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Lanne's T-functor and Hypersurfaces

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Abstract Through discussing the transformation of the invariant ideals, we firstly prove that the **T**-functor can only decrease the embedding dimension in the category of unstable algebras over the Steenrod algebra. As a corollary we obtain that the **T**-functor preserves the hypersurfaces in the category of unstable algebras. Then with the applications of these results to invariant theory, we provide an alternative proof that if the invariant of a finite group is a hypersurface, then so are its stabilizer subgroups. Moreover, by several counter-examples we demonstrate that if the invariants of the stabilizer subgroups or Sylow p-subgroups are hypersurfaces, the invariant of the group itself is not necessarily a hypersurface.

Keywords **T**-functor; hypersurface; pointwise stabilizers

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1. Introduction

The **T**-functor is a remarkable functor, which takes an unstable module over the Steenrod algebra to another object of the same type. It was first built in [1] by Lannes to study the cohomology of function spaces. This functor has played important roles in many problems and continue to yield exciting applications, such as the Sullivan conjecture [2] and the original proof of the Landweber-Stong conjecture [3]. One of the strengths of **T**-functor lies in the fact that it preserves many homological properties. For example, the polynomail algebra, the complete intersection and the Cohen-Macaulay ring [3,4]. It is Dwyer and Wilkerson who realized the significance of the **T**-functor for invariant theory [4] and gave a new proof of Steinberg's [5] and Nakajama's [6] result. But their proof is confined in the prime field. Then the work of Larry Smith constructed the **T**-functor for any finite field and provided more properties on the generalized **T**-functor [3,7,8]. In this paper, we will make use of Smith's reconstruction of the **T**-functor for all finite fields and continue to study the **T**-functor and invariant rings. One of our main results is that:

The **T**-functor preserves the hypersurfaces in the category of unstable algebras over the Steenrod algebra.

Generally speaking, a functor does not always maintain this property, even the embedding dimension is not inherited. However, we claim that the **T**-functor not only preserves the

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hypersurfaces but also does not increase the embedding dimension in the category of unstable algebras. The detailed description about these could be found in Section 3. As corollaries of our conclusions, we also get Gregor Kemper's interesting results about the hypersurfaces [9].

Comparing with the other homological properties, there are many differences about the hypersurfaces. For instance, if the invariants of all the Sylow-p subgroups are Cohen-Macaulay, then so is the invariant of the group itself [3]. However, the example 4.10 in this article demonstrates that this is not true for the hypersurfaces. We also construct another example 4.7 to illustrate that when the invariants of the stabilizer subgroups are hypersurfaces, it does not mean that the invariant of the group itself is. In the non-modular case, Nakajima has studied those representations, for which their invariants are hypersurfaces [10,11]. For the modular case, the results in [12–14] extend the work of Nakajima, discussing whether the invariants of certain subgroups of the group with polynomial invariant are hypersurfaces. Another criterion for testing when the invariants are hypersurfaces, in the case its Cohen-Macaulay, was provided in [3, Proposition 5.5.8]. And for the invariant rings of the modular reflection groups which are hypersurfaces could be found in the calculation in [15].

The organization of the paper: Section 2 is devoted to list some basic conceptions and results that are needed for this paper. In Section 3, we discuss several rules about the transformation of the invariant ideals and prove that the **T**-functor does not increase the embedding dimension of the unstable algebra. As a corollary, we claim that **T**-functor preserves the hypersurfaces in the category of unstable algebras over the Steenrod algebra. Section 4 looks into the application of the results in Section 3, which leads to an alternative proof of a known result in [9], that when the invariant of a finite group is hypersurface, so are its stabilizer subgroups. At the same time, we give several examples to explain that if the invariants of the stabilizer subgroups or Sylow p-subgroups are hypersurfaces, the invariant of the group itself is not necessarily a hypersurface.

2. Preliminaries

We suppose that the readers are familiar with the basic facts concerning the Steenrod algebra of a Galois field. However, we will briefly describe the work of Larry Smith about the reconstruction of the **T**-functor for any finite group [1], because it is indispensable for this article. More details concerning the Steenrod algebra and the **T**-functor could be found in [2,3,19–21] and the other knowledge on the Homological algebra and the Representation theory in [17,18].

Let $\rho : G \hookrightarrow \mathrm{GL}(n, F_q)$ be a faithful representation of a finite group G over the field F_q, where F_q is a Galois field with $\mathrm{Char}\, F_q = p$ and $|F_q| = p^\nu$, where p is a positive fixed prime number. Then, via ρ, G acts on the left of the vector space $V = F_q^n$. There is an induced action on the symmetric algebra $F_q[V] := S(V^*)$ given by $g(f) = f \circ g^{-1}$ for $g \in G$ and $f \in F_q[V]$. By definition, the ring, or algebra, of invariants, is the fixed subalgebra as follows:

$$F_q[V]^G = \{f \in F_q[V] \mid g \circ f = f,\ \forall\, g \in G\}.$$

Theorem 2.1 ([3]) *Let $\rho : G \hookrightarrow \mathrm{GL}(n, F)$ be a faithful representation of a finite group G over the field F. Suppose $F[V]^G$ contains a system of parameters $f_1, \ldots, f_n$, such that*

$\deg(f_1)\cdots\deg(f_n) = |G|$. *Then* $F[V]^G = F[f_1, \ldots, f_n]$.

Definition 2.2 ([3]) *A graded commutative algebra H over F_q is called an algebra over the Steenrod algebra if it is a left $\mathscr{P}^*$-module satisfying the Cartan formula, where the $\mathscr{P}^*$ denotes the Steenrod algebra of F_q. If, in addition, the unstability condition holds, then we say that H is an unstable algebra over the Steenrod algebra, or simply an unstable algebra.*

The Steenrod operations acting on H satisfy unstability condition means:

$$\mathscr{P}^i(f) = \begin{cases} f^q & \text{if} \quad i = \deg(f), \\ 0 & \text{if} \quad i > \deg(f), \end{cases} \qquad \forall f \in H.$$

And the formula

$$\mathscr{P}^k(f'f'') = \sum_{i+j=k} \mathscr{P}^i(f')\mathscr{P}^j(f''), \quad \forall f', f'' \in H$$

is called the Cartan formula for the Steenrod operations.

Let $\mathscr{K}$ denote the category of unstable algebra of F_q. Here we point out the facts that the algebra $F_q[V]$ and the algebra invariant $F_q[V]^G$ are both in the category of $\mathscr{K}$.

The category of unstable modules over the Steenrod algebra of F_q is denoted by $\mathscr{U}$. If U is a finite-dimensional F_q-vector space, then based on the adjoint functor theorem, the functor $\mathscr{U} \rightsquigarrow \mathscr{U}$, $M \rightsquigarrow F_q[U] \otimes_{F_q} M$ has a left adjoint denoted by

$$T_U \; : \; \mathscr{U} \rightsquigarrow \mathscr{U},$$

which is characterized by

$$\mathrm{Hom}_{\mathscr{U}}(M, F_q[U] \otimes_{F_q} N) \cong \mathrm{Hom}_{\mathscr{U}}(T_U(M), N).$$

At the same time, as described in [3, Section 10.1], the T_U could also denote the functor from $\mathscr{K}$ to $\mathscr{K}$, because computing $T_U(H)$ as an object of $\mathscr{K}$ or $\mathscr{U}$, H in both $\mathscr{K}$ and $\mathscr{U}$, the result is the same. So without conflict, we refer to the functors T_U for $U = F_q^m$, m in the set of positive integer Z^+, as **T**-functor.

Proposition 2.3 ([2]) *Let M be a locally finite unstable $\mathscr{P}^*$-module, which means that if each $x \in M$ lies in a finite $\mathscr{P}^*$-submodule of M. Then $T_U(M)$ is isomorphic to M.*

From the adjointness relation $\mathrm{Hom}_{\mathscr{K}}(H, F_q[U]) \cong \mathrm{Hom}_{\mathscr{K}}(T_U(H), F_q)$, we could separate $T_U(H)$ into components. For a map $\varphi \in \mathrm{Hom}_{\mathscr{K}}(H, F_q[U])$, there is a corresponding map $\tilde{\varphi}$: $T_U(H) \to F_q$, which amounts to a homomorphism $\tilde{\varphi} : T_U(H)_0 \to F_q$. Then we define the **component of φ in** $T_U(H)$ by $T_{U,\varphi}(H) = T_U(H) \otimes_{T_U(H)_0} F_q$. Because the component functors $T_{U,\varphi}$ preserve connectedness, they behave better algebraically than T_U.

Definition 2.4 ([3]) *An object M that is both an unstable $\mathscr{P}^*$-module and a module over an unstable $\mathscr{P}^*$-algebra H such that the Cartan formula*

$$\mathscr{P}^k(h \cdot x) = \sum_{i+j=k} \mathscr{P}^i(h)\mathscr{P}^j(x), \quad \forall h \in H, x \in M$$

*is satisfied is called an **unstable** $H \odot \mathscr{P}^*$-module.*

We use $\mathscr{U}_H$ to denote the category whose objects are unstable $H \odot \mathscr{P}^*$-modules and morphisms are both H- and $\mathscr{P}^*$-module homomorphisms. Because T_U preserves tensor products, we have

$$T_{U,\varphi}(M) = T_U(M) \otimes_{T_U(H)_0} F_q \cong T_U(M) \otimes_{T_U(H)} T_{U,\varphi}(H).$$

These $T_{U,\varphi}$ are called the components of $T_U(M)$ for M regarded as an H-module. In the next parts of this article, we will discuss the functor

$$T_{U,\varphi} \ : \ \mathscr{U}_H \rightsquigarrow \mathscr{U}_{T_{U,\varphi}(H)}.$$

Definition 2.5 ([3]) *Let H be an unstable algebra over the Steenrod algebra. An ideal $I \subseteq H$ is called $\mathscr{P}^*$-invariant ideal or just invariant ideal if it is closed under the action of the Steenrod algebra, i.e., $P(f) \in I, \ \forall f \in I, P \in \mathscr{P}^*$.*

It is apparent that the invariant ideals $I \subseteq H$ are equipped with the structure of $H \odot \mathscr{P}^*$-module.

3. Lanne's T-functor and hypersurfaces

In this section, the claims that **T**-functor does not increase the embedding dimension of the unstable algebra and the **T**-functor preserves the hypersurfaces are proved. These facts do not hold for general functors. For example, we could define a functor $F[-] : \mathscr{K} \rightsquigarrow \mathscr{K}$ as $F[-] = F_q[x] \otimes_{F_q} -$, where x is an indeterminate. Choose an object $F_q[y]$ in $\mathscr{K}$, where y is another indeterminate. The embedding dimension of $F_q[y]$ is obviously one. But the embedding dimension of $F_q[x] \otimes_{F_q} F_q[y] = F_q[x, y]$ is two. Here we firstly cite several conclusions as follows.

Proposition 3.1 ([19]) *Let us suppose that $\varphi : H \to F_q[U]$ is a $\mathscr{K}^*$-map. Then the functor $T_{U,\varphi}(-)$ is exact and preserves tensor products in the sense that if M and N are objects of $\mathscr{U}_H$, there is a natural isomorphism*

$$T_{U,\varphi}(M \otimes_H N) \cong T_{U,\varphi}(M) \otimes_{T_{U,\varphi}(H)} T_{U,\varphi}(N).$$

Proposition 3.2 ([3]) *Let H be a connected unstable integral domain over $\mathscr{P}^*$. Suppose that H is Noetherian and that $\varphi : H \to F_q[U]$ is a morphism in $\mathscr{K}$, where $U = F_q^m$ is a finite-dimensional vector space over F_q. Then $\dim(T_{U,\varphi}(H)) = \dim(H)$.*

Proposition 3.3 ([3]) *Let H be an unstable $\mathscr{P}^*$-algebra, $\varphi : H \to F_q[U]$ a map of $\mathscr{P}^*$-algebra, and M an $H \odot \mathscr{P}^*$-module that is free as an H-module. Then $T_{U,\varphi}(M)$ is free as a $T_{U,\varphi}(H)$-module.*

Then, discussing the transformation of the invariant ideals leads to the next lemma.

Lemma 3.4 *Let H denote a graded connected commutative algebra over F_q and $I, \ I_1, \ I_2$ are invariant ideals of H. If $\varphi : H \to F_q[U]$ is a map of unstable algebras, then*

(1) $T_{U,\varphi}(I)$ is an invariant ideal of $T_{U,\varphi}(H)$;

(2) $T_{U,\varphi}(I_1 \cap I_2) \subseteq T_{U,\varphi}(I_1) \cap T_{U,\varphi}(I_2)$;

(3) $T_{U,\varphi}(I_1 \cdot I_2) \cong T_{U,\varphi}(I_1) \cdot T_{U,\varphi}(I_2)$;

(4) If I is a principal ideal of H, then $T_{U,\varphi}(I)$ is a principal ideal of $T_{U,\varphi}(H)$;

(5) If I is the augmentation ideal of H, then $T_{U,\varphi}(I)$ is the augmentation ideal of $T_{U,\varphi}(H)$;

(6) If $I_2 \subseteq I_1$, then $T_{U,\varphi}(I_1/I_2) \cong T_{U,\varphi}(I_1)/T_{U,\varphi}(I_2)$.

Proof (1) and (2) are directly from the exactness of the functor.

(3) Since $I_1 \cdot I_2 \cong I_1 \otimes_H I_2$, the equation follows from Proposition 3.1.

(4) Because I is a principal ideal, it is free as $H \odot \mathscr{P}^*$-module. Consequently, by Proposition 3.3, $T_{U,\varphi}(I)$ is a free $T_{U,\varphi}(H) \odot \mathscr{P}^*$-module. Since $T_{U,\varphi}(I)$ is a submodule of $T_{U,\varphi}(H)$, it must have rank one. So as an ideal in $T_{U,\varphi}(H)$ it is also principal.

(5) Let I denote the unique maximal ideal of H. We have the $H \odot \mathscr{P}^*$-module exact sequence

$$0 \to I \to H \to F_q \to 0.$$

Because of the exactness of $T_{U,\varphi}$, we obtain the next exact sequence

$$0 \to T_{U,\varphi}(I) \to T_{U,\varphi}(H) \to T_{U,\varphi}(F_q) \to 0.$$

Based on Proposition 2.3 and the definition of the component of the **T**-functor $T_{U,\varphi}(F_q) \cong F_q$, therefore, $T_{U,\varphi}(H)/T_{U,\varphi}(I) \cong F_q$. So $T_{U,\varphi}(I)$ is also the unique maximal ideal of $T_{U,\varphi}(H)$.

(6) We are familiar with the exact sequence as follows:

$$0 \to I_2 \to I_1 \to I_1/I_2 \to 0.$$

By Proposition 3.1, under the transformation of $T_{U,\varphi}(-)$, the sequence

$$0 \to T_{U,\varphi}(I_2) \to T_{U,\varphi}(I_1) \to T_{U,\varphi}(I_1/I_2) \to 0$$

is still exact. So $T_{U,\varphi}(I_1/I_2) \cong T_{U,\varphi}(I_1)/T_{U,\varphi}(I_2)$. □

Corollary 3.5 *If the graded commutative connected algebra $H = F_q[h]$, where h is a linear form or a monomial. Then the form of $T_{U,\varphi}(F_q[h])$ could be taken as $F_q[h']$ for some $h' \in T_{U,\varphi}(F_q[h])$.*

Proof According to the (4) and (5), we could draw the conclusion that the form of $T_{U,\varphi}(F_q[h])$ is $F_q[h']$ for some $h' \in T_{U,\varphi}(F_q[h])$. □

Definition 3.6 ([17]) *Let H be a commutative Noetherian graded connected ring over F_q with maximal ideal $\bar{H}$. Then the embedding dimension of H is* $\mathrm{edim}(H) := \dim_{H/\bar{H}}(\bar{H}/(\bar{H})^2)$.

Theorem 3.7 *Let H be a graded connected commutative algebra over F_q with* $\mathrm{edim}(H) = m$ $(m \in Z^+)$. *If $\varphi : H \to F_q[U]$ is a map of unstable algebras, then* $\mathrm{edim}(T_{U,\varphi}(H)) \leq m$.

Proof Suppose that the maximal ideal of H is $\bar{H}$.

Then due to the results (3), (5) and (6) of Lemma 3.4, we obtain

$$T_{U,\varphi}(\bar{H}/\bar{H}^2) \cong T_{U,\varphi}(\bar{H})/(T_{U,\varphi}(\bar{H}))^2.$$

Because of $\mathrm{edim}(H) = m$, we could say that as an F_q vector space, $\dim(\bar{H}/\bar{H}^2) = m$.

Meanwhile, $T_U(\bar{H}/\bar{H}^2) \cong \bar{H}/(\bar{H}^2)$ by Proposition 2.1. And the definition of $T_{U,\varphi}(-)$ tells us that as F_q vector spaces, $\dim_{F_q}(T_U(\bar{H}/\bar{H}^2)) \leq \dim(T_{U,\varphi}(\bar{H}/\bar{H}^2))$.

As a result, we say that $\mathrm{edim}(T_{U,\varphi}(H)) \le m$.

Definition 3.8 ([16]) *A commutative Noetherian graded connected ring H over F_q is called a hypersurface ring or just a hypersurface if* $\mathrm{edim}(H) \le \dim(H) + 1$, *where* $\dim(H)$ *denotes the Krull dimension.*

It is known that the embedding dimension of a finitely generated F_q-algebra H is equal to the Krull dimension of H if and only if H is a polynomial ring. So in the next part of this paper, we will only take the case $\mathrm{edim}(H) = \dim(H) + 1$ into consideration.

Corollary 3.9 *Let H be a graded connected commutative algebra that is a hypersurface over F_q with* $\dim(H) = n$. *If $\varphi : H \to F_q[U]$ is a map of unstable algebras, then $T_{U,\varphi}(H)$ is also a hypersurface over F_q.*

Proof Combining Proposition 3.2 and Theorem 3.7, we have

$$n \le \mathrm{edim}(T_{U,\varphi}(H)) \le n + 1,$$

which means it is also a hypersurface. □

4. Invariants of stabilizer subgroups

Next, we will apply the results on the unstable algebra to the invariant theory and discuss when the invariants of finite group are hypersurfaces.

Definition 4.1 ([3]) *Let G_U denote the pointwise stabilizer subgroup of U in G, which is defined by*

$$G_U = \{g \in G | g(u) = u, \ \ \forall u \in U, \ U \text{ is a linear subspace of } V\}.$$

Proposition 4.2 ([3]) *Let $\rho : G \hookrightarrow \mathrm{GL}(n, F_q)$ be a representation of a finite group G over the Galois field F_q and $i : U \hookrightarrow V = F_q^n$ a linear subspace. Let $\alpha : F_q[V]^G \hookrightarrow F_q[V] \xrightarrow{i^*} F_q[U]$ be the composition of i^* with the canonical inclusion $\lambda : F_q[V]^G \hookrightarrow F_q[V]$. Then $T_{U,\alpha}(F_q[V]^G) \cong F_q[V]^{G_U}$.*

Theorem 4.3 *Let $\rho : G \hookrightarrow \mathrm{GL}(n, F_q)$ be a faithful representation of a finite group G over the field F_q and $i : U \le V = F_q^n$ a linear subspace. If* $\mathrm{edim}(F_q[V]^G) = m$, *then* $\mathrm{edim}(F_q[V]^{G_U}) \le m$, *where G_U is the pointwise stabilizer of U in G.*

Proof Let $\alpha : F_q[V]^G \to F_q[U]$ be the composition of i^* with the canonical inclusion $F_q[V]^G \hookrightarrow F_q[V]$. By Theorem 3.7, the **T**-functor can only decrease the embedding dimension. By Proposition 4.2, $T_{U,\alpha}(F_q[V]^G) \cong F_q[V]^{G_U}$. Then we get the result as desired. □

Theorem 4.4 *Let $\rho : G \hookrightarrow \mathrm{GL}(n, F_q)$ be a faithful representation of a finite group G over the field F_q and $i : U \le V = F_q^n$ a linear subspace. If $F_q[V]^G$ is a hypersurface, then so is $F_q[V]^{G_U}$, where G_U is the pointwise stabilizer of U in G.*

Proof This result comes from Theorem 4.2 and Corollary 3.9. □

Remark 4.5 As corollaries of our results in the last section, Theorems 4.3 and 4.4 are two special cases of the beautiful work of Gregor Kemper in [9].

The following example is concerning the calculating and comparing of the invariant of a finite group and its stabilizer subgroups.

Example 4.6 Let F_p be the prime field of characteristic p and consider the group G which is isomorphic to C_p^3 in $\mathrm{GL}_4(F_p)$, where C_p is the cyclic group of order p. It is generated by the three elements α, β, γ, where

$$\alpha^{-1} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \beta^{-1} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},\ \gamma^{-1} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Let $\{x_1, x_2, x_3, x_4\}$ denote the basis of V^* dual to the canonical basis $\{e_1, e_2, e_3, e_4\}$ of V. The act of C_3 on V^* is consistent with the definition at the beginning of Section 2. Quoting [16, Example 10.0.11], it is known that

$$\begin{aligned} F_p[V]^G &= F_p[x_1, x_2, f_1, f_2, h], \\ f_1 &= (x_3^p - x_1^{p-1}x_3)(x_3^p - (x_1+x_2)^{p-1}x_3), \\ f_2 &= (x_4^p - x_2^{p-1}x_4)(x_4^p - (x_1+x_2)^{p-1}x_4), \\ h &= x_2(x_4^p - x_2^{p-1}x_4) + x_1(x_3^p - x_1^{p-1}x_3). \end{aligned}$$

It is a hypersurface, but not a polynomial algebra. Then we discuss the invariants of the stabilizer subgroups of G.

Let $V_J = \mathrm{Span}_{F_p}\{e_j | j \in J\}$ denote the subspace of V, where $I = \{1,2,3,4\}$ and $J \subseteq I$. We have the facts that $G_{V_{\{3,4\}}} = G_{V_{\{2,3,4\}}} = G_{V_{\{1,3,4\}}} = G_{V_{\{1,2,3,4\}}} = 1$, $G_{V_{\{4\}}} = G_{V_{\{1,4\}}} = G_{V_{\{2,4\}}} = G_{V_{\{1,2,4\}}} = \langle\alpha\rangle$, $G_{V_{\{3\}}} = G_{V_{\{1,3\}}} = G_{V_{\{2,3\}}} = G_{V_{\{1,2,3\}}} = \langle\beta\rangle$, $G_{V_{\{1\}}} = G_{V_{\{2\}}} = G_{V_{\{1,2\}}} = G$. Through calculating the top Chern Orbit Classes and using Theorem 2.1, it is easy to gain that

$$\begin{aligned} F_p[V]^{\langle\alpha\rangle} &= F_p[x_1, x_2, x_3^p - x_1^{p-1}x_3, x_4], \\ F_p[V]^{\langle\beta\rangle} &= F_p[x_1, x_2, x_3, x_4^p - x_2^{p-1}x_4]. \end{aligned}$$

They are polynomial algebra, of course, they are hypersurfaces.

Remark 4.7 It is natural to ask whether $F_q[V]^G$ is a hypersurface, provided the invariants of every pointwise stabilizer subgroup is. The answer is negative, as demonstrated by the following example.

Example 4.8 Let A_3 and C_3 denote alternating group and cyclic group of order 3. The ground field is F_3 of characteristic 3. We consider the group as follows:

$$G = A_3 \oplus C_3 = \{g = \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix} \mid \sigma \in A_3,\ \tau \in C_3\}.$$

V^* denotes the dual space of $V = F_3^6 = X \oplus Y$ with basis $\{x_1, x_2, x_3, y_1, y_2, y_3\}$. Suppose the action of σ is determined by the tautological representation of A_3 and τ on Y by the matrix

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Because A_3, C_3 are both simple groups, the only non-trivial subgroups of G are $I \oplus C_3$ and $A_3 \oplus I$, where I is the identity matrix. We can check the stabilizer subgroups of G are $G_{Span_{F_3}\{y_1\}} = I \oplus C_3$ and $G_{Span_{F_3}\{x_1+x_2+x_3\}} = A_3 \oplus I$. According to the calculation of the invariant of C_3 and A_3 (see [16, Sections 4.4 and 4.10]), we get

$$\begin{aligned} F_3[X \oplus Y]^{I \oplus C_3} &= F_3[X]^I \otimes F_3[Y]^{C_3} = F_3[x_1, x_2, x_3, y_1, f_1, f_2, f_3]. \\ f_1 &= y_2^3 - y_1^2 y_2, \\ f_2 &= y_3^3 - y_2^2 y_3, \\ f_3 &= y_2^2 - 2y_1 y_3 - y_1 y_2. \\ F_3[X \oplus Y]^{A_3 \oplus I} &= F_3[X]^{A_3} \otimes F_3[Y]^I = F_3[h_1, h_2, h_3, h_4, y_1, y_2, y_3]. \\ h_1 &= x_1 + x_2 + x_3, \\ h_2 &= x_1 x_2 + x_2 x_3 + x_1 x_3, \\ h_3 &= x_1 x_2 x_3, \\ h_4 &= x_3^2 x_2 + x_1 x_2^2 - x_2^2 x_3 - x_1 x_3^2 - x_1^2 x_2 + x_1^2 x_3. \end{aligned}$$

They are both hypersurfaces. Meanwhile,

$$F_3[X \oplus Y]^{A_3 \oplus C_3} = F_3[X]^{A_3} \otimes F_3[Y]^{C_3} = F_3[h_1, h_2, h_3, h_4, y_1, f_1, f_2, f_3].$$

Because $f_2^2 \in F_3[y_1, f_3, f_4]$ and $h_4^2 \in F_3[h_1, h_2, h_3]$, there are two relations among the generators. So the number of generators minus the number of relations is equal to the Krull dimension, i.e., $F_3[X \oplus Y]^{A_3 \oplus C_3}$ is a complete intersection, but not a hypersurface.

Remark 4.9 It is known that if the invariants of all the Sylow-p subgroups are Cohen-Macaulay, then so is the invariant of the group itself. Nevertheless, another example could be provided to explain that when the invariants of Sylow-p subgroups are all hypersurfaces, it is not necessary for the invariants of the group itself to be hypersurface.

Example 4.10 Let C_3 still denote cyclic group of order 3 and C_2 the cyclic group of order 2. The ground field is F_3 of characteristic 3. We consider the group as follows:

$$G = C_2 \oplus C_3 = \{g = \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix} \mid \sigma \in C_2,\ \tau \in C_3\}.$$

The V^* denotes the of $V = F_3^5 = X \oplus Y$ with basis $\{x_1, x_2, y_1, y_2, y_3\}$. Suppose the action of σ

and τ on X and Y are respectively determined by the matrices

$$\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Because the order of the group G is 6, the sylow-p subgroups of G is just $I \oplus C_3$, where I is the identity matrix. Taking the same method as Example 4.7, we obtain

$$\begin{aligned} F_3[X \oplus Y]^{I \oplus C_3} = F_3[X]^I \otimes F_3[Y]^{C_3} &= F_3[x_1, x_2, x_3, y_1, f_1, f_2, f_3], \\ f_1 &= y_2^3 - y_1^2 y_2, \\ f_2 &= y_3^3 - y_2^2 y_3, \\ f_3 &= y_2^2 - 2y_1 y_3 - y_1 y_2. \end{aligned}$$

It is a hypersurface. Meanwhile, the invariant of $C_2 \oplus I$ is easy to calculate

$$F_3[X \oplus Y]^{C_2 \oplus I} = F_3[X]^{C_2} \otimes F_3[Y]^I = F_3[x_1^2, x_2^2, x_1 x_2, y_1, y_2, y_3].$$

So we gain the equations as follows

$$F_3[X \oplus Y]^{C_2 \oplus C_3} = F_3[X]^{C_2} \otimes F_3[Y]^{C_3} = F_3[x_1^2, x_2^2, x_1 x_2, y_1, f_1, f_2, f_3].$$

This means the invariant of the group G is complete intersection, but not a hypersurface. However, the invariants of the sylow-p subgroups are hypersurfaces.

Go on with the way of the examples above, we point out the obvious method to construct groups whose invariants are hypersurfaces.

Proposition 4.11 *Suppose the group $G(G_1, G_2)$ is a direct sum defined as follows:*

$$G(G_1, G_2) = \{g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} | A \in G_1, B \in G_2\} \cong (G_1 \oplus G_2),$$

where $G_1 < \mathrm{GL}_r(F_q)$, $G_2 < \mathrm{GL}_{n-r}(F_q)$. Let $V = F_q^n$ and the dual space $V^* = X \oplus Y$, where $X = \mathrm{Span}_{F_q}\{x_1, \ldots, x_r\}$, $Y = \mathrm{Span}_{F_q}\{x_{r+1}, \ldots, x_n\}$ are the dual space of V_1 and V_2. If $F_q[V_1]^{G_1}$ is a hypersurface and $F_q[V_2]^{G_2}$ is a polynomial. Then $F_q[V]^{G_1 \oplus G_2}$ is a hypersurface.

Proof The conclusion is apparent from the fact that

$$F_q[V]^{G_1 \oplus G_2} = F_q[V_1 \oplus V_2]^{G_2 \oplus G_2} = F_q[V_1]^{G_1} \otimes F_q[V_2]^{G_2}. \quad \square$$

References

[1] J. LANNES. *Sur les espaces fonctionnels dont la source est le classifiant d'un p -groupe abélien élémentaire.* Inst. Hautes Études Sci. Publ. Math., 1992, **75**: 135–244. (in French)

[2] L. SCHWARTZ. *Unstable Modules Over the Steenrod Algebra and Sullivan's Fixed Point Set Conjecture.* University of Chicago Press, Chicago, IL, 1994.

[3] M. D. NEUSEL, L. SMITH. *Invariant Theory of finite groups.* American Mathematical Society, Providence, RI, 2002.

[4] W. G. DWYER, C. W. WILKERSON. *Kähler Differentials, the T-functor, and a Theorem of Steinberg.* Trans. Amer. Math. Soc., 1998, **350**(12): 4919–4930.

[5] R. STEINBERG. *Invariants of finite reflection groups*. Canad. J. Math., 1960, **12**: 616–618.
[6] H. NAKAJIMA. *Regular Rings of Invariants of Unipotent Groups*. J. Algebra, 1983, **85**(2): 253–286.
[7] L. SMITH. *Lannes **T**-functor and invariants of pointwise stabilizers*. Forum Math., 2000, **12**(4): 461–476.
[8] D. M. MEYER, L. SMITH. *Lannes' **T**-functor, pointwise stabilizers, and degree bounds*. Comm. Algebra, 2010, **38**(1): 302–307.
[9] G. KEMPER. *Loci in quotients by finite groups, pointwise stabilizers and the Buchsbaum property*. J. Reine Angew. Math., 2002, **547**: 69–96.
[10] H. NAKAJIMA. *Rings of invariants of finite groups which are hypersurfaces*. J. Algebra, 1983, **80**(2): 279–294.
[11] H. NAKAJIMA. *Rings of invariants of finite groups which are hypersurfaces (II)*. Adv. in Math., 1987, **65**(1): 39–64.
[12] H. E. A. CAMPBELL, I. P. HUGHES. *Rings of Invariants of Certain p-Groups over the Field F_q*. J. Algebra, 1999, **211**(2): 549–561.
[13] I. P. HUGHES, N. KECHAGIAS. *Rings of invariants of modular p-groups which are hypersurfaces*. J. Algebra, 2005, **291**(1): 72–89.
[14] A. BROER. *Hypersurfaces in modular invariant theory*. J. Algebra, 2006, **306**(2): 576–590.
[15] H. DERKSEN, G. KEMPER. *Computational Invariant Theory*. Springer-Verlag, Berlin, 2002.
[16] H. E. A. Eddy CAMPBELL, D. L. WEHLAU. *Modular Invariant Theory*. Springer-Verlag, Berlin, 2011.
[17] D. EISENBUD. *Commutative Algebra: with a View Toward Algebraic Geometry*. Springer-Verlag, Berlin Heidelberg, 1999.
[18] D. J. BENSON. *Representations and Cohomolgy*. Cambridge University Press, United Kingdom, 1995.
[19] W. G. DWYER, C. W. WILKERSON. *Smith Theory and the Functor T*. Commentarii Mathematici Helvetici, 1991, **66**: 1–17.
[20] I. KARACA. *Unstable modules over the Steenrod Algebra*. J. Pure Appl. Algebra, 2002, **171**(2-3): 249–255.
[21] I. KARACA. *On the nilclosure in a category of unstable modules*. J. Math. Sci. Adv. Appl., 2012, **13**(1): 55–70.