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Maps to Ordered Topological Vector Spaces and Stratifiable Spaces

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Abstract In the paper [Monotone countable paracompactness and maps to ordered topological vector spaces, Top. Appl., 2014, 169(3): 51–70], Yamazaki initiated the study on maps with values into ordered topological vector spaces. Characterizations of monotonically countably paracompact spaces and some other spaces in terms of maps to ordered topological vector spaces were obtained. In this paper, following Yamazaki's method, we present some characterizations of stratifiable spaces and k-semi-stratifiable spaces in terms of maps with values into ordered topological vector spaces.

Keywords stratifiable spaces; k-semi-stratifiable spaces; maps to ordered topological vector spaces; lower (upper) semi-continuity

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1. Introduction and preliminaries

Throughout, all spaces are assumed to be T_1 topological spaces. A vector space always means a real vector space. The origin of a vector space is denoted by $\mathbf{0}$. The set of all positive integers is denoted by $\mathbb{N}$. $\mathcal{T}_X$ and $\mathcal{F}_X$ denote the topology and the family of all closed subsets of a space X, respectively. For a space X and $A \subset X$, we use $\operatorname{int} A$ and $\overline{A}$ to denote the interior and the closure of A in X, respectively. Also, χ_A denotes the characteristic function of A.

A partially ordered vector space $(Y, \leq)$ is called an ordered vector space if

(1) For each $x, y, z \in Y$, if $x \leq y$, then $x + z \leq y + z$;

(2) For each $x, y \in Y$ and $r \in \mathbb{R}$ with $r \geq 0$, if $x \leq y$, then $rx \leq ry$.

For $y_1, y_2 \in Y$, $y_1 \leq y_2$ will be sometimes written as $y_2 \geq y_1$. Also, we write $y_1 < y_2$ if $y_1 \leq y_2$ and $y_1 \neq y_2$.

A topological vector space Y is called an ordered topological vector space if Y is an ordered vector space and the positive cone $Y^+ = \{y \in Y : y \geq \mathbf{0}\}$ is closed in Y.

Let Y be an ordered topological vector space and $e \in Y^+$. Then e is called an interior point of Y^+ if $e \in \operatorname{int}_Y(Y^+)$. If e is an interior point of Y^+ and $e > \mathbf{0}$, then e is called a positive interior point. It is clear (see [1]) that if e is an interior point of Y^+, then $-re + Y^+$ and $re - Y^+$ are both $\mathbf{0}$-neighborhoods for each $r > 0$.

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Recall that a real-valued function f on a space X is called lower (resp., upper) semi-continuous if for any real number r, the set $\{x \in X : f(x) > r\}$ (resp., $\{x \in X : f(x) < r\}$) is open. In [2], the notion of real-valued semi-continuous functions was generalized to the semi-continuous maps with values into ordered topological vector spaces. Let X be a topological space and Y an ordered topological vector space. A map $f : X \to Y$ is called lower semi-continuous [2] if the set-valued mapping $\varphi : X \to 2^Y$, defined by letting $\varphi(x) = f(x) - Y^+$ for each $x \in X$, is lower semi-continuous. f is upper semi-continuous if $-f$ is lower semi-continuous. We write $C(X, [\mathbf{0}, e]_Y)$ (resp., $L(X, [\mathbf{0}, e]_Y)$, $U(X, [\mathbf{0}, e]_Y)$) for the set of all continuous (resp., lower semi-continuous, upper semi-continuous) maps from X to Y with values in $Y^+ \cap (e - Y^+)$. Let f and g be two maps from a space X to an ordered topological vector space. We write $f \le g$ if $f(x) \le g(x)$ for all $x \in X$.

Definition 1.1 ([3]) *A space X is called stratifiable if there is a map $\rho : \mathbb{N} \times \mathcal{T}_X \to \mathcal{T}_X$ such that*

(1) $U = \bigcup_{n\in\mathbb{N}} \rho(n, U) = \bigcup_{n\in\mathbb{N}} \overline{\rho(n, U)}$ for each $U \in \mathcal{T}_X$;

(2) If $U \subset V$, then $\rho(n, U) \subset \rho(n, V)$ for all $n \in \mathbb{N}$.

Definition 1.2 ([4]) *A space X is k-semi-stratifiable if and only if there is a map $\varrho : \mathbb{N} \times \mathcal{T}_X \to \mathcal{F}_X$, such that*

(1) $\bigcup_{n\in\mathbb{N}} \varrho(n, U) = U$ for each $U \in \mathcal{T}_X$;

(2) If $U \subset V$, then $\varrho(n, U) \subset \varrho(n, V)$ for each $n \in \mathbb{N}$;

(3) For each compact subset K of X and $U \in \mathcal{T}_X$ with $K \subset U$, there is $n \in \mathbb{N}$ such that $K \subset \varrho(n, U)$.

Recall that a sequence $\langle f_n \rangle$ of functions on a space X is said to be weakly locally uniformly convergent to a function f on X if for each $x \in X$ and $\varepsilon > 0$, there exists an open neighborhood U of x and $m \in \mathbb{N}$ such that $|f_n(y) - f(y)| < \varepsilon$ for all $n \ge m$ and $y \in U$ (see [5]). In [6], the notion of the weakly locally uniform convergence of real-valued functions was generalized to maps with values into topological vector spaces as follows.

Definition 1.3 ([6]) *Let X be a topological space and Y a topological vector space. A sequence $\langle f_n \rangle$ of maps from X to Y is said to be weakly locally uniformly convergent to a map $f : X \to Y$ on X if for each $x \in X$ and any $\mathbf{0}$-neighborhood V in Y, there exists an open neighborhood U of x and $m \in \mathbb{N}$ such that $f_n(x') - f(x') \in V$ for all $n \ge m$ and $x' \in U$.*

In [1], Yamazaki generalized real-valued functions in some insertion theorems to maps with values into ordered topological vector spaces. Characterizations of monotonically countably paracompact spaces and some other spaces in terms of maps to ordered topological vector spaces were obtained. In this paper, we shall carry on the similar study for stratifiable spaces and k-semi-stratifiable spaces. Some characterizations of stratifiable spaces and k-semi-stratifiable spaces in terms of maps with values into ordered topological vector spaces are obtained. These results generalize real-valued functions in some known results to maps with values into ordered topological vector spaces.

2. Some basic lemmas

In this section, we list some basic lemmas which will be used in the sequel.

Lemma 2.1 ([1]) *Let X be a topological space and Y an ordered topological vector space. For a map $f : X \to Y$, the following (1) and (2) are equivalent, and (1) implies (3).*

(1) f is lower (resp., upper) semi-continuous.

(2) For each $x \in X$ and each $\mathbf{0}$-neighborhood V, there exists a neighborhood O_x of x such that $f(O_x) \subset f(x) + V + Y^+$ (resp., $f(O_x) \subset f(x) + V - Y^+$).

(3) $f^{-1}(y - Y^+)$ (resp., $f^{-1}(y + Y^+)$) is closed in X for each $y \in Y$.

Lemma 2.2 ([7]) *Let X be a topological space and Y an ordered topological vector space and f a real-valued function on X. If f is continuous, then for each $y \in Y$, the map $g : X \to Y$ defined by letting $g(x) = f(x)y$ for each $x \in X$ is continuous. If f is lower (resp., upper) semi-continuous, then for each $y \in Y^+$, the map $g : X \to Y$ defined by letting $g(x) = f(x)y$ for each $x \in X$ is lower (resp., upper) semi-continuous.*

Lemma 2.3 *Let Y be an ordered topological vector space and $\langle a_n \rangle$ a sequence of Y such that $\langle a_n \rangle$ converges to a. If $b \le a_n$ (resp., $a_n \le b$) for each $n \in \mathbb{N}$, then $b \le a$ (resp., $a \le b$).*

Proof Assume that $b \le a_n$ for each $n \in \mathbb{N}$ and $a - b \notin Y^+$. Since Y^+ is closed, there exists an open neighborhood V of $a - b$ such that $V \cap Y^+ = \emptyset$. Then $V + b$ is an open neighborhood of a. Since $\langle a_n \rangle$ converges to a, there exists $m \in \mathbb{N}$ such that $a_n \in V + b$ for all $n > m$. Thus $a_n - b \notin Y^+$, a contradiction. □

Lemma 2.4 ([6]) *Let X be a topological space and Y an ordered topological vector space. If a sequence $\langle f_n \rangle$ of real valued functions on X weakly locally uniformly converges to a function f on X, then for each $y \in Y$, the sequence $\langle f_n y \rangle$ weakly locally uniformly converges to fy.*

3. Stratifiable spaces

In this section, we shall give some characterizations of stratifiable spaces in terms of maps to ordered topological vector spaces.

Recall that a space X is called monotonically normal [8] if for each pair (F, H) of disjoint closed subsets of X, one can assign an open set $G(F, H)$ such that $F \subset G(F, H) \subset \overline{G(F, H)} \subset X \setminus H$ and $G(F, H) \subset G(F', H')$ whenever $F \subset F'$, $H \supset H'$.

The following characterization of monotonically normal spaces with real-valued functions is due to Kubiak.

Lemma 3.1 ([9]) *A space X is monotonically normal if and only if there exists an operator Λ assigning to each pair (f, g) of real-valued functions with f upper semi-continuous, g lower semi-continuous and $f \le g$, a continuous function $\Lambda(f, g)$ such that $f \le \Lambda(f, g) \le g$ and $\Lambda(f, g) \le \Lambda(f', g')$ whenever $f \le f'$, $g \le g'$.*

Theorem 3.2 *Let X be a topological space and Y an ordered topological vector space with a positive interior point e of Y^+. Then the following are equivalent.*

(a) X is stratifiable.

(b) There exists a map $\varphi : \mathcal{T}_X \to C(X, [\mathbf{0}, e]_Y)$ such that $X \setminus U = \varphi(U)^{-1}(\mathbf{0})$ for each $U \in \mathcal{T}_X$ and $\varphi(U) \le \varphi(V)$ whenever $U \subset V$.

(c) There exist two maps $\phi : \mathcal{T}_X \to L(X, [\mathbf{0}, e]_Y)$ and $\psi : \mathcal{T}_X \to U(X, [\mathbf{0}, e]_Y)$ such that $\phi(U) \le \psi(U)$ and $X \setminus U = \phi(U)^{-1}(\mathbf{0}) = \psi(U)^{-1}(\mathbf{0})$ for each $U \in \mathcal{T}_X$ and $\phi(U) \le \phi(V)$, $\psi(U) \le \psi(V)$ whenever $U \subset V$.

Proof (a) $\Rightarrow$ (b). Let ρ be the map in Definition 1.1. For each $U \in \mathcal{T}_X$ and each $n \in \mathbb{N}$, let $\alpha(n, U) = \chi_{\overline{\rho(n,U)}}$. Then $\alpha(n, U)$ is upper semi-continuous and $\alpha(n, U) \le \chi_U$ for each $n \in \mathbb{N}$. Since X is stratifiable, it is monotonically normal. Let Λ be the operator in Lemma 3.1 and let $\phi(n, U) = \Lambda(\alpha(n, U), \chi_U)$ for each $n \in \mathbb{N}$. Then let

$$\varphi(U) = \sum_{n=1}^{\infty} \frac{1}{2^n} \phi(n, U) e.$$

By Lemma 2.2, $\varphi(U) \in C(X, [\mathbf{0}, e]_Y)$.

If $U \subset V$, then $\chi_U \le \chi_V$ and $\alpha(n, U) \le \alpha(n, V)$ for each $n \in \mathbb{N}$ from which it follows that $\phi(n, U) \le \phi(n, V)$. Thus $\varphi(U) \le \varphi(V)$.

For each $x \in X$, if $x \in X \setminus U$, then $\chi_U(x) = 0$ from which it follows that $\phi(n, U)(x) = 0$ for all $n \in \mathbb{N}$. Thus $\varphi(U)(x) = \mathbf{0}$. Conversely, If $\varphi(U)(x) = \mathbf{0}$, then $\alpha(n, U)(x) = 0$ and so $x \notin \overline{\rho(n, U)}$ for all $n \in \mathbb{N}$. Thus $x \in X \setminus U$.

(b) $\Rightarrow$ (c) is clear, since a continuous map is both lower semi-continuous and upper semi-continuous.

(c) $\Rightarrow$ (a). For each $U \in \mathcal{T}_X$ and $n \in \mathbb{N}$, let

$$\rho(n, U) = X \setminus \phi(U)^{-1}(\frac{1}{2^n} e - Y^+), \quad F(n, U) = X \setminus \operatorname{int}(\psi(U)^{-1}(\frac{1}{2^n} e - Y^+)).$$

Since $\phi(U)$ is lower semi-continuous, by Lemma 2.1, $\phi(U)^{-1}(\frac{1}{2^n} e - Y^+)$ is a closed subset of X. Thus $\rho(n, U)$ is open in X. It is clear that if $U \subset V$, then $\rho(n, U) \subset \rho(n, V)$ for each $n \in \mathbb{N}$. Since $\phi(U) \le \psi(U)$, we have $\rho(n, U) \subset F(n, U)$ and thus $\overline{\rho(n, U)} \subset F(n, U)$.

Let $x \in X \setminus U$. Then $\psi(U)(x) = \mathbf{0}$. For each $n \in \mathbb{N}$, since $\psi(U)$ is upper semi-continuous and $\frac{1}{2^n} e - Y^+$ is a $\mathbf{0}$-neighborhood, by Lemma 2.1, there exists an open neighborhood O_x of x such that

$$\psi(U)(O_x) \subset \psi(U)(x) + \frac{1}{2^n} e - Y^+ - Y^+ = \frac{1}{2^n} e - Y^+.$$

Thus $x \in \operatorname{int}(\psi(U)^{-1}(\frac{1}{2^n} e - Y^+))$ and so $x \notin F(n, U) \supset \overline{\rho(n, U)}$. This implies that $\overline{\rho(n, U)} \subset U$ for each $n \in \mathbb{N}$. If $x \notin \bigcup_{n \in \mathbb{N}} \rho(n, U)$, then $\phi(U)(x) \le \frac{1}{2^n} e$ for each $n \in \mathbb{N}$ and thus $\phi(U)(x) = \mathbf{0}$. This implies that $x \in X \setminus U$ and so $U \subset \bigcup_{n \in \mathbb{N}} \rho(n, U)$.

By Definition 1.1, X is a stratifiable space. $\square$

Lemma 3.3 ([10]) *A space X is stratifiable if and only if there exists an operator Φ assigning to each lower semi-continuous function $f : X \to [0, 1]$, a continuous function $\Phi(f) : X \to [0, 1]$ with*

$\Phi(f) \leq f$ *such that* $\Phi(f) \leq \Phi(f')$ *whenever* $f \leq f'$ *and* $0 < \Phi(f)(x) < h(x)$ *whenever* $f(x) > 0$.

Remark 3.4 Let Φ be the operator in Lemma 3.3. It follows that if $f(x) > 0$, then $\Phi(f)(x) > 0$ and so $\Phi(f)(x) > r$ for some $r > 0$. Put $O_x = \Phi(f)^{-1}((r, 1])$. Since $\Phi(f)$ is continuous, O_x is an open neighborhood of x and $\Phi(f)(x') > r$ for each $x' \in O_x$.

Theorem 3.5 *Let* X *be a topological space and* Y *an ordered topological vector space with a positive interior point* e *of* Y^+. *Then the following are equivalent.*

(a) X *is stratifiable.*

(b) For each $U \in \mathcal{T}_X$, *there is an increasing sequence* $\{\delta_{nU} \in C(X, [\mathbf{0}, e]_Y) : n \in \mathbb{N}\}$ *of maps such that*

(b1) $\langle \delta_{nU} \rangle$ *weakly locally uniformly converges to* $\chi_U e$ *on* U *and pointwise converges to* $\chi_U e$ *on* $X \setminus U$*;*

(b2) If $U \subset V$, *then* $\delta_{nU} \leq \delta_{nV}$ *for each* $n \in \mathbb{N}$.

(c) For each $U \in \mathcal{T}_X$, *there exist two increasing sequences* $\{\delta_{nU} \in L(X, [\mathbf{0}, e]_Y) : n \in \mathbb{N}\}$ *and* $\{\eta_{nU} \in U(X, [\mathbf{0}, e]_Y) : n \in \mathbb{N}\}$ *of maps such that*

(c1) $\langle \delta_{nU} \rangle$ *and* $\langle \eta_{nU} \rangle$ *pointwise converge to* $\chi_U e$ *on* X*;*

(c2) If $U \subset V$, *then* $\delta_{nU} \leq \delta_{nV}$ *for each* $n \in \mathbb{N}$*;*

(c3) For each $U \in \mathcal{T}_X$ *and* $n \in \mathbb{N}$, $\delta_{nU} \leq \eta_{nU}$.

Proof (a) $\Rightarrow$ (b). Let Φ be the operator in Lemma 3.3. For each $n \in \mathbb{N}$ and $U \in \mathcal{T}_X$, let $\zeta_{nU} = \min\{1, n\Phi(\chi_U)\}$ and $\delta_{nU} = \zeta_{nU} e$. Then $\{\delta_{nU} \in C(X, [\mathbf{0}, e]_Y) : n \in \mathbb{N}\}$ is an increasing sequence of maps which satisfies (b2).

By Lemma 2.4, to show that $\langle \delta_{nU} \rangle$ weakly locally uniformly converges to $\chi_U e$ on U, it suffices to show that $\langle \zeta_{nU} \rangle$ weakly locally uniformly converges to χ_U on U. Let $\varepsilon > 0$. For each $x \in U$, $\chi_U(x) = 1$. By Remark 3.4, there exist an open neighborhood O_x of x and $m \in \mathbb{N}$ such that $m\Phi(\chi_U)(y) > 1$ for each $y \in O_x$. Let $U_x = O_x \cap U$. Then U_x is an open neighborhood of x in U. For each $n \geq m$ and $y \in U_x$, $n\Phi(\chi_U)(y) > 1$ from which it follows that $\zeta_{nU}(y) = 1$ and thus $|\zeta_{nU}(y) - \chi_U(y)| = 0 < \varepsilon$. This implies that $\langle \zeta_{nU} \rangle$ weakly locally uniformly converges to χ_U on U. If $x \in X \setminus U$, then $\Phi(\chi_U)(x) \leq \chi_U(x) = 0$. Thus $\delta_{nU}(x) = \zeta_{nU}(x)e = \mathbf{0}$ for each $n \in \mathbb{N}$. Therefore, $\langle \delta_{nU}(x) \rangle$ converges to $\chi_U(x)e = \mathbf{0}$.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a). For each $n \in \mathbb{N}$ and $U \in \mathcal{T}_X$, let

$$\rho(n, U) = X \setminus \delta_{nU}^{-1}(\frac{1}{2}e - Y^+)$$

and

$$F(n, U) = X \setminus \mathrm{int}(\eta_{nU}^{-1}(\frac{1}{2}e - Y^+)).$$

By Lemma 2.1, $\{\rho(n, U) : n \in \mathbb{N}\}$ is an increasing sequence of open subsets of X. From (c3) it follows that $\rho(n, U) \subset F(n, U)$ and so $\overline{\rho(n, U)} \subset F(n, U)$.

If $U \subset V$, then it follows from (c2) that $\rho(n, U) \subset \rho(n, V)$.

If $x \notin \bigcup_{n\in\mathbb{N}} \rho(n,U)$, then $\delta_{nU}(x) \le \frac{1}{2}e$ for each $n \in \mathbb{N}$. By (c1) and Lemma 2.3, $\chi_U(x)e \le \frac{1}{2}e$ from which it follows that $\chi_U(x) \le \frac{1}{2}$. This implies that $x \notin U$. Hence, $U \subset \bigcup_{n\in\mathbb{N}} \rho(n,U)$. If $x \notin U$, then by (c1), $\langle \eta_{nU}(x)\rangle$ converges to $\chi_U(x)e = \mathbf{0}$. Since $\frac{1}{4}e - Y^+$ is a $\mathbf{0}$-neighborhood, there exists $m \in \mathbb{N}$ such that $\eta_{nU}(x) \in \frac{1}{4}e - Y^+$ for each $n > m$. Since η_{nU} is upper semi-continuous and $\frac{1}{4}e - Y^+$ is a $\mathbf{0}$-neighborhood, by Lemma 2.1, there exists a neighborhood O_x of x such that

$$\eta_{nU}(O_x) \subset \eta_{nU}(x) + \frac{1}{4}e - Y^+ \subset \frac{1}{2}e - Y^+.$$

Thus $x \in \mathrm{int}(\eta_{nU}^{-1}(\frac{1}{2}e - Y^+))$ from which it follows that $x \notin F(n,U) \supset \overline{\rho(n,U)}$ for each $n > m$. This indicates that $\bigcup_{n\in\mathbb{N}} \overline{\rho(n,U)} \subset U$.

Consequently, X is a stratifiable space. □

Comparing with the above characterizations of stratifiable spaces, it is a natural question that whether the real-valued functions in Lemma 3.3 can be generalized to maps with values into ordered topological vector spaces. We pose it as a question as follows.

Question 3.6 Let X be a topological space and Y an ordered topological vector space with a positive interior point of Y^+. Are the following conditions equivalent?

(1) X is a stratifiable space.

(2) There exists an operator Φ assigning to each map $f \in L(X,[\mathbf{0},e]_Y)$, a map $\Phi(f) \in C(X,[\mathbf{0},e]_Y)$ with $\Phi(f) \le f$ such that $\Phi(f) \le \Phi(f')$ whenever $f \le f'$, and $\mathbf{0} < \Phi(f)(x) < f(x)$ whenever $f(x) > \mathbf{0}$.

4. k-semi-stratifiable spaces

In this section, we prove some analogous results for k-semi-stratifiable spaces to those for stratifiable spaces.

Theorem 4.1 *Let X be a topological space and Y an ordered topological vector space with a positive interior point e of Y^+. Then X is a k-semi-stratifiable space if and only if there exists a map $\varphi : \mathcal{T}_X \to U(X,[\mathbf{0},e]_Y)$ such that*

(1) For each $U \in \mathcal{T}_X$, $X \setminus U = \varphi(U)^{-1}(\mathbf{0})$;

(2) If $U \subset V$, then $\varphi(U) \le \varphi(V)$;

(3) For each compact $K \subset X$ and $U \in \mathcal{T}_X$ with $K \subset U$, there is $r > 0$ such that $\varphi(U)(x) \ge re$ for each $x \in K$.

Proof Let ϱ be the map in Definition 1.2 which is increasing with respect to n. For each $U \in \mathcal{T}_X$, let

$$\varphi(U) = \sum_{n=1}^{\infty} \frac{1}{2^n}\chi_{\varrho(n,U)}e.$$

Then $\varphi(U) \in U(X,[\mathbf{0},e]_Y)$ and it is easy to verify that $X \setminus U = \varphi(U)^{-1}(\mathbf{0})$ and $\varphi(U) \le \varphi(V)$ whenever $U \subset V$.

Let K be a compact subset of X and $U \in \mathcal{T}_X$ with $K \subset U$. By (c) of Definition 1.2, there exists $n \in \mathbb{N}$ such that $K \subset \varrho(n,U)$. Let $m = \min\{n \in \mathbb{N} : K \subset \varrho(n,U)\}$. Then for each $x \in K$,

$$\varphi(U)(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{\varrho(n,U)}(x)e \geq \sum_{n=m}^{\infty} \frac{1}{2^n} \chi_{\varrho(n,U)}(x)e = \sum_{n=m}^{\infty} \frac{1}{2^n} e = \frac{1}{2^{m-1}} e.$$

Conversely, for each $U \in \mathcal{T}_X$ and $n \in \mathbb{N}$, let

$$\varrho(n,U) = X \setminus \mathrm{int}(\varphi(U)^{-1}(\frac{1}{2^n}e - Y^+)).$$

By (2), it is clear that if $U \subset V$, then $\varrho(n,U) \subset \varrho(n,V)$ for each $n \in \mathbb{N}$.

Let $x \in X \setminus U$. Then $\varphi(U)(x) = \mathbf{0}$. For each $n \in \mathbb{N}$, since $\varphi(U)$ is upper semi-continuous and $\frac{1}{2^n}e - Y^+$ is a $\mathbf{0}$-neighborhood, there exists an open neighborhood O_x of x such that

$$\varphi(U)(O_x) \subset \varphi(U)(x) + \frac{1}{2^n}e - Y^+ - Y^+ = \frac{1}{2^n}e - Y^+.$$

Thus $x \in \mathrm{int}(\varphi(U)^{-1}(\frac{1}{2^n}e - Y^+))$ and so $x \notin \varrho(n,U)$. This implies that $\bigcup_{n\in\mathbb{N}} \rho(n,U) \subset U$. If $x \notin \bigcup_{n\in\mathbb{N}} \varrho(n,U)$, then $\varphi(U)(x) \in \frac{1}{2^n}e - Y^+$ for each $n \in \mathbb{N}$ from which it follows that $\varphi(U)(x) = \mathbf{0}$. Thus $x \in X \setminus U$. This implies that $U \subset \bigcup_{n\in\mathbb{N}} \varrho(n,U)$.

Now, let K be a compact subset of X and $U \in \mathcal{T}_X$ with $K \subset U$. By (3), there exists $n \in \mathbb{N}$ such that $\varphi(U)(x) \geq \frac{1}{2^{n-1}}e$ for each $x \in K$. Thus $x \notin \varphi(U)^{-1}(\frac{1}{2^n}e - Y^+))$ from which it follows that

$$x \in X \setminus \mathrm{int}(\varphi(U)^{-1}(\frac{1}{2^n}e - Y^+)) = \varrho(n,U).$$

Therefore, $K \subset \varrho(n,U)$. By Definition 1.2, X is a k-semi-stratifiable space. □

Proposition 4.2 *Let X be a topological space and Y an ordered topological vector space with a positive interior point e of Y^+. Then X is a k-semi-stratifiable space if and only if for each $U \in \mathcal{T}_X$, there is an increasing sequence $\{\delta_{nU} \in U(X, [\mathbf{0}, e]_Y) : n \in \mathbb{N}\}$ of maps such that*

(1) $\langle \delta_{nU} \rangle$ pointwise converges to $\chi_U e$ on X;

(2) If $U \subset V$, then $\delta_{nU} \leq \delta_{nV}$ for each $n \in \mathbb{N}$;

(3) For each compact $K \subset X$ and $U \in \mathcal{T}_X$ with $K \subset U$, there is $n \in \mathbb{N}$ such that $\delta_{nU}(x) = e$ for each $x \in K$.

Proof Let ϱ be the map in Definition 1.2 and assume that ϱ is increasing with respect to n. For each $n \in \mathbb{N}$ and $U \in \mathcal{T}_X$, let $\delta_{nU} = \chi_{\varrho(n,U)} e$. Then $\{\delta_{nU} \in U(X, [\mathbf{0}, e]_Y) : n \in \mathbb{N}\}$ is an increasing sequence of maps as required.

Conversely, for each $n \in \mathbb{N}$ and $U \in \mathcal{T}_X$, let $\varrho(n,U) = X \setminus \mathrm{int}(\delta_{nU}^{-1}(\frac{1}{2}e - Y^+))$. Then with a similar argument as that in the proof of (c) $\Rightarrow$ (a) in Theorem 3.5, one readily shows that $\bigcup_{n\in\mathbb{N}} \varrho(n,U) = U$ and $\varrho(n,U) \subset \varrho(n,V)$ for each $n \in \mathbb{N}$ whenever $U \subset V$.

Now, let K be a compact subset of X and $U \in \mathcal{T}_X$ with $K \subset U$. By (3), there exists $n \in \mathbb{N}$ such that $\delta_{nU}(x) = e$ for each $x \in K$. Thus $x \in \varrho(n,U)$ from which it follows that $K \subset \varrho(n,U)$.

Consequently, X is a k-semi-stratifiable space. □

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