac{u}{k(u)}|^{p(x)-2}\frac{uv}{k(u)}\mathrm{d}x.\end{aligned} \tag{3.9}$$

The proof of Theorem 3.2 is completed. □

Definition 3.3 *(1) A pair $(u, \lambda) \in W^{1,p(x)}(\Omega) \times \mathbb{R}$ is called a weak solution to the following*

Robin boundary value problem

$$\begin{cases} -\mathrm{div}(|\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\nabla u}{K(u)}) = \lambda S(u)|\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)} & \text{in } \Omega, \\ |\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\partial u}{\partial \nu} + \beta(x)T(u)K(u)|\frac{u}{H(u)}|^{p(x)-2}\frac{u}{H(u)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.10}$$

if

$$\begin{aligned} &\int_\Omega |\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\nabla u\cdot\nabla v}{K(u)}\mathrm{d}x + T(u)\int_{\partial\Omega}\beta(x)|\frac{u}{H(u)}|^{p(x)-2}\frac{uv}{H(u)}\mathrm{d}\sigma \\ &= \lambda S(u)\int_\Omega |\frac{u}{k(u)}|^{p(x)-2}\frac{uv}{k(u)}\mathrm{d}x, \quad \forall v\in W^{1,p(x)}(\Omega). \end{aligned} \tag{3.11}$$

(2) If $(u,\lambda)\in W^{1,p(x)}(\Omega)\times\mathbb{R}$ is a weak solution to problem (3.10), then u and λ are called eigenfunction and eigenvalue, respectively. Notice that u is a nontrivial eigenfunction corresponding to λ in the problem (3.10).

Remark 3.4 If $p(x)\equiv p$, the problem (3.10) can be written

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = (\frac{|\nabla u|_{L^p(\Omega)}}{|u|_{L^p(\Omega)}})^{p-1}\lambda|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} + (\frac{|\nabla u|_{L^p(\Omega)}}{|u|_{L^p_{\beta(x)}(\partial\Omega)}})^{p-1}\beta(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Theorem 3.5 *λ_1 is the smallest eigenvalue of problem (3.10) and the corresponding eigenfunctions are strictly positive. Moreover, they are called the first eigenvalue and the first eigenfunction of problem (3.10), respectively.*

Proof (1) If u is an eigenfunction corresponding to the eigenvalue λ of problem (3.10), then (3.11) holds. Taking $v=u$ in (3.11) and using (3.2), we obtain

$$\lambda = \frac{K(u)+H(u)}{k(u)} \ge \lambda_1,$$

then λ_1 is the smallest eigenvalue of problem (3.10).

(2) Obviously, if u is an eigenfunction corresponding to λ_1, so is $|u|$, thus the first eigenfunctions are non-negative.

Assume $W=\frac{u}{K(u)}$. Then $W\ge 0$, $\nabla W=\frac{\nabla u}{K(u)}$, $k(u)=k(W)K(u)$ and $K(W)=1$. From

$$-\mathrm{div}(|\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\nabla u}{K(u)}) = \lambda_1 S(u)|\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)},$$

it follows that

$$-\mathrm{div}(|\nabla W|^{p(x)-2}\nabla W) = \lambda_1\mathrm{d}(x)|W|^{p(x)-2}W, \tag{3.13}$$

where

$$d(x) = (\frac{1}{k(w)})^{p(x)-1}\frac{\int_\Omega|\nabla W|^{p(x)}\mathrm{d}x}{\int_\Omega|\frac{W}{k(W)}|^{p(x)}\mathrm{d}x} \ge 0.$$

Multiplying both sides of (3.13) by any test function $\varphi\in C_0^\infty(\Omega)$ ($\varphi\ge 0$) and integrating over Ω, we have

$$\int_\Omega |\nabla W|^{p(x)-2}\nabla W\nabla\varphi\mathrm{d}x = \lambda_1\int_\Omega \mathrm{d}(x)|W|^{p(x)-2}W\varphi\mathrm{d}x \ge 0, \tag{3.14}$$

so that W is a weak supersolution of problem (3.13). By virtue in [30, Theorem 4.1], we obtain $W(x) = W^*(x)$ for almost every $x \in \Omega$, where $W^*(x)$ is a lower semi-continuous representative of $W(x)$.

Since $\varphi = W + \varphi - W$, from (3.14), we infer that

$$\int_\Omega |\nabla W|^{p(x)-2}\nabla W(\nabla(W+\varphi) - \nabla W)\mathrm{d}x \geq 0. \tag{3.15}$$

Applying well-known Young's inequality, we get

$$\begin{aligned}\int_\Omega |\nabla W|^{p(x)}\,\mathrm{d}x &\leq \int_\Omega |\nabla W|^{p(x)-2}\,\nabla W\nabla(W+\varphi)\mathrm{d}x \\ &\leq \int_\Omega |\nabla W|^{p(x)-1}\,|\nabla(W+\varphi)|\,\mathrm{d}x \\ &\leq \int_\Omega \frac{|\nabla(W+\varphi)|^{p(x)}}{p(x)}\mathrm{d}x - \int_\Omega \frac{|\nabla W|^{p(x)}}{p(x)}\mathrm{d}x + \int_\Omega |\nabla W|^{p(x)}\,\mathrm{d}x, \end{aligned} \tag{3.16}$$

namely,

$$\int_\Omega |\nabla W|^{p(x)}\,\mathrm{d}x \leq \frac{p^+}{p_-}\int_\Omega |\nabla(W+\varphi)|^{p(x)}\,\mathrm{d}x,$$

and W is a quasisuperminimizer in Ω.

In [30, Theorem 4.1] and [31, Theorem 5.3], we obtain $W > 0$ in Ω, thus $u > 0$ in Ω. This completes the proof. □

Remark 3.6 According to [32, Theorem 4.4], the weak solutions to problem (3.10) are locally Hölder continuous if $p(x)$ is Hölder continuous.

Indeed, let

$$A(x,u,\nabla u) = |\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\nabla u}{K(u)}, \quad B(x,u,\nabla u) = \lambda S(u)|\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)},$$

then it follows that

$$A(x,u,\nabla u)\cdot\nabla u \geq \min\{(\frac{1}{K(u)})^{p^+-1}, (\frac{1}{K(u)})^{p^--1}\}|\nabla u|^{p(x)},$$

$$|A(x,u,\nabla u)| \leq \max\{(\frac{1}{K(u)})^{p^+-1}, (\frac{1}{K(u)})^{p^--1}\}|\nabla u|^{p(x)-1}$$

and

$$|B(x,u,\nabla u)| \leq \lambda p^+\max\{(\frac{1}{k(u)})^{p^+-1}, (\frac{1}{k(u)})^{p^--1}\}|u|^{p(x)-1}.$$

This shows that the assumptions in [32, Theorem 4.4] hold, and consequently the weak solutions to problem (3.10) are locally Hölder continuous.

4. Existence of infinitely many eigenvalue sequences for Robin problem

In this section we discuss the variable exponent eigenvalue problem for Robin boundary condition.

Obviously, the problem (3.10) is equivalent to the following problem

$$\begin{cases} -\mathrm{div}(|\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\nabla u}{K(u)})+S(u)|\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)}=\mu S(u)|\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)} & \text{in } \Omega,\\ |\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\partial u}{\partial \nu}+\beta(x)T(u)K(u)|\frac{u}{H(u)}|^{p(x)-2}\frac{u}{H(u)}=0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with $\lambda=\mu-1$.

For any $u\in W^{1,p(x)}(\Omega)$, let $F(u)=K(u)+k(u)+H(u)$, where $K(u)$, $k(u)$ and $H(u)$ are seen in (3.4). Set

$$M_1=\{u\in W^{1,p(x)}(\Omega):k(u)=1\}$$

and

$$\Sigma=\{A\subset M_1: A \text{ is compact and } -A=A\}.$$

We denote $\gamma(A)$ (see [33–35]) the genus of $A\in\Sigma$, which is defined as

$$\gamma(A)=\inf\{k:\text{there exists } h:\ A\to\mathbb{R}^k\setminus\{0\} \text{ such that } h \text{ is continuous and odd}\},$$

if there is no integer in the above definition, we set $\gamma(A)=+\infty$ and $\gamma(\emptyset)=0$.

Applying Ljusternik-Schnirelman principle [34], we obtain that the problem (3.10) has infinitely many eigenvalue sequences $\{\lambda_n\}$ such that

$$\lambda_n=\inf_{A\in\Sigma,\gamma(A)\geq n}\sup_{u\in A}(|\nabla u|_{p(x)}+|u|_{(p(x),\ \beta(x))}),\quad n=1,2,\ldots. \tag{4.2}$$

The following Theorem is the main result in this section.

Theorem 4.1 *We have* $0<\lambda_n\leq\lambda_{n+1}$ *and* $\lambda_n\to+\infty$ *as* $n\to\infty$*. Moreover, the problem (3.10) has infinitely many solution pairs* $\{\pm u_n,\ \lambda_n\}$ *such that* $k(\pm u_n)=1$, $F(\pm u_n)-1=\lambda_n$ *and* $\lambda_n=K(u_n)+H(u_n)$*. In particular,* λ_1 *is the same as in (3.2).*

In order to prove the Theorem 4.1, we need the following statement.

Lemma 4.2 Suppose $F,k:W^{1,p(x)}(\Omega)\to\mathbb{R}$ *are even functionals and* $F,k\in C^1(W^{1,p(x)}(\Omega),\mathbb{R})$ *with* $F(0)=k(0)=0$*. Then, for any* $v\in W^{1,p(x)}(\Omega)$*, we have*

$$\begin{aligned}\langle F'(u),v\rangle=&\frac{\int_\Omega|\frac{\nabla u}{K(u)}|^{p(x)-2}\frac{\nabla u}{K(u)}\nabla v\mathrm{d}x}{\int_\Omega|\frac{\nabla u}{K(u)}|^{p(x)}\mathrm{d}x}+\frac{\int_\Omega|\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)}v\mathrm{d}x}{\int_\Omega|\frac{u}{k(u)}|^{p(x)}\mathrm{d}x}+\\ &\frac{\int_{\partial\Omega}\beta(x)|\frac{u}{H(u)}|^{p(x)-2}\frac{u}{H(u)}v\mathrm{d}\sigma}{\int_{\partial\Omega}\beta(x)|\frac{u}{H(u)}|^{p(x)}\mathrm{d}\sigma}\end{aligned}$$

and

$$\langle k'(u),v\rangle=\frac{\int_\Omega|\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)}v\mathrm{d}x}{\int_\Omega|\frac{u}{k(u)}|^{p(x)}\mathrm{d}x}.$$

Proof (1) According to the assumption of Lemma 4.2, the terms $\langle F'(u),v\rangle$ and $\langle k'(u),v\rangle$ can be deduced from (3.6)–(3.8).

(2) Proving that $F'(u)\in\big(W^{1,p(x)}(\Omega)\big)^*$ and $k'(u)\in\big(W^{1,p(x)}(\Omega)\big)^*$.

Using Young's inequality, we have

$$\int_\Omega|\frac{u}{k(u)}|^{p(x)-1}\frac{v}{k(v)}\mathrm{d}x\leq\int_\Omega|\frac{u}{k(u)}|^{p(x)}\mathrm{d}x-\int_\Omega|\frac{u}{k(u)}|^{p(x)}\frac{\mathrm{d}x}{p(x)}+\int_\Omega|\frac{v}{k(v)}|^{p(x)}\frac{\mathrm{d}x}{p(x)}.$$

The last two terms on the right hand side of the above inequality are equal to 1, then we have

$$|\langle k'(u), v\rangle| \le k(v)\frac{\int_\Omega |\frac{u}{k(u)}|^{p(x)}\mathrm{d}x}{\int_\Omega |\frac{u}{k(u)}|^{p(x)}\mathrm{d}x} = |v|_{p(x)} \le \|v\|_{p(x)}.$$

Similarly,

$$|\langle K'(u), v\rangle| \le |\nabla v|_{p(x)}, \quad |\langle H'(u), v\rangle| \le |v|_{(p(x),\beta(x))}.$$

It follows that

$$\begin{aligned}|\langle F'(u), v\rangle| \le & |v|_{p(x)} + |\nabla v|_{p(x)} + |v|_{(p(x),\ \beta(x))} = ||v||_{p(x)} + |v|_{(p(x),\ \beta(x))}\\ \le & (1+C)||v||_{p(x)},\end{aligned}$$

where $C>0$ is a Sobolev constant from the embedding of $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}_{\beta(x)}(\partial\Omega)$. Therefore, $F'(u) \in \left(W^{1,p(x)}(\Omega)\right)^*$ and $k'(u) \in \left(W^{1,p(x)}(\Omega)\right)^*$.

(3) Showing that $F'(u), k'(u) : W^{1,p(x)}(\Omega) \to \left(W^{1,p(x)}(\Omega)\right)^*$ are continuous.

Set

$$f(u) = \int_\Omega |\frac{u}{k(u)}|^{p(x)}\mathrm{d}x, \quad \forall u \in W^{1,p(x)}(\Omega),$$

we observe that $f(u) \ge \int_\Omega |\frac{u}{k(u)}|^{p(x)}\frac{\mathrm{d}x}{p(x)} = 1$.

Using Hölder's inequality, then for any $v \in W^{1,p(x)}(\Omega)$, we have

$$\begin{aligned}&|\langle k'(u_n) - k'(u), v\rangle| = |\frac{\int_\Omega |\frac{u_n}{k(u_n)}|^{p(x)-2}\frac{u_n}{k(u_n)}v\mathrm{d}x}{f(u_n)} - \frac{\int_\Omega |\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)}v\mathrm{d}x}{f(u)}|\\ &\le \int_\Omega |v||\frac{1}{f(u_n)}|\frac{u_n}{k(u_n)}|^{p(x)-2}\frac{u_n}{k(u_n)} - \frac{1}{f(u)}|\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)}|\mathrm{d}x\\ &\le \int_\Omega |v||\frac{1}{f(u_n)} - \frac{1}{f(u)}||\frac{u}{k(u)}|^{p(x)-1}\mathrm{d}x +\\ &\quad \int_\Omega \frac{|v|}{f(u_n)}||\frac{u_n}{k(u_n)}|^{p(x)-2}\frac{u_n}{k(u_n)} - |\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)}|\mathrm{d}x\\ &\le I_1 + 2CI_2,\end{aligned}$$

where

$$I_1 = \int_\Omega |v||\frac{1}{f(u_n)} - \frac{1}{f(u)}||\frac{u}{k(u)}|^{p(x)-1}\mathrm{d}x,$$

$$I_2 = ||v||_{p(x)}||\frac{u_n}{k(u_n)}|^{p(x)-2}\frac{u_n}{k(u_n)} - |\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)}|_{\frac{p(x)}{p(x)-1}}$$

and $C>0$ is a Sobolev constant from the embedding of $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

Since $u_n \to u$ in $W^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $|u_n|_{p(x)} \to |u|_{p(x)}$, i.e., $k(u_n) \to k(u)$ as $n \to \infty$.

Using [29, Proposition 2.67] and noting the fact that $u_n \to u$ in $L^{p(x)}(\Omega)$, then there exists a subsequence (which is still denoted $\{u_n\}$), and a function $v \in L^{p(x)}(\Omega)$ such that $u_n \to u$ and $|u_n| \le |v|$ a.e. in Ω for all n. Then, for n sufficiently large and $\varepsilon \in (0, k(u))$, we have

$$|\frac{u_n}{k(u_n)}|^{p(x)} \to |\frac{u}{k(u)}|^{p(x)} \text{ a.e. in } \Omega,$$

$$|\frac{u_n}{k(u_n)}|^{p(x)} \le \max\{(\frac{1}{k(u)-\varepsilon})^{p^+}, (\frac{1}{k(u)-\varepsilon})^{p^-}\}|v|^{p(x)} \in L^1(\Omega) \text{ a.e. in } \Omega.$$

In view of dominated convergence theorem, we get

$$\int_\Omega |\frac{u_n}{k(u_n)}|^{p(x)}\mathrm{d}x \to \int_\Omega |\frac{u}{k(u)}|^{p(x)}\mathrm{d}x, \text{ as } n\to\infty.$$

From dominated convergence theorem, we can deduce that $I_1 \to 0$, as $n\to\infty$. Let

$$W_n = |\frac{u_n}{k(u_n)}|^{p(x)-2}\frac{u_n}{k(u_n)}, \quad W = |\frac{u}{k(u)}|^{p(x)-2}\frac{u}{k(u)}.$$

According to the above discussion, we can prove that

$$W_n \to W \text{ a.e. in } \Omega,$$

$$|W_n| = |\frac{u_n}{k(u_n)}|^{p(x)-1} \le \max\{(\frac{1}{k(u)-\varepsilon})^{p^+}, (\frac{1}{k(u)-\varepsilon})^{p^-}\}|v|^{p(x)-1} \in L^{\frac{p(x)}{p(x)-1}}(\Omega) \text{ a.e. in } \Omega.$$

Consequently, from Proposition 2.5, we get $W_n \to W$ in $L^{\frac{p(x)}{p(x)-1}}(\Omega)$, i.e., $I_2 \to 0$. Thus, $|\langle k'(u_n) - k'(u), v\rangle| \to 0$, as $n\to\infty$.

Similarly, we infer that $|\langle F'(u_n) - F'(u), v\rangle| \to 0$, as $n\to\infty$.

Hence, $F'(u), k'(u) : W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^*$ are continuous.

Remark 4.3 According to the proof of Lemma 4.2, we have

(1) $\forall u, v \in L^{p(x)}(\Omega)$, $|\langle k'(u), v\rangle| \le |v|_{p(x)}$.

(2) $\forall u, v \in L^{p(x)}(\Omega)$, $|\langle K'(u), v\rangle| \le |\nabla v|_{p(x)}$.

(3) $\forall u, v \in L^{p(x)}_{\beta(x)}(\partial\Omega)$, $|\langle H'(u), v\rangle| \le |v|_{(p(x),\ \beta(x))}$.

As $u \in M_1$, we can deduce that $k'(u) \ne 0$, and so M_1 is a C^1-submanifold of $W^{1,p(x)}(\Omega)$ with codimension 1. We denote $T_u(M_1)$ the tangent space at $u \in M_1$, i.e., $T_u(M_1) = \ker k'(u) = \{v \in W^{1,p(x)}(\Omega) : \langle k'(u), v\rangle = 0\}$, $\tilde F : M_1 \to \mathbb{R}$ the restriction of F on M_1, and $\tilde F'(u)$ the derivative of $\tilde F$ at $u \in M_1$, i.e., the restriction of $F'(u)$ on $T_u(M_1)$ (see [11]).

It is well known that if u is a critical point of $\tilde F$ on M_1, then (u,λ) is a solution to problem (4.1) (see [35]).

Lemma 4.4 Suppose $\tilde F$ *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* $W^{1,p(x)}(\Omega)$ *and* $\limsup_{n\to\infty}\langle \tilde F'(u_n), u_n - u\rangle \le 0$, *then* $u_n \to u$ *in* $W^{1,p(x)}(\Omega)$.

Proof Since

$$\langle \tilde F'(u_n), u_n\rangle = K(u_n) + k(u_n) + H(u_n) = ||u_n||_{p(x)} + |u_n|_{(p(x),\beta(x))}$$

and

$$\begin{aligned}\langle \tilde F'(u_n), u\rangle =& \langle K'(u_n), u\rangle + \langle k'(u_n), u\rangle + \langle H'(u_n), u\rangle \\ \le& |\nabla u|_{p(x)} + |u|_{p(x)} + |u|_{(p(x),\ \beta(x))} \\ =& ||u||_{p(x)} + |u|_{(p(x),\ \beta(x))}\end{aligned}$$

by using Remark 4.2, we have $\langle \tilde F'(u_n), u_n - u\rangle \ge ||u_n||_{p(x)} + |u_n|_{(p(x),\ \beta(x))} - ||u||_{p(x)} - |u|_{(p(x),\beta(x))}$ and

$$\limsup_{n\to\infty}(||u_n||_{p(x)} + |u_n|_{(p(x),\ \beta(x))} - ||u||_{p(x)} - |u|_{(p(x),\ \beta(x))}) \le \limsup_{n\to\infty}\langle \tilde F'(u_n), u_n - u\rangle \le 0.$$

Using Propositions 2.4 and 2.7 and noting that $u_n \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$, we have $u_n \to u$ in $L^{p(x)}(\Omega)$, $\nabla u_n \rightharpoonup \nabla u$ in $L^{p(x)}(\Omega)$ and $u_n \to u$ in $L^{p(x)}_{\beta(x)}(\partial\Omega)$. Therefore, together with the weak lower semi-continuity of the norm, we get

$$\begin{aligned}
&\limsup_{n\to\infty}(||u_n||_{p(x)} + |u_n|_{(p(x),\beta(x))}) \le |u|_{p(x)} + |\nabla u|_{p(x)} + |u|_{(p(x),\ \beta(x))} \\
&\le \liminf_{n\to\infty}\left(|u_n|_{p(x)} + |\nabla u_n|_{p(x)} + |u_n|_{(p(x),\ \beta(x))}\right) \\
&= \liminf_{n\to\infty}\left(||u_n||_{p(x)} + |u_n|_{(p(x),\ \beta(x))}\right).
\end{aligned}$$

Thus it follows that $\lim_{n\to\infty}||u_n||_{p(x)} = ||u||_{p(x)}$. Consequently, the conclusion holds as $W^{1,p(x)}(\Omega)$ is uniformly convex.

Lemma 4.5 *For all $c \in \mathbb{R}$, $\tilde{F}$ satisfies the $(PS)_c$ condition, i.e., for every sequences $\{u_n\} \subset M_1$ such that $\tilde{F}(u_n) \to c$ and $\tilde{F}'(u_n) \to 0$ has a convergent subsequence.*

Proof Suppose that $u \in M_1$ and $w = \frac{u}{\langle k'(u),u\rangle}$, it is easy to see that $k'(u) \neq 0$ and $w \notin T_u(M_1)$. Thus

$$W^{1,p(x)}(\Omega) = T_u(M_1) \oplus \{\alpha w : \alpha \in \mathbb{R}\}.$$

Let $P : W^{1,p(x)}(\Omega) \to T_u(M_1)$ be a natural projection. Then for any $v \in W^{1,p(x)}(\Omega)$, there exists a unique $\alpha \in \mathbb{R}$ such that $v = Pv + \alpha w$. Since $\langle k'(u), Pv\rangle = 0$, it follows $\alpha = \langle k'(u), v\rangle$. Consequently,

$$\begin{aligned}
\langle \tilde{F}'(u), v\rangle &= \langle F'(u), Pv\rangle = \langle F'(u), v\rangle - \alpha\langle F'(u), w\rangle \\
&= \langle F'(u), v\rangle - \langle k'(u), v\rangle \cdot \frac{\langle F'(u), u\rangle}{\langle k'(u), u\rangle},
\end{aligned}$$

and

$$\tilde{F}'(u) = F'(u) - \frac{\langle F'(u), u\rangle}{\langle k'(u), u\rangle} \cdot k'(u).$$

For any sequences $\{u_n\} \subset M_1$ such that $\tilde{F}(u_n) \to c$ and $\tilde{F}'(u_n) \to 0$, then $F(u_n) \to c$ and there is a sequence $\{c_n\} \subset \mathbb{R}$ such that $F'(u_n) - c_n k'(u_n) \to 0$, where

$$c_n = \frac{\langle F'(u_n), u_n\rangle}{\langle k'(u_n), u_n\rangle} \to c, \text{ as } n \to \infty.$$

Noting that $F(u_n) \to c$, we conclude that $\{u_n\}$ is bounded in $W^{1,p(x)}(\Omega)$, and there exists a subsequence that we still denote $\{u_n\}$ such that $u_n \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$ and $u_n \to u$ in $L^{p(x)}(\Omega)$. Then, from Remark 4.3, we deduce that

$$0 \le |\langle k'(u_n), u_n - u\rangle| \le |u_n - u|_{p(x)} \to 0.$$

Since

$$\begin{aligned}
0 \le&|\langle F'(u_n), u_n - u\rangle - c_n\langle k'(u_n), u_n - u\rangle| \\
\le&\|F'(u_n) - c_n k'(u_n)\|_{\left(W^{1,p(x)}(\Omega)\right)^*}\|u_n - u\|_{W^{1,p(x)}(\Omega)} \to 0,
\end{aligned}$$

we have, $\langle F'(u_n), u_n - u\rangle \to 0$. Since Lemma 4.4 holds, we deduce that $u_n \to u$ in $W^{1,p(x)}(\Omega)$. By virtue of $\{u_n\} \subset M_1$ and $u_n \to u$ in $L^{p(x)}(\Omega)$, we obtain $u \in M_1$ and the proof is achieved.

Let M_1, F, k and $\tilde{F}$ be defined above. We consider

$$c_n = \inf_{A\in\Sigma,\gamma(A)\geq n} \sup_{u\in A} \tilde{F}(u). \tag{4.3}$$

The fact that $W^{1,p(x)}(\Omega)$ is a separable and reflexive Banach space, then there exist $\{e_n\}_{n=1}^{\infty} \subset W^{1,p(x)}(\Omega)$ and $\{f_n\}_{n=1}^{\infty} \subset \big(W^{1,p(x)}(\Omega)\big)^*$ such that

$$f_n(e_m) = \begin{cases} 1, & \text{if } n = m,\\ 0, & \text{if } n \neq m, \end{cases}$$

$$W^{1,p(x)}(\Omega) = \overline{\text{span}}\{e_n : n = 1, 2, \ldots\}, \quad (W^{1,p(x)}(\Omega))^* = \overline{\text{span}}^{W^*}\{f_n : n = 1, 2, \ldots\}.$$

Set

$$(W^{1,p(x)}(\Omega))_n = \text{span}\{e_i,\ 1 \leq i \leq n\},\ Y_n = \bigoplus_{j=1}^{n}(W^{1,p(x)}(\Omega))_j,\ Z_n = \overline{\bigoplus_{j=n}^{\infty}(W^{1,p(x)}(\Omega))_j}.$$

Then we have the following conclusions.

Lemma 4.6 *According to the definitions of M_1 and Z_n, we have*

$$\lim_{n\to\infty} \inf_{u\in Z_n\cap M_1} \|u\|_{p(x)} = +\infty.$$

Proof Since $k(u) = |u|_{p(x)}$, $k(0) = 0$.

If $u_n \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$, then the embedding theorem implies $u_n \to u$ in $L^{p(x)}(\Omega)$. So that we can deduce that $|u_n|_{p(x)} \to |u|_{p(x)}$. Hence, $k : W^{1,p(x)}(\Omega) \to \mathbb{R}$ is weakly-strongly continuous.

Assume that there exist $c_0 > 0$ and $\{u_n\} \subset Z_n \cap M_1$ such that $\|u_n\|_{p(x)} \leq c_0$ for any n. Then,

$$\lim_{n\to\infty} \sup_{u\in Z_n,\|u\|_{p(x)}\leq c_0} |k(u)| \geq \lim_{n\to\infty} \sup_{u\in Z_n\cap M_1,\|u\|_{p(x)}\leq c_0} |k(u)| \geq \lim_{n\to\infty} |k(u_n)| = 1 > 0.$$

In [17, Lemma 3.3], it is a contradiction.

Lemma 4.7 *Suppose that (4.3) holds, for any natural number n, then $1 < c_n \leq c_{n+1}$ and $\lim_{n\to\infty} c_n = +\infty$.*

Proof (1) For each n, as $\gamma(A) \geq n+1$, then also $\gamma(A) \geq n$, thus by using (4.3), $c_n \leq c_{n+1}$. Note that for any $u \in \sum$, $\tilde{F}(u) = |\nabla u|_{p(x)} + 1 + |u|_{(p(x),\ \beta(x))} > 1$. In fact, if $\tilde{F}(u) = 1$, $|\nabla u|_{p(x)} = 0$ and $|u|_{(p(x),\beta(x))} = 0$, then $u \equiv 0$, it is a contradiction with $|u|_{p(x)} = 1$. Hence $c_n > 1$.

(2) Lemma 4.6, for any $c > 0$, implies that there exists n_0 such that, as $n > n_0$, $\|u\|_{p(x)} > c$ and $u \in Z_n \cap M_1$. On one hand, [34, Proposition 2.3] and statement (b) imply that for any $A \in \sum$, $\gamma(A \cap Y_{n-1}) \leq n-1$; on the other hand, the codimension of Z_n is less than or equal to $n-1$, and so for each $A \in \Sigma$ with $\gamma(A) \geq n$, $A \cap Z_n$ is nonempty by using (g) of [34, Proposition 2.3]. Then,

$$\begin{aligned} c_n &= \inf_{A\in\Sigma} \sup_{u\in A,\gamma(A)\geq n} \tilde{F}(u) = \inf_{A\in\Sigma} \max\{\sup_{u\in A\cap(X\setminus Y_{n-1}),\gamma(A)\geq n} \tilde{F}(u), \sup_{u\in A\cap Y_{n-1},\gamma(A)\geq n} \tilde{F}(u)\}\\ &= \inf_{A\in\Sigma} \max\{\sup_{u\in A\cap(X\setminus Y_{n-1}\setminus Z_n),\gamma(A)\geq n} \tilde{F}(u), \sup_{u\in A\cap Z_n,\gamma(A)\geq n} \tilde{F}(u), \sup_{u\in A\cap Y_{n-1},\gamma(A)\geq n} \tilde{F}(u)\} \end{aligned}$$

$$= \inf_{A\in\Sigma} \max\{\sup_{u\in A\cap(X\setminus Y_{n-1}\setminus Z_n),\gamma(A)\geq n} \tilde{F}(u), \sup_{u\in A\cap Z_n,\gamma(A)\geq n} \tilde{F}(u)\}$$

$$\geq \inf_{A\in\Sigma} \sup_{u\in A\cap Z_n,\gamma(A)\geq n} \tilde{F}(u) \geq c.$$

Due to the arbitrariness of c, the assertion is proved. $\square$

From Lemmas 4.5, 4.7 and the Ljusternik-Schnirelmann principle [34], we can easily deduce the statement given by the following theorem.

Theorem 4.8 *Each c_n $(n = 1, 2, \ldots)$ defined by (4.3) is a critical value of $\tilde{F}$, $1 < c_n \leq c_{n+1}$ and $c_n \to +\infty$ as $n \to \infty$. Moreover, the problem (3.2) has infinitely many solution pairs $\{\pm u_n,\ \mu_n\}$ such that $k(\pm u_n) = 1$, $F(\pm u_n) = c_n$ and $\mu_n = \frac{\langle F'(u_n),u_n\rangle}{\langle k'(u_n),u_n\rangle} = K(u_n) + H(u_n) + 1$.*

Remark 4.9 According to Theorem 4.8, we have

$$\mu_n = c_n = \inf_{A\in\Sigma,\gamma(A)\geq n} \sup_{u\in A} \tilde{F}(u)\ (n = 1, 2, \ldots).$$

Indeed, $\mu_n = \mu_n \cdot 1 = \mu_n \cdot k(u_n) = \mu_n \cdot \langle k'(u_n), u_n\rangle = \langle F'(u_n), u_n\rangle = F(u_n) = c_n$.

Therefore, from Theorem 4.8, Eq. (3.2) and Remark 4.9, we infer that Theorem 4.1 holds.

Remark 4.10 Using the same method, we can discuss the Neumann eigenvalue problem.

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