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Reduced Polynomial of Framed Links in Thickened Torus

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Abstract Framed links in thickened torus is studied. We give an expression of torus knot in the Kauffman bracket skein algebra. From this expression and using the theory of Gröbner bases, we drive the reduced polynomial of a framed link, which is an ambient isotopic invariant and can be computed feasibly.

Keywords reduced polynomial; framed link; thickened torus; Kauffman bracket skein algebra

MR(2010) Subject Classification 57M27

1. Introduction

We are concerned with framed links in thickened torus by using skein theory and Gröbner bases theory. We will extend the Kauffman bracket skein algebra to the Kauffman bracket skein commutative algebra and obtain the reduced polynomials of framed links in thickened torus.

Skein modules were introduced by Przytycki in [1]. The skein module based on Kauffman bracket skein relation is one of the most extensively studied object of the algebraic topology based on framed links, it is also an important invariant of 3-manifolds. There have been extensive study and application of Kauffman bracket skein module [2–6]. Specially, the Kauffman bracket skein module of thickened surface is a free module generated by links (simple closed curves) on surface with no trivial component (but including the empty knot) [6]. More specifically, if the thickened surface is thickened torus, then the link on torus is the disjoint union of torus knots, while torus knot was a kind of knot that had been investigated and used widely [7]. So we want to search for new ambient isotopic invariant of framed links in thickened torus depending on the special generators of Kauffman bracket skein module of thickened torus.

Note that the subject investigated is the framed link. A convenient way of representing a framed link in an orientable 3-manifold M is in the form of smoothly embedded closed bands $(\sqcup_{j=1}^{k} S_j^1 \times I \hookrightarrow M)$, such that bands for different components do not intersect. For a framed link L in a thickened surface $F\times I$, suppose $r: \sqcup_{j=1}^{k} S_j^1 \times \{0\} \to \sqcup_{j=1}^{k} S_j^1 \times I$, $p: F\times I \to F\times \{0\}$,

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we call the composition mapping $p \cdot L \cdot r : \sqcup_{j=1}^{k} S_j^1 \to F$ a projection of L onto F, denoted by ℓ. In general, we consider the projection which is regular [8].

One method in this paper is based on the Gröbner bases [9]. The centerpiece of Gröbner bases theory is the Buchberger algorithm, which provides a common generalization of the Euclidean algorithm and the Gaussian elimination algorithm to multivariate polynomial rings. It can be extended to commutative algebra, so we can apply Gröbner bases to Kauffman bracket skein commutative algebra of thickened torus.

This paper is organized as follows: In Section 2, we cover the necessary definitions and lemmas; The main results are summarized in Section 3; Section 4 provides the lengthy proof of Theorem 3.1.

2. Preliminary

We give the definition of the Kauffman bracket skein module of $F \times I$ for an oriented surface F and an interval I as follows:

Definition 2.1 ([1,10]) *The Kauffman bracket skein module of 3-manifold $F \times I$, $S_{2,\infty}(F \times I; R, A)$ is defined as follows: Let $\mathcal{L}$ be the set of unoriented framed links in $F \times I$ (including the empty knot $\emptyset$), R any commutative ring with identity and A an invertible element in R. $R\mathcal{L}$ be the free R-module generated by $\mathcal{L}$, $S_{2,\infty}$ be the submodule of $R\mathcal{L}$ generated by two skein relations:*

(1) $L_+ - AL_0 - A^{-1}L_\infty$;

(2) $L \sqcup T_1 + (A^2 + A^{-2})L$,

where the triple L_+, L_0 and L_∞ as presented by their regular projections ℓ_+, ℓ_0 and ℓ_∞ on F are shown in Fig. 1, which can be ambient isotopy except within the neighborhood shown, and T_1 denotes the trivial framed knot. Set $S_{2,\infty}(F \times I; R, A) = R\mathcal{L}/S_{2,\infty}$. Notation is shortened for special case: $S_{2,\infty}(F \times I) = S_{2,\infty}(F \times I; \mathbb{Z}[A^{\pm 1}], A)$.

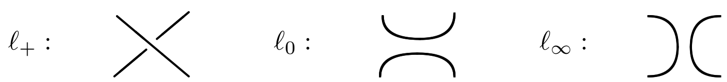

Figure 1 Link projections

We are concerned in this paper with the torus knot, which is defined as below:

Definition 2.2 ([7]) *Given two generators x_1, x_2 in $\pi_1(T^2)$, where $x_1 : S^1 \hookrightarrow T^2$, $x_1(e^{i\theta}) = (e^{i\theta}, 1)$, $x_2 : S^1 \hookrightarrow T^2$, $x_2(e^{i\theta}) = (1, e^{i\theta})$, and consider the closed curve $\gamma : S^1 \hookrightarrow T^2$, $\gamma(e^{i\theta}) = x_1^p x_2^q$. If $(p,q) = (0,0)$ or p, q are relatively prime, then γ is called a (p,q) knot in T^2, denoted by $K_{(p,q)}$. Obviously, $K_{(1,0)} = x_1$, $K_{(0,1)} = x_2$.*

The following lemmas about the Kauffman bracket skein module will be useful later on.

Lemma 2.3 ([6]) *$S_{2,\infty}(F \times I; R, A)$ is a free R-module with a basis $B(F)$ consisting of links in F without contractible components (but including the empty knot).*

Lemma 2.4 ([3]) *$S_{2,\infty}(F\times I;R,A)$ is an R algebra with $\emptyset$ as a unit element and $L_1\cdot L_2$ defined by placing L_1 above L_2.*

Lemma 2.5 ([8]) *For a framed link L in $F\times I$, its expression in $S_{2,\infty}(F\times I)$ is an ambient isotopic invariant of L.*

Lemma 2.6 ([2]) *With curves $x_1=K_{(1,0)}, x_2=K_{(0,1)}, x_3=K_{(1,1)}\in T^2$, the Kauffman bracket skein algebra $S_{2,\infty}(T^2\times I)$ is presented as $\{x_1,x_2,x_3|(2.1)(2.2)(2.3)(2.4)\}$, where*

$$Ax_1x_2-A^{-1}x_2x_1=(A^2-A^{-2})x_3 \tag{2.1}$$

$$Ax_2x_3-A^{-1}x_3x_2=(A^2-A^{-2})x_1 \tag{2.2}$$

$$Ax_3x_1-A^{-1}x_1x_3=(A^2-A^{-2})x_2 \tag{2.3}$$

$$A^2x_1^2+A^{-2}x_2^2+A^2x_3^2-Ax_1x_2x_3=2A^2+2A^{-2}. \tag{2.4}$$

It is necessary to describe the ingredients of Gröbner bases of commutative algebra.

Definition 2.7 ([9]) *Let R be a commutative algebra with finite generators, J be a nonzero idea in R, $G=\{g_1,\ldots,g_t|g_i\in J\}$ be a finite set. If we put an arbitrary term order on J and for every nonzero polynomial $f\in J$, there exists $i, 1\le i\le t$, such that $lp(g_i)|lp(f)$, where $lp(g_i)$ and $lp(f)$ denote the leading power product of g_i and f under the known term order, respectively, then G is called the Gröbner bases of J with regard to the above term order.*

Definition 2.8 ([9]) *A polynomial $r\in R$ is called reduced with respect to Gröbner bases G if $r=0$ or no power product that appear in r is divisible by anyone of $lp(g_i)$. In other words, r cannot be reduced modulo G. If the polynomial f reduces to r module G and r is not reduced modulo G anymore, we call r the reduced polynomial of f in G, denoted by $N_G(f)$.*

Lemma 2.9 ([9]) *Let R be a commutative algebra with finite generators $\{x_1,\ldots,x_k\}$, and J a nonzero idea in R. Suppose G is the Gröbner bases of J in R with regard to a certain term order, and f,g are any two polynomials in variables $x_1,\ldots,x_k$ in R, then we have $f\equiv g \pmod J$ iff $N_G(f)=N_G(g)$.*

3. Main results

Theorem 3.1 *In the Kauffman bracket skein algebra of $T^2\times I$, $K_{(p,1)}$ in T^2 has an expression in variables x_1,x_2,x_3 with $\mathbb{Z}[A^{\pm1}]$ coefficients:*

When p ($p\ge 8$) is even,

$$K_{(p,1)}=\frac{x_3(A^{-1}x_1)^{p-2}}{2^{p-1}}+\frac{x_3A^{-2}}{2^{p-3}}\sum_{k=1}^{\frac{p-2}{2}}C_{p-2}^{2k}(A^{-1}x_1)^{p-2-2k}(A^{-2}x_1^2-4A^{-2})^k-$$
$$\frac{x_3}{2^{p-1}}\sum_{k=1}^{\frac{p}{2}}C_p^{2k}(A^{-1}x_1)^{p-2k}(A^{-2}x_1^2-4A^{-2})^k-\frac{x_2A^{-2}(A^{-1}x_1)^{p-3}}{2^{p-2}}-$$

$$\frac{x_2A^{-4}}{2^{p-4}}[\sum_{k=1}^{\frac{p-6}{2}} C_{p-3}^{2k}(A^{-1}x_1)^{p-3-2k}(A^{-2}x_1^2-4A^{-2})^k+$$
$$(p-3)A^{-1}x_1(A^{-2}x_1^2-4A^{-2})^{\frac{p-4}{2}}]+$$
$$\frac{x_2A^{-2}}{2^{p-2}}[\sum_{k=1}^{\frac{p-4}{2}} C_{p-1}^{2k}(A^{-1}x_1)^{p-2k-1}(A^{-2}x_1^2-4A^{-2})^k+$$
$$(p-1)A^{-1}x_1(A^{-2}x_1^2-4A^{-2})^{\frac{p-2}{2}}];$$

When p ($p \geq 7$) is odd,

$$\begin{aligned} K_{(p,1)} =& \frac{x_3(A^{-1}x_1)^{p-2}}{2^{p-1}} + \frac{x_3A^{-2}}{2^{p-3}}[\sum_{k=1}^{\frac{p-5}{2}} C_{p-2}^{2k}(A^{-1}x_1)^{p-2-2k}(A^{-2}x_1^2-4A^{-2})^k+ \\ & (p-2)A^{-1}x_1(A^{-2}x_1^2-4A^{-2})^{\frac{p-3}{2}}]- \\ & \frac{x_3}{2^{p-1}}[\sum_{k=1}^{\frac{p-3}{2}} C_p^{2k}(A^{-1}x_1)^{p-2k}(A^{-2}x_1^2-4A^{-2})^k + pA^{-1}x_1(A^{-2}x_1^2-4A^{-2})^{\frac{p-1}{2}}]- \\ & \frac{x_2A^{-2}(A^{-1}x_1)^{p-3}}{2^{p-2}} - \frac{x_2A^{-4}}{2^{p-4}}\sum_{k=1}^{\frac{p-3}{2}} C_{p-3}^{2k}(A^{-1}x_1)^{p-2k-3}(A^{-2}x_1^2-4A^{-2})^k+ \\ & \frac{x_2A^{-2}}{2^{p-2}}\sum_{k=1}^{\frac{p-1}{2}} C_{p-1}^{2k}(A^{-1}x_1)^{p-2k-1}(A^{-2}x_1^2-4A^{-2})^k. \end{aligned}$$

In addition,

$$\begin{aligned} & K_{(0,1)} = x_2, \quad K_{(1,1)} = x_3, \quad K_{(2,1)} = A^{-1}x_1x_3 - A^{-2}x_2, \\ & K_{(3,1)} = A^{-2}x_1^2x_3 - A^{-3}x_1x_2 - A^{-2}x_3, \\ & K_{(4,1)} = A^{-3}x_1^3x_3 - A^{-4}x_1^2x_2 - 2A^{-3}x_1x_3 + A^{-4}x_2, \\ & K_{(5,1)} = A^{-4}x_1^4x_3 - A^{-5}x_1^3x_2 - 3A^{-4}x_1^2x_3 + 2A^{-5}x_1x_2, \\ & K_{(6,1)} = A^{-5}x_1^5x_3 - A^{-6}x_1^4x_2 - 4A^{-5}x_1^3x_3 + 3A^{-6}x_1^2x_2 + 2A^{-5}x_1x_3 - A^{-6}x_2. \end{aligned}$$

Theorem 3.2 *In the Kauffman bracket skein algebra of $T^2 \times I$, $K_{(p,q)}$ in T^2 has an expression in variables x_1, x_2, x_3 with $\mathbb{Z}[A^{\pm 1}]$ coefficients:*

When q ($q \geq 6$) is even,

$$\begin{aligned} K_{(p,q)} =& \frac{K_{(p,1)}(Ax_2)^{q-2}}{2^{q-1}} + \frac{K_{(p,1)}(-A^2)}{2^{q-3}}\sum_{k=1}^{\frac{q-2}{2}} C_{q-2}^{2k}(Ax_2)^{q-2-2k}(A^2x_2^2-4A^2)^k- \\ & \frac{K_{(p,1)}}{2^{q-1}}\sum_{k=1}^{\frac{q}{2}} C_q^{2k}(Ax_2)^{q-2k}(A^2x_2^2+4A^2)^k - \frac{x_1^p(-A^2)(Ax_2)^{q-3}}{2^{q-2}}- \\ & \frac{x_1^pA^4}{2^{q-4}}[\sum_{k=1}^{\frac{q-5}{2}} C_{q-3}^{2k}(Ax_2)^{q-3-2k}(A^2x_2^2+4A^2)^k + (q-3)Ax_2(A^2x_2^2+4A^2)^{\frac{q-4}{2}}]+ \end{aligned}$$

$$\frac{x_1^p(-A^2)}{2^{q-2}}[\sum_{k=1}^{\frac{q-3}{2}} C_{q-1}^{2k}(Ax_2)^{q-2k-1}(A^2x_2^2+4A^2)^k+(q-3)Ax_2(A^2x_2^2+4A^2)^{\frac{q-2}{2}}];$$

When q ($q \geq 5$) is odd,

$$\begin{aligned}K_{(p,q)} =&\frac{K_{(p,1)}(Ax_2)^{q-2}}{2^{q-1}}+\frac{K_{(p,1)}(-A^2)}{2^{q-3}}[\sum_{k=1}^{\frac{q-4}{2}} C_{q-2}^{2k}(Ax_2)^{q-2-2k}(A^2x_2^2+4A^2)^k+\\ &(q-2)Ax_2(A^2x_2^2+4A^2)^{\frac{q-3}{2}}]-\\ &\frac{K_{(p,1)}}{2^{q-1}}[\sum_{k=1}^{\frac{q-2}{2}} C_{q}^{2k}(Ax_2)^{q-2k}(A^2x_2^2+4A^2)^k+qAx_2(A^2x_2^2+4A^2)^{\frac{q-1}{2}}]-\\ &\frac{x_1^p(-A^2)(Ax_2)^{q-3}}{2^{q-2}}-\frac{x_1^pA^4}{2^{q-4}}[\sum_{k=1}^{\frac{q-3}{2}} C_{q-3}^{2k}(Ax_2)^{q-3-2k}(A^2x_2^2+4A^2)^k+\\ &\frac{x_1^p(-A^2)}{2^{q-2}}\sum_{k=1}^{\frac{q-1}{2}} C_{q-1}^{2k}(Ax_2)^{q-2k-1}(A^2x_2^2+4A^2)^k,\end{aligned}$$

where $K_{(p,1)}$ is expressed as above.

In addition,

$$\begin{aligned}&K_{(p,0)}=x_1^p,\quad K_{(p,2)}=Ax_2K_{(p,1)}-A^2x_1^p,\\ &K_{(p,3)}=A^2x_2^2K_{(p,1)}-A^3x_2x_1^p-A^2K_{(p,1)},\\ &K_{(p,4)}=A^3x_2^3K_{(p,1)}-A^4x_2^2x_1^p-2A^3x_2K_{(p,1)}+A^4x_1^p.\end{aligned}$$

Definition 3.3 *Let the set $J=\{Ax_1x_2-Bx_2x_1-(A^2-B^2)x_3, Ax_2x_3-ABx_3x_2-(A^2-B^2)x_1, Ax_3x_1-Bx_1x_3-(A^2-B^2)x_2, A^2x_1^2+B^2x_2^2+A^2x_3^2-Ax_1x_2x_3-2A^2-2B^2, AB-1\}$ be an idea of the free commutative algebra $<x_1,x_2,x_3,A,B>$ generated by $\{x_1,x_2,x_3,A,B\}$, where $x_1=K_{(1,0)}$, $x_2=K_{(0,1)}$, $x_3=K_{(1,1)}\in T^2$. We call $<x_1,x_2,x_3,A,B>/J$ the Kauffman bracket skein commutative algebra of $T^2\times I$, denoted by $\overline{S_{2,\infty}(T^2\times I)}$.*

Proposition 3.4 *The Gröbner bases G of the idea J with regard to the term order $x_1>x_2>x_3>A>B$ is the set presented as $\{-1+B^2+B^6-B^{10}-B^{14}+B^{16}, A-B-B^5+B^9+B^{13}-B^{15}, -x_3+B^2x_3+B^4x_3-B^8x_3-B^{10}x_3+B^{12}x_3, 2+B^2-B^6-2B^8-B^{10}+B^{14}-x_3^2+B^4x_3^2, 3x_3-2B^2x_3-2B^6x_3+B^8x_3-x_3^3+B^2x_3^3, -x_2+B^2x_2+B^4x_2-B^8x_2-B^{10}x_2+B^{12}x_2, -x_2x_3+2B^2x_2x_3-B^4x_2x_3+B^6x_2x_3-2B^8x_2x_3+B^{10}x_2x_3, 2x_2-B^4x_2-B^6x_2-B^8x_2+B^{10}x_2-x_2x_3^2+B^2x_2x_3^2, 2+B^2-B^6-2B^8-B^{10}+B^{14}-x_2^2+B^4x_2^2, 2x_3-B^4x_3-B^6x_3-B^8x_3+B^{10}x_3-x_2^2x_3+B^2x_2^2x_3, 3x_2-2B^2x_2-2B^6x_2+B^8x_2-x_2^3+B^2x_2^3, -x_1+B^4x_1+Bx_2x_3-B^3x_2x_3, Bx_2-B^7x_2-B^9x_2+B^{11}x_2-x_1x_3+B^2x_1x_3, -x_1x_2+B^2x_1x_2+Bx_3-B^7x_3-B^9x_3+B^{11}x_3, -4-B^2-2B^4+B^6+2B^8+B^{10}-B^{14}+x_1^2+x_2^2-Bx_1x_2x_3+x_3^2\}$.*

Theorem 3.5 *Suppose L is the framed link in $T^2\times I$, $L(x_1,x_2,x_3,A,B)$ is the polynomial of L in $\overline{S_{2,\infty}(T^2\times I)}$, and G is the Gröbner bases of J, then the reduced polynomial $N_G(L(x_1,x_2,x_3,A,B))$ of L is an ambient isotopic invariant of L.*

Proof Let L be an arbitrary framed link in $T^2 \times I$. By Lemma 2.3, L has a unique expression:

$$L = \sum_i f_i(A) \sqcup_{k=1}^{n_i} K_{(p_i,q_i)} \in S_{2,\infty}(T^2 \times I), \tag{3.1}$$

where $f_i(A) \in \mathbb{Z}[A^{\pm 1}]$, and $\forall i, K_{(p_i,q_i)}$ denotes (p_i, q_i) knot in T^2; By Theorem 3.2, $\forall i, K_{(p_i,q_i)}$ has an expression in variables $x_1,\ x_2,\ x_3$ with $\mathbb{Z}[A^{\pm 1}]$ coefficients in the Kauffman bracket skein algebra. Substituting this expression into (3.1), we have a polynomial $L(x_1, x_2, x_3, A, B)$ in variables $x_1, x_2, x_3, A, B(= A^{-1})$; Finally, we can drive the reduced polynomial $N_G(L(x_1, x_2, x_3, A, B))$ of $L(x_1, x_2, x_3, A, B)$ in Gröbner bases G. By Lemma 2.5, the expression (3.1) is the ambient isotopic invariant of L. Moreover, $N_G(L(x_1, x_2, x_3, A, B))$ is the ambient isotopic invariant of L due to Lemma 2.9. $\square$

Let us illustrate this method with an example:

Example 3.6 Suppose L is a framed link in $T^2 \times I$ with blackboard framing, and its regular projection is shown in Figure 2. Observe that L has two crossings. By skein relation (1) and (2), we obtain $L = (-A^4 - A^{-4})K_{(3,2)}$, and by Theorem 3.2, we have

$$K_{(3,2)} = A^{-1}x_2x_1^2x_3 + A^2x_1^3 - A^{-2}x_2x_1x_2 - A^{-1}x_2x_3 \in S_{2,\infty}(T^2 \times I),$$

so

$$\begin{aligned}
L &= (-A^4 - A^{-4})K_{(3,2)} \\
&= (-A^4 - A^{-4})(A^{-1}x_2x_1^2x_3 + A^2x_1^3 - A^{-2}x_2x_1x_2 - A^{-1}x_2x_3) \\
&= A^6x_1^3 - B^5x_1^2x_2x_3 + B^6x_1x_2^2 - A^3x_1^2x_2x_3 + B^5x_2x_3 + A^2x_1^3 + A^2x_1x_2^2 + A^3x_2x_3 \\
&\in \overline{S_{2,\infty}(T^2 \times I)}.
\end{aligned}$$

By Proposition 3.4, the reduced polynomial of L in G is

$$N_G(L(x_1, x_2, x_3, A, B)) = 8B^2x_1 + 16Bx_2x_3 - 14B^3x_2x_3 + 8B^5x_2x_3 - 18B^7x_2x_3 + 10B^9x_2x_3 - 2x_1x_3^2.$$

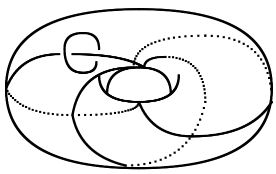

Figure 2 The regular projection of the framed link in Example 3.6

4. Proof of Theorem 3.1

Here we shall treat the details of the proof of the Theorem 3.1.

Proof Assume that the consequence is true for $n \leq p - 1$, and it suffices to prove the theorem in the case that $n = p$. We only consider the case when p is even, the proof adopts the same procedure when p is odd. By the skein relation (1) in Definition 2.1, we have

$$K_{(p,1)} = A^{-1}x_1(p-1,1) - A^{-2}(p-2,1). \tag{4.1}$$

Suppose $A^{-1}x_1 = b, A^{-2} = c$, then simplifying the above equation in Theorem 3.1 gives

$$\begin{aligned}K_{(p,1)} =& \frac{x_3 b^{p-2}}{2^{p-1}} + \frac{x_3 c}{2^{p-3}} \sum_{k=1}^{\frac{p-2}{2}} C_{p-2}^{2k} b^{p-2-2k}(b^2-4c)^k - \\ & \frac{x_3}{2^{p-1}} \sum_{k=1}^{\frac{p}{2}} C_p^{2k} b^{p-2k}(b^2-4c)^k - \frac{x_2 c b^{p-3}}{2^{p-2}} - \\ & \frac{x_2 c^2}{2^{p-4}}\Big[\sum_{k=1}^{\frac{p-6}{2}} C_{p-3}^{2k} b^{p-3-2k}(b^2-4c)^k + (p-3)b(b^2-4c)^{\frac{p-4}{2}}\Big] + \\ & \frac{x_2 c}{2^{p-2}}\Big[\sum_{k=1}^{\frac{p-4}{2}} C_{p-1}^{2k} b^{p-2k-1}(b^2-4c)^k + (p-1)b(b^2-4c)^{\frac{p-2}{2}}\Big].\end{aligned}$$

Before approaching further, let us remark on the denotes $b = f_1 + f_2$, $c = f_1 f_2$. Thus, when n is even,

$$f_1^n + f_2^n = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n}{2}} C_n^{2k} b^{n-2k}(b^2-4c)^k,$$

when n is odd,

$$f_1^n + f_2^n = \frac{1}{2^{n-1}} \sum_{k=0}^{\frac{n-1}{2}} C_n^{2k} b^{n-2k}(b^2-4c)^k.$$

$$\begin{aligned}& x_3 f_1 f_2(f_1^{p-2} + f_2^{p-2}) - x_3(f_1^p + f_2^p) \\ & = (2f_1f_2 - f_1^2 - f_2^2)\{\frac{x_3 b^{p-2}}{2^{p-1}} + \frac{x_3 c}{2^{p-3}} \sum_{k=1}^{\frac{p-2}{2}} C_{p-2}^{2k} b^{p-2-2k}(b^2-4c)^k - \\ & \quad \frac{x_3}{2^{p-1}} \sum_{k=1}^{\frac{p}{2}} C_p^{2k} b^{p-2k}(b^2-4c)^k\}, \\ & - x_2 f_1^2 f_2^2 (f_1^{p-3} + f_2^{p-3}) + x_2 f_1 f_2 (f_1^{p-1} + f_2^{p-1}) \\ & = (2f_1f_2 - f_1^2 - f_2^2)\{-\frac{x_2 c b^{p-3}}{2^{p-2}} - \\ & \quad \frac{x_2 c^2}{2^{p-4}}[\sum_{k=1}^{\frac{p-6}{2}} C_{p-3}^{2k} b^{p-3-2k}(b^2-4c)^k + (p-3)b(b^2-4c)^{\frac{p-4}{2}}] + \\ & \quad \frac{x_2 c}{2^{p-2}}[\sum_{k=1}^{\frac{p-4}{2}} C_{p-1}^{2k} b^{p-2k-1}(b^2-4c)^k + (p-1)b(b^2-4c)^{\frac{p-2}{2}}]\}.\end{aligned}$$

To prove this theorem, it is enough to show that

$$\begin{aligned}(2f_1f_2 - f_1^2 - f_2^2)K_{(p,1)} =& x_3 f_1 f_2 (f_1^{p-2} + f_2^{p-2}) - x_3(f_1^p + f_2^p) - \\ & x_2 f_1^2 f_2^2 (f_1^{p-3} + f_2^{p-3}) + x_2 f_1 f_2 (f_1^{p-1} + f_2^{p-1}),\end{aligned}$$

that is

$$f_1f_2(2f_1f_2 - f_1^2 - f_2^2)K_{(p,1)} = x_3f_1^2f_2^2(f_1^{p-2} + f_2^{p-2}) - x_3f_1f_2(f_1^p + f_2^p) - x_2f_1^3f_2^3(f_1^{p-3} + f_2^{p-3}) + x_2f_1^2f_2^2(f_1^{p-1} + f_2^{p-1}),$$

that is

$$(f_1^2 - f_1f_2)(f_2^2 - f_1f_2)K_{(p,1)} = (x_3 - x_2f_2)(f_2^2 - f_1f_2)f_1^p + (x_3 - x_2f_1)(f_1^2 - f_1f_2)f_2^p.$$

By induction, we obtain

$$(f_1^2 - f_1f_2)(f_2^2 - f_1f_2)K_{(p-1,1)} = (x_3 - x_2f_2)(f_2^2 - f_1f_2)f_1^{p-1} + (x_3 - x_2f_1)(f_1^2 - f_1f_2)f_2^{p-1}, \tag{4.2}$$

$$(f_1^2 - f_1f_2)(f_2^2 - f_1f_2)K_{(p-2,1)} = (x_3 - x_2f_2)(f_2^2 - f_1f_2)f_1^{p-2} + (x_3 - x_2f_1)(f_1^2 - f_1f_2)f_2^{p-2}. \tag{4.3}$$

Finally, if we plug (4.2), (4.3) back into (4.1), we have

$$(f_1^2 - f_1f_2)(f_2^2 - f_1f_2)K_{(p,1)} = (x_3 - x_2f_2)(f_2^2 - f_1f_2)f_1^p + (x_3 - x_2f_1)(f_1^2 - f_1f_2)f_2^p.$$

We complete the proof. □

Notice that the proof of Theorem 3.2 is similar to that of Theorem 3.1, so it will be omitted.

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