

Some Identities for Palindromic Compositions Without 2's

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Abstract In this paper, we study the palindromic compositions of even integers when no 2's are allowed in a composition and its conjugate. We show that the number of these palindromes is equal to $2F_{n-1}$, where, F_n is the n -th Fibonacci number. Consequently, we obtain several identities between the number of these palindromes, the number of compositions into parts equal to 1's or 2's, the number of compositions into odd parts and the number of compositions into parts greater than 1.

Keywords palindrome; the Fibonacci number; identity; combinatorial proof

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1. Introduction

A composition of a positive integer n is a representation of n as a sequence of positive integers called parts which sum to n . For example, the compositions of 4 are: (4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1). A palindromic composition [1] or palindrome of n is one for which the sequence is the same from left to right as from right to left. For example, there are 3 palindromes of 4, namely (4), (1, 2, 1), (1, 1, 1, 1).

It is well known that there are 2^{n-1} unrestricted compositions of n (see [2–5]). MacMahon's [4] study of compositions was influenced by his pioneering work in partitions. For instance, he devised a graphical representation of a composition, called a *zig-zag graph*, which resembles the partition Ferrers graph except that the first dot of each part is aligned with the last part of its predecessor. The *zig-zag graph* of the composition (6, 3, 1, 2, 2) is shown in Figure 1.

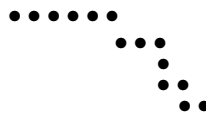


Figure 1 zig-zag graph

The conjugate of a composition is obtained by reading its graph by columns from left to right. The Figure 1 gives the conjugate of the composition (6, 3, 1, 2, 2) as (1, 1, 1, 1, 1, 2, 1, 3, 2, 1).

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Munagi [6] presented five methods to obtain the conjugate of a composition including the zig-zag graph. He also introduced some primary classes of compositions. Now we recall some terminologies from [6] herein. Let $C = (c_1, c_2, \dots, c_k)$ denote the composition of n with k parts. Then the conjugate of C is denoted by C' , the inverse of C is the reversal composition $\overline{C} = (c_k, c_{k-1}, \dots, c_1)$. C is inverse-conjugate if its inverse coincides with its conjugate: $C' = \overline{C}$.

In 2003, Chinn and Heubach [7] studied the compositions of n without 2's. They gave the following recurrence relation for the number of palindromic compositions of n without 2's.

Theorem 1.1 ([7]) *Let $P_{\neq 2}(n)$ be the number of palindromic compositions of n without 2's. Then*

$$P_{\neq 2}(1) = 1, \quad P_{\neq 2}(3) = 2, \quad P_{\neq 2}(5) = 3,$$

$$P_{\neq 2}(2k + 1) = 2P_{\neq 2}(2(k - 1) + 1) - P_{\neq 2}(2(k - 2) + 1) + P_{\neq 2}(2(k - 3) + 1).$$

In this paper, we will study the palindromic compositions of even integers without 2's. In Section 2, we obtain some properties for the number of the palindromes of $2n$ without 2's in which no 2's are allowed in the conjugate composition. And we find the number of the palindromes of $2n$ when no 2's are allowed in a composition and its conjugate composition be equal to $2F_{n-1}$. Consequently, we obtain several identities between the number of these palindromic compositions, the number of compositions into parts equal to 1's or 2's, the number of compositions into odd parts and the number of compositions into parts greater than 1.

2. Main results

We consider the palindromes of even integers when no 2's are allowed in a composition and its conjugate composition. For example, the composition $(3, 1, 1, 3)$ of 8 is a relevant palindrome. $(4, 4)$, however, is not an allowed palindrome because the conjugate of $(4, 4)$ is $(1, 1, 1, 2, 1, 1, 1)$ in which 2 appears. We first present the following recurrence relation for the number of these palindromes.

Theorem 2.1 *Let $P'_{\neq 2}(m)$ be the number of palindromes of even integer m when no 2's are allowed in a composition and its conjugate composition. Then*

$$P'_{\neq 2}(2) = 1, \quad P'_{\neq 2}(4) = 2, \quad P'_{\neq 2}(6) = 2,$$

$$P'_{\neq 2}(2n) = P'_{\neq 2}(2n - 2) + P'_{\neq 2}(2n - 4), \quad n > 3.$$

Proof We split the relevant compositions of $2n$ into four classes.

- (a) the parts on both ends are 3's;
- (b) the parts on both ends are d 's, $d > 3$;
- (c) the parts on both ends are "1, 1";
- (d) the parts on both ends are " $\underbrace{1, 1, \dots, 1}_l$ ", where $l > 2$.

Given any composition α in class (a), we replace 3 on both ends by 1, hence we have the palindromic composition of $2n - 4$ in which the parts on both ends are 1's and no 2's are allowed

in both the composition and its conjugate composition. And vice versa.

Given any composition α in class (b), we replace d on both ends by $d - 1$, and then we get the palindromic composition of $2n - 2$ in which the parts on both ends are > 1 and no 2's are allowed in both the composition and its conjugate composition. And vice versa.

Given any composition α in class (c), by deleting "1, 1" on both ends, we obtain the palindromic composition of $2n - 4$ in which the parts on both ends are > 1 and no 2's are allowed in both the composition and its conjugate composition. And vice versa.

Given any composition α in class (d), by deleting 1 on the left end and the right end of α , we get the palindromic composition of $2n - 2$ in which the parts on both ends are 1's and no 2's are allowed in both the composition and its conjugate composition. And vice versa.

Hence, we know that compositions in both class (a) and (c) correspond to the relevant compositions of $2n - 4$, while compositions in both class (b) and (d) correspond to the relevant compositions of $2n - 2$.

This completes the proof. \square

The following table gives some values of $P'_{\neq 2}(2n)$:

$2n$	2	4	6	8	10	12	14	16	18	20	22	24	26	28
$P'_{\neq 2}(2n)$	1	2	2	4	6	10	16	26	42	68	110	178	288	466

Table 1 The number of relevant palindromes of $2n$ without 2's

We notice that the sequence $P'_{\neq 2}(2n)$ gives twice the Fibonacci sequence with $F(0) = 1$. This sequence $P'_{\neq 2}(2n)$ appears as A055389 in [8]. Naturally, we have the following corollary.

Corollary 2.2 For integers n , let $P'_{\neq 2}(2n)$ be the number of palindromes of $2n$ when no 2's are allowed in a composition and its conjugate composition. Then

$$P'_{\neq 2}(2n) = 2F_{n-1}, \quad n \geq 2.$$

Where F_n is the n -th Fibonacci number.

Proof From Theorem 2.1, we have $P'_{\neq 2}(2n) = P'_{\neq 2}(2n-2) + P'_{\neq 2}(2n-4)$ with $P'_{\neq 2}(4) = P'_{\neq 2}(6) = 2$. So $P'_{\neq 2}(2n) = 2F_{n-1}$, $n \geq 2$. \square

Further, we obtain the following corollary.

Corollary 2.3 For integers m , let $P'_{\neq 2}(m)$ be the number of palindromes of m when no 2's are allowed in a composition and its conjugate composition. Then

$$P'_{\neq 2}(2n) = P'_{\neq 2}(2n-1), \quad n \geq 1.$$

We provide a combinatorial bijective proof below.

Proof Given any relevant composition α of $2n$, the middle part of α is either even $h > 2$ or "1, 1". So we replace h by $h - 1$ when the middle part is $h > 2$, or we delete one 1 when the middle two parts are "1, 1". Consequently, we obtain the relevant composition of $2n - 1$.

Conversely, for each relevant composition β of $2n - 1$, the middle part of β is either odd $t \geq 3$ or $t = 1$. Thereupon we replace t by $t + 1$ when the middle part is $t \geq 3$, or we replace 1 by "1, 1" when the middle part is 1. Therefore, we get the relevant composition of $2n$.

This completes the proof. \square

The following table gives some values of $P'_{\neq 2}(n)$:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$P'_{\neq 2}(n)$	1	1	2	2	2	2	4	4	6	6	10	10	16	16	26	26

Table 2 The number of relevant palindromes of n without 2's

We notice that the sequence $P'_{\neq 2}(n)$ is formed by the Fibonacci sequence for $n > 2$ interlaced with itself. The sequence of $P'_{\neq 2}(n)$ appears as A214927 except $n = 0, 1, 2$ in [8]. A214927 is the number of n -digit numbers N that do not end with 0 and are such that the reversal of N divides N but is different from N .

Not unnaturally, we get the following corollary by Corollary 2.3 and Theorem 2.1.

Corollary 2.4 For integers m , let $P'_{\neq 2}(m)$ be the number of palindromes of m when no 2's are allowed in a composition and its conjugate composition. Then

$$P'_{\neq 2}(1) = 1, \quad P'_{\neq 2}(3) = 2, \quad P'_{\neq 2}(5) = 2, \quad \text{and}$$

$$P'_{\neq 2}(2n + 1) = P'_{\neq 2}(2n - 1) + P'_{\neq 2}(2n - 3), \quad n > 2.$$

We know that the number of compositions of n into odd parts is F_n , the number of compositions of n into parts equal to 1's or 2's is F_{n+1} , and the number of compositions of n into parts greater than 1 is F_{n-1} . Consequently, Corollary 2.2 leads to the identities in the following three theorems.

We give combinatorial proofs for palindrome when no 2's are allowed in a composition and its conjugate composition in which the first part is 1. The proofs for compositions with the first part > 1 are similar.

Theorem 2.5 Let $C_{>1}(n)$ be the number of compositions of n into parts greater than 1. Then

$$P'_{\neq 2}(2n) = 2C_{>1}(n), \quad n \geq 2.$$

Proof Given any relevant palindrome α of $2n$, we first obtain a composition β of n by cutting off the right half of α when the number of parts of α is an even integer, otherwise, cutting off the right half of α and replace the middle part m by $\frac{m}{2}$. Next, a composition γ is obtained by summing all the adjacent 1's to produce new parts > 1 from left to right in composition β . Lastly, the parts "s, 1" ($s > 2$) are replaced by "2, (s - 1)" from left to right in composition γ . Hence, we obtain the composition of n into parts greater than 1.

For example, the palindromes (1, 1, 1, 3, 1, 3, 1, 1, 3, 1, 3, 1, 1, 1) and (1, 1, 1, 4, 1, 6, 1, 4, 1, 1, 1) of 22 produce the composition (3, 2, 2, 2, 2) and (3, 2, 3, 3) of 11 as follows:

$$(1, 1, 1, 3, 1, 3, 1, 1, 3, 1, 3, 1, 1, 1) \longrightarrow (1, 1, 1, 3, 1, 3, 1)$$

$$\longrightarrow (3, 3, 1, 3, 1) \longrightarrow (3, 2, 2, 2, 2)$$

$$(1, 1, 1, 4, 1, 6, 1, 4, 1, 1, 1) \longrightarrow (1, 1, 1, 4, 1, 3) \longrightarrow (3, 4, 1, 3) \longrightarrow (3, 2, 3, 3).$$

Conversely, for each composition $\gamma = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k)$ of n into parts greater than 1, we first split λ_1 into " $\underbrace{1, 1, \dots, 1}_{\lambda_1}$ " from left to right, while $\lambda_2, \lambda_3, \dots, \lambda_k$ stays the same. Next, we split λ_3 into " $\underbrace{1, 1, \dots, 1}_{\lambda_3}$ " when $\lambda_2 \neq 2$. If $\lambda_2 = 2$, we split λ_3 into two parts: " $(\lambda_3 - 1), 1$ ", and we have $\lambda_2 + (\lambda_3 - 1) = 2 + (\lambda_3 - 1) = \lambda_3 + 1$ as a new part, while 1 is another part. Repeat these steps for the remaining parts to produce a composition π of n without 2's except the last part h of π . Lastly, we extend π into a palindrome of $2n$ with middle part equal to $2h$ when $h > 1$. If $h = 1$, the middle two parts are " $1, 1$ ".

For example, the composition $(3, 2, 3, 3)$ of 11 produces the palindrome $(1, 1, 1, 4, 1, 6, 1, 4, 1, 1, 1)$ of 22 as follows:

$$(3, 2, 3, 3) \longrightarrow (1, 1, 1, 2, 2, 1, 3) \longrightarrow (1, 1, 1, 4, 1, 3) \longrightarrow (1, 1, 1, 4, 1, 6, 1, 4, 1, 1, 1).$$

This completes the proof. \square

Theorem 2.6 *Let $C_{\text{odd}}(n)$ be the number of compositions of n into odd parts. Then*

$$P'_{\neq 2}(2n) = 2C_{\text{odd}}(n-1), \quad n \geq 2.$$

Proof Given any relevant palindrome α of $2n$, using the proof of Theorem 2.5 we have a composition C of n into parts > 1 . And then obtain the conjugate C' of C . Since the parts of C are greater than 1, C' is a composition of n into parts equal to 1's or 2's where the first and last parts equal to 1. Next we delete the last part 1 of C' to obtain a composition g of $n-1$, then replace the leftmost 1 and all 2's to the right of it by its sum to produce new parts from left to right in g . As a result, we get a composition of $n-1$ into odd parts.

For example, the palindrome $(1, 1, 3, 1, 6, 1, 3, 1, 1)$ of 18 produces the composition $(7, 1)$ of 8 as follows:

$$\begin{aligned} (1, 1, 3, 1, 6, 1, 3, 1, 1) &\longrightarrow (1, 1, 3, 1, 3) \longrightarrow (2, 2, 2, 3) \\ &\longrightarrow (1, 2, 2, 2, 1, 1) \longrightarrow (1, 2, 2, 2, 1) \longrightarrow (7, 1). \end{aligned}$$

In a similar manner to the proof of Theorem 2.5 we know this correspondence is one-to-one, and vice versa.

This completes the proof. \square

Theorem 2.7 *Let $C_{1-2}(n)$ be the number of compositions of n into parts equal to 1's or 2's. Then*

$$P'_{\neq 2}(2n) = 2C_{1-2}(n-2), \quad n > 2.$$

Proof Given any relevant palindrome α of $2n$, using the proof of Theorem 2.5, we have a composition C of n into parts > 1 and then obtain the conjugate C' of C . Since the parts of C are greater than 1, C' is a composition of n into parts equal to 1 or 2 with the first and last parts

equal to 1's. Next, deleting both the first part 1 and the last part 1 of C' , we obtain composition of $n - 2$ into parts equal to 1 or 2.

For example, the palindrome $(1, 1, 3, 1, 6, 1, 3, 1, 1)$ of 18 produces the composition $(2, 2, 2, 1)$ of 7 as follows:

$$\begin{aligned} (1, 1, 3, 1, 6, 1, 3, 1, 1) &\longrightarrow (1, 1, 3, 1, 3) \longrightarrow (2, 2, 2, 3) \\ &\longrightarrow (1, 2, 2, 2, 1, 1) \longrightarrow (2, 2, 2, 1). \end{aligned}$$

In a similar way to the proof of Theorem 2.5 we know this correspondence is one-to-one, and vice versa.

This completes the proof. \square

From Corollary 2.3 and Theorems 2.5–2.7, we also have several analogous identities for the number of palindromes of odd integers when no 2's are allowed in both composition and its conjugate composition. We present the following corollaries.

Corollary 2.8 For integers n , let $P'_{\neq 2}(2n + 1)$ be the number of palindromes of $2n + 1$ when no 2's are allowed in a composition and its conjugate composition, and $C_{\text{odd}}(n)$ be the number of compositions of n into odd parts. Then

$$P'_{\neq 2}(2n + 1) = 2C_{\text{odd}}(n), \quad n \geq 1.$$

Corollary 2.9 For integers n , let $P'_{\neq 2}(2n + 1)$ be the number of palindromes of $2n + 1$ when no 2's are allowed in a composition and its conjugate composition, and $C_{1-2}(n)$ be the number of compositions of n into parts equal to 1's or 2's. Then

$$P'_{\neq 2}(2n + 1) = 2C_{1-2}(n - 1), \quad n \geq 2.$$

Corollary 2.10 For integers n , let $P'_{\neq 2}(2n + 1)$ be the number of palindromes of $2n + 1$ when no 2's are allowed in a composition and its conjugate composition, and $C_{>1}(n)$ be the number of compositions of n into parts greater than 1. Then

$$P'_{\neq 2}(2n + 1) = 2C_{>1}(n + 1), \quad n \geq 1.$$

The combinatorial proofs of these corollaries are similar to the proofs of Theorems 2.5–2.7, so we omit them. We only cite an example of Corollary 2.10 as follows.

Example 2.11 Let $n = 6$. The corresponding relation between the relevant compositions of 13 and 7 are as follows.

$$\begin{aligned} (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) &\longleftrightarrow (7), \quad (1, 1, 1, 1, 5, 1, 1, 1, 1) \longleftrightarrow (4, 3), \\ (1, 1, 1, 3, 1, 3, 1, 1, 1, 1) &\longleftrightarrow (3, 2, 2), \quad (1, 1, 1, 1, 1, 3, 1, 1, 1, 1, 1) \longleftrightarrow (5, 2). \\ (1, 1, 1, 7, 1, 1, 1) &\longleftrightarrow (3, 4), \quad (1, 1, 9, 1, 1) \longleftrightarrow (2, 5), \\ (1, 1, 3, 1, 1, 1, 3, 1, 1) &\longleftrightarrow (2, 3, 2), \quad (1, 1, 4, 1, 4, 1, 1) \longleftrightarrow (2, 2, 3). \end{aligned}$$

And for the conjugate compositions counted in $P'_{\neq 2}(13)$

$$(13) \longleftrightarrow (7), \quad (5, 1, 1, 1, 5) \longleftrightarrow (4, 3),$$

$$\begin{aligned}
(4, 1, 3, 1, 4) &\longleftrightarrow (3, 2, 2), & (6, 1, 6) &\longleftrightarrow (5, 2). \\
(4, 1, 1, 1, 1, 1, 4) &\longleftrightarrow (3, 4), & (3, 1, 1, 1, 1, 1, 1, 3) &\longleftrightarrow (2, 5), \\
(3, 1, 5, 1, 3) &\longleftrightarrow (2, 3, 2), & (3, 1, 1, 3, 1, 1, 3) &\longleftrightarrow (2, 2, 3).
\end{aligned}$$

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