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Results on the Hadamard Factorization Theorem for Analytic Functions in the Finite Disc

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Abstract In this paper, we investigate the Hadamard factorization theorem of analytic functions in the finite disc $D_R = \{z \in \mathbf{C} : |z| = r < R < \infty\}$, where $\mathbf{C}$ is the whole complex plane. Examples are given to show that our results are sharp.

Keywords analytic function; meromorphic function; finite disc; Hadamard factorization theorem

MR(2010) Subject Classification 30D35

1. Introduction and main results

Set that $E(u,0) = 1 - u$ and

$$E(u,p) = (1-u)\exp\{u + \frac{1}{2}u^2 + \cdots + \frac{1}{p}u^p\}, \quad p = 1, 2, \ldots.$$

It is well-known that Weierstrass [1] obtained the following factorization theorem of entire function with infinite many zeros on the whole complex plane $\mathbf{C}$.

Theorem 1.1 (Weierstrass factorization theorem) ([1]) *Let f be an entire function with infinite many zeros $0, z_1, z_2, \ldots, z_n, \ldots$. Then*

$$f(z) = z^m \Phi(z) e^{g(z)},$$

where

$$\Phi(z) = \prod_{n=1}^{\infty} E(\frac{z}{z_n}, p_n - 1),$$

p_n is a positive integer number depending on n, the number m is the order of zeros of $f(z)$ at $z = 0$, and g is an entire function.

Later, Hadamard improved Theorem 1.1 and obtained the following well-known theorem by making use of the concept of the growth of order of entire function.

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Theorem 1.2 (Hadamard factorization theorem) ([1,2]) *An entire function $f(z)$ of finite order $\sigma(f)$ is factorized by*

$$f(z)=z^m\Phi(z)e^{g(z)}$$

which satisfies that $g(z)$ is a polynomial of degree $\deg(g)\le\sigma(f)$, and that $\Phi(z)$ is the canonical product [1,2] of non-zero zeros of $f(z)$, and that m is the order of zeros of $f(z)$ at $z=0$.

From the Riemann Mapping Theorem, we know that there are essential differences between the finite open disc $D_R=\{z\in\mathbf{C}:|z|=r<R<\infty\}$ and the whole complex plane $\mathbf{C}$. Thus it is very interesting to consider the Weierstarss factorization theorem and Hadamard factorization theorem of analytic or meromorphic functions in the finite disc $D_R=\{z\in\mathbf{C}:|z|=r<R<\infty\}$. In [3], there exists a factorization theorem for meromorphic functions with bounded characteristic (Nevanlinna class) in the unit disc as the following form

$$f(z)=z^m\frac{\pi_1(z)}{\pi_2(z)}e^{h(z)},$$

where m is an integer, π_1, π_2 are Blaschke products, and $h(z)$ is analytic in the unit disc. Using $u=\frac{1-|z_k|^2}{1-\bar{z}_kz}$ into canonical product, it is known that if f is an analytic function in the unit disc $\mathbb{D}$ with finite M-order $\sigma_M(f)$, then $f(z)=P(z)q(z)$, where P is a canonical product formed using zeros of f, and both P and q are analytic and of M-order $\sigma_M(P)$ and $\sigma_M(q)$ at most $\sigma_M(f)$ in $\mathbb{D}$ (see [4]). If f is a meromorphic function with finite order $\sigma(f)$ in $\mathbb{D}$, then $f(z)=\frac{f_1(z)}{f_2(z)}$, where f_1 and f_2 are analytic functions in $\mathbb{D}$ with order at most $\sigma(f)$ (see [5, p.227]).

Recently, by making use of

$$u=\frac{R\frac{z_k}{|z_k|}-z_k}{R\frac{z_k}{|z_k|}-z}$$

into canonical product, Xu [6] obtained the disc version of Weierstrass theorem.

Theorem 1.3 ([6]) *Let f be an analytic function in D_R with infinite many zeros $0,z_1,z_2,\dots,z_n$, $\dots$, where $0<|z_n|<R$. Then*

$$f(z)=z^mP(z)e^{g(z)},\tag{1.1}$$

where

$$P(z)=\prod_{n=1}^{\infty}E\Big(\frac{R\frac{z_n}{|z_n|}-z_n}{R\frac{z_n}{|z_n|}-z},p_n-1\Big),\tag{1.2}$$

p_n is a positive integer number depending on n, the number m is the order of zero of $f(z)$ at $z=0$, and g is an analytic function in D_R.

In this paper, we will investigate the Hadamard factorization theorem in the disc D_R by also using

$$u=\frac{R\frac{z_k}{|z_k|}-z_k}{R\frac{z_k}{|z_k|}-z}$$

into canonical product. Let us use the standard notations of the Nevanlinna's theory in D_R (see [3,7]). We introduce some definitions which are similar to those in the unit disc as follows.

Definition 1.4 *Let f be a meromorphic function in D_R, and*

$$D(f)=\overline{\lim_{r\to R^-}}\frac{T(r,f)}{\log\frac{1}{R-r}}=b. \tag{1.3}$$

If $b<\infty$, we say f is non-admissible; if $b=\infty$, we say f is admissible.

Definition 1.5 *Let f be an analytic function in D_R. If*

$$D_M(f)=\overline{\lim_{r\to R^-}}\frac{\log^+ M(r,f)}{\log\frac{1}{R-r}}=b<\infty \ (or\ =\infty), \tag{1.4}$$

then we say f is of finite b degree (or infinite degree).

Remark 1.6 We have the inequality $D(f)\le D_M(f)$ if f is an analytic function in D_R. (see Theorem 2.3(v) in the next section). However, we cannot get that $D(f)<\infty$ if and only if $D_M(f)<\infty$ (see [8, Proposition 1(i)]), since there exists a non-admissible analytic function f in D_R which satisfies $D_M(f)=\infty$. For example, if

$$f(z)=e^{g(z)}=\exp\{\frac{1}{1-z}\},\quad z\in\Delta,$$

then one can get that $D_M(f)=\infty$ and $D(f)<\infty$.

Definition 1.7 *The order of a meromorphic function f in D_R is defined by*

$$\sigma(f)=\overline{\lim_{r\to R^-}}\frac{\log^+ T(r,f)}{\log\frac{1}{R-r}}. \tag{1.5}$$

For an analytic function f in D_R, we also define

$$\sigma_M(f)=\overline{\lim_{r\to R^-}}\frac{\log^+\log^+ M(r,f)}{\log\frac{1}{R-r}}, \tag{1.6}$$

where $M(r,f)$ is the maximum modulus function.

Remark 1.8 We have the inequalities $\sigma(f)\le\sigma_M(f)\le\sigma(f)+1$ if f is an analytic function in D_R. (see Theorem 2.3(v) in the next section). Thus the order $\sigma(f)$ of an analytic function in D_R is finite if and only if $\sigma_M(f)$ is finite. It is well known that $\sigma(f)=\sigma_M(f)$ if f is an entire function in $\mathbf{C}$. However, there exists an analytic function f in D_R which satisfies $\sigma(f)\neq\sigma_M(f)$. For example [5, p.205], if

$$f(z)=e^{g(z)}=\exp\{\frac{1}{(1-z)^k}\}\text{ with }k>1,\ z\in\Delta,$$

then $\sigma_M(f)=k$ and $\sigma(f)=k-1$ which satisfies $\sigma_M(f)=\sigma(f)+1$.

Now we give the concept of the convergence exponent in D_R.

Definition 1.9 *The convergence exponent of a ($a\in\mathbf{C}$) of a meromorphic function f in D_R is defined by*

$$\lambda(f-a)=\overline{\lim_{r\to R^-}}\frac{\log n(r,\frac{1}{f-a})}{\log\frac{1}{R-r}}; \tag{1.7}$$

if $a=\infty$, then

$$\lambda(\frac{1}{f})=\overline{\lim_{r\to R^-}}\frac{\log n(r,f)}{\log\frac{1}{R-r}}. \tag{1.8}$$

Remark 1.10 It is easy to see that

$$\lambda(f-a)=\overline{\lim_{r\to R^-}}\frac{\log N(r,\frac{1}{f-a})}{\log\frac{1}{R-r}};\quad \lambda(\frac{1}{f})=\overline{\lim_{r\to R^-}}\frac{\log N(r,f)}{\log\frac{1}{R-r}}.$$

For the convergence exponent, we have the following results.

Theorem 1.11 *Let f be a meromorphic function in D_R which has zeros $z_1,z_2,\ldots,z_k,\ldots,|z_k|=r_k<R$, and $r_i\le r_j$ $(i\le j)$. The multiple zeros count those multiplicity. Denote*

$$\rho(f)=\inf\{\tau:\sum_{j=1}^{\infty}(R-r_j)^{\tau}<\infty,\tau>0\}. \tag{1.9}$$

Then $\rho(f)=\lambda(f)$.

From Theorem 1.11, one can easily get the following corollary.

Corollary 1.12 *(i) $\rho(f)=\lambda(f)\le\sigma(f)$; (ii) when $\tau>\sigma(f)$, the sum $\sum_{j=1}^{\infty}(R-r_j)^{\tau}$ converges.*

Let f be an analytic function of finite order in D_R which has zeros $z_1,z_2,\ldots,z_n,\ldots,|z_n|=r_n<R$. Set $p_n=p+1$ in the sum (1.2). By Corollary 1.12, the sum $\sum_{j=1}^{\infty}(R-r_j)^{p+1}$ converges if $p+1>\sigma(P)$. Hence there exists some integer number p which is independent of n and satisfies that the product

$$P(z)=\prod_{n=1}^{\infty}E(\frac{R\frac{z_n}{|z_n|}-z_n}{R\frac{z_n}{|z_n|}-z},p) \tag{1.10}$$

converges uniformly in any domain in D_R (similar discussion as to the proof of Lemma 2.11, see [6]). Thus we can give the definition as follows.

Definition 1.13 *Canonical product formed by non-zero zeros of an analytic function $f(z)$ in D_R is defined by the function $P(z)$ of (1.10), if p is the maximum integer number which satisfies that the sum $\sum_{j=1}^{\infty}(R-r_j)^{p+1}$ converges. And p is called the genus of the canonical product.*

For the canonical product, we have the following result.

Theorem 1.14 *The order $\sigma(P)$ and $\sigma_M(P)$ of canonical product $P(z)$ are equal to the convergence exponent $\lambda(P)$ of its zeros, namely, $\sigma_M(P)=\lambda(P)=\sigma(P)$.*

Remark 1.15 For example [6, Example 2.2], if $f(z)$ is analytic in Δ which has non-zero zeros $z_n=1-\frac{1}{n^2}$ $(n\in\mathbf{N}\setminus\{1\})$ in Δ, then one can get that

$$P(z)=\prod_{n=2}^{\infty}E(\frac{1}{n^2(1-z)},0),$$

where $P(z)$ is the the canonical product formed by non-zero zeros of f, and that $\sigma_M(P)=\sigma(P)=\lambda(P)=\lambda(f)=\frac{1}{2}$.

Now we give factorization theorems of functions of finite order in D_R. For analytic functions of finite order in D_R, we obtain the result as follows.

Theorem 1.16 *An analytic function $f(z)$ of finite order in D_R is factorized by*

$$f(z) = z^m P(z) e^{g(z)} \tag{1.11}$$

which satisfies that

$$D_M(g) \leq \sigma(f) + 2, \tag{1.12}$$

where $g(z)$ is an analytic function of finite degree in D_R, $P(z)$ is the canonical product of non-zero zeros of $f(z)$ in D_R, and m is the order of zeros of $f(z)$ at $z = 0$.

For a meromorphic function of finite order in D_R, we obtain the following result.

Theorem 1.17 *A meromorphic function $f(z)$ of finite order in D_R is factorized by*

$$f(z) = z^m \frac{P_1(z)}{P_2(z)} e^{g(z)} \tag{1.13}$$

which satisfies the inequality

$$D_M(g) \leq \sigma(f) + 2, \tag{1.14}$$

where $g(z)$ is an analytic function of finite degree in D_R, $P_1(z), P_2(z)$ are respectively the canonical products of non-zero zeros of $f(z)$ and $\frac{1}{f(z)}$ in D_R, and m is the order of zeros or poles of $f(z)$ at $z = 0$.

Remark 1.18 If $f(z) = e^{g(z)} = \exp\{\frac{-1}{(1-z)^2}\}$, $z \in \Delta$, then one can get that $D_M(g) = 2$ and $\sigma(f) = \sigma_M(f) = 0$ which satisfies $D_M(g) = 2 + \sigma(f)$. The example shows that Theorems 1.16 and 1.17 are sharp.

2. Some basic results in disc

For describing accurately the infinite order, we give the definition of the hyper order as follows. We omit the details of proofs of the following results since they are very similar to the case in the unit disc [4,9,11]

Definition 2.1 *The hyper order of a meromorphic function f in D_R is defined by*

$$\sigma_2(f) = \overline{\lim_{r \to R^-}} \frac{\log^+ \log^+ T(r,f)}{\log \frac{1}{R-r}}, \tag{2.1}$$

for an analytic function f in D_R, we also define

$$\sigma_{2,M}(f) = \overline{\lim_{r \to R^-}} \frac{\log^+ \log^+ \log^+ M(r,f)}{\log \frac{1}{R-r}}. \tag{2.2}$$

Now we give the results related to the order and hyper order in D_R as follows.

Theorem 2.2 *If f and g are meromorphic functions in D_R, then we have*

(i) $\sigma(f) = \sigma(\frac{1}{f}), \sigma(a \cdot f) = \sigma(f), (a \in \boldsymbol{C} \setminus \{0\})$;

(ii) $\sigma(f) = \sigma(f')$;

(iii) $\max\{\sigma(f+g), \sigma(f \cdot g)\} \leq \max\{\sigma(f), \sigma(g)\}$;

(iv) *If $\sigma(f) < \sigma(g)$, then $\sigma(f+g) = \sigma(g), \sigma(f \cdot g) = \sigma(g)$.*

Theorem 2.3 *If f and g are analytic functions in D_R, then we have*

(i) $\sigma_M(a\cdot f)=\sigma_M(f)$ $(a\in \mathbf{C}\setminus\{0\})$;

(ii) $\sigma_M(f)=\sigma_M(f')$; $D_M(f')=D_M(f)+1$ *when* $D_M(f)>1$;

(iii) $\max\{\sigma_M(f+g),\sigma_M(f\cdot g)\}\le\max\{\sigma_M(f),\sigma_M(g)\}$;

(iv) *If* $\sigma_M(f)<\sigma_M(g)$, *then* $\sigma_M(f+g)=\sigma_M(g)$;

(v) $D(f)\le D_M(f)$, $\sigma(f)\le\sigma_M(f)\le\sigma(f)+1$, $\sigma_2(f)=\sigma_{2,M}(f)$.

Remark 2.4 It is very interesting that we have the result $\sigma_2(f)=\sigma_{2,M}(f)$ in Theorem 2.3 if f is an analytic function in D_R, which is the same result as the entire function in $\mathbf{C}$.

Theorem 2.5 *Let $f(z)=e^{g(z)}$, where $g(z)$ is an analytic function in D_R. Then $\sigma_M(f)\le D_M(g)\le\sigma_M(f)+1\le\sigma(f)+2$, and $\sigma_M(g)=\sigma_{2,M}(f)=\sigma_2(f)$.*

Corollary 2.6 *Let $f(z)=e^{g(z)}$, where $g(z)$ is an analytic function in D_R. If f is of finite order of growth, then g is non-admissible.*

Remark 2.7 From the functions f in Remarks 1.6 and 1.8, we can see that $D_M(g)=\sigma_M(f)$. However, there exists $f(z)=e^{g(z)}$ in D_R which satisfies $D_M(g)\ne\sigma_M(f)$. For example, if

$$f(z)=e^{g(z)}=\exp\{\frac{z}{z-1}\}\quad\text{or}\quad f(z)=e^{g(z)}=\exp\{\frac{1}{z-1}\},\quad z\in\Delta,$$

then one can see that $D_M(g)=1$ and $\sigma_M(f)=\sigma(f)=0$ which satisfies $D_M(g)=1+\sigma_M(f)=1+\sigma(f)$. From the function f in Remark 1.18, it shows that $D_M((g))=2+\sigma(f)$. Thus it is sharp of $\sigma_M(f)\le D_M(g)\le\sigma_M(f)+1\le\sigma(f)+2$.

Lemma 2.8 ([6]) *The general form of an analytic function f in D_R with no zeros in D_R is $e^{g(z)}$, where $g(z)$ is also an analytic function in D_R.*

Lemma 2.9 ([1,2]) *Let $k>1$. Then for $|u|\le\frac{1}{k}$, we have*

$$|\log E(u,p)|\le\frac{k}{k-1}|u|^{p+1}. \tag{2.3}$$

Lemma 2.10 ([6]) *Let the positive real sequences $\{r_n\}$ satisfy*

$$0<r_1\le r_2\le\cdots\le r_n\le\cdots<R, R<\infty$$

and let $n\to\infty$, $r_n\to R^-$. Then there exists a positive integer series p_n satisfying that

$$\sum_{n=1}^{\infty}(\frac{R-r_n}{R-r})^{p_n} \tag{2.4}$$

is convergent for any $r\in(0,R)$.

Lemma 2.11 ([6]) *For any given sequences $\{z_n\}$ such that $|z_n|<R$, its unique limit point is just at R, then there exists an analytic function in D_R which has and only has zeros z_n, $n=1,2,\ldots$.*

3. Proofs of Theorems

In this section we give the proofs of our results.

Proof of Theorem 1.11 We use similar reasoning as in the proof of [2, Theorem 2.1] to get the theorem. Let $n(r)$ be the number of non-zero zeros of $f(z)$ in $|z| \leq r < R$, and set

$$N(r) = \int_0^r \frac{n(t)}{t} \mathrm{d}t,$$

then

$$n(r, \frac{1}{f}) = n(r) + n(0, \frac{1}{f}).$$

Set $\lambda(f) = \lambda$. Now we consider the following two cases.

Case 1 $\lambda < \infty$. By Definition 1.9, for any given $\varepsilon(> 0)$, when j is large enough, we have

$$\frac{\log n(r_j, \frac{1}{f})}{\log \frac{1}{R - r_j}} < \lambda + \frac{\varepsilon}{2},$$

namely,

$$n(r_j, \frac{1}{f}) < \{(\frac{1}{R - r_j})^{\lambda + \frac{\varepsilon}{2}}\}.$$

Since the multiple zeros count those multiplicity, then we have $n(r_j, \frac{1}{f}) \geq n(r_j) \geq j$. Thus

$$j < \{(\frac{1}{R - r_j})^{\lambda + \frac{\varepsilon}{2}}\},$$

$$(R - r_j)^{\lambda + \varepsilon} \leq (R - r_j)^{(\lambda + \frac{\varepsilon}{2}) \cdot (\frac{\lambda + \varepsilon}{\lambda + \frac{\varepsilon}{2}})} < j^{-\frac{\lambda + \varepsilon}{\lambda + \frac{\varepsilon}{2}}}.$$

Since $\frac{\lambda+\varepsilon}{\lambda+\frac{\varepsilon}{2}} > 1$, then $\sum_{j=1}^{\infty}(R - r_j)^{\lambda+\varepsilon}$ converges.

On the other hand, for any given ε (> 0), one can prove that

$$\sum_{j=1}^{\infty} (R - r_j)^{\lambda - \varepsilon} \tag{3.1}$$

is not convergent. In fact, if (3.1) is convergent, then

$$\lim_{j \to \infty} (j \cdot (R - r_j)^{\lambda - \varepsilon}) = 0.$$

Thus, when j is large enough, $n(r_j) = j < \{(\frac{1}{(R - r_j)})^{\lambda - \varepsilon}\}$. Set $r_k \leq r < r_{k+1}$, then $n(r_k) = n(r) < n(r_{k+1})$. Therefore, when j is large enough, we have

$$n(r, \frac{1}{f}) - n(0, \frac{1}{f}) = n(r) < \{(\frac{1}{R - r})^{\lambda - \varepsilon}\}$$

Hence

$$\lambda = \overline{\lim_{r \to R^-}} \frac{\log n(r, \frac{1}{f})}{\log \frac{1}{R - r}} \leq \lambda - \varepsilon.$$

This is a contradiction. Hence (3.1) is not convergent.

By the arbitrariness of ε, we obtain the convergence exponent $\lambda(f)$ of zeros of $f(z)$ is equal to $\rho(f)$.

Case 2 $\lambda = \infty$. Then for any $\tau > 0$, the sum $\sum\limits_{j=1}^{\infty} (R - r_j)^{\tau}$ is not convergent. Otherwise, if there exists some τ such that $\sum_{j=1}^{\infty}(R - r_j)^{\tau}$ converges, then using the same method as above, we have $\lambda=\tau$. This contradicts $\lambda=\infty$. Thus $\rho(f)=\infty$ when $\lambda(f)=\infty$. $\Box$

Proof of Theorem 1.14 By Theorem 2.3 (v) and Corollary 1.12, $\lambda=\lambda(P)\le\sigma(P)\le\sigma_M(P)$. Now we prove $\lambda\ge\sigma_M(P)$. We set

$$\log|P(z)|=\sum_{\frac{R-r_n}{R-r}\ge\frac12}|\log E(\frac{R\frac{z_n}{|z_n|}-z_n}{R\frac{z_n}{|z_n|}-z},p)|+\sum_{\frac{R-r_n}{R-r}<\frac12}|\log E(\frac{R\frac{z_n}{|z_n|}-z_n}{R\frac{z_n}{|z_n|}-z},p)|=\sum_1+\sum_2.$$

(i) For $\sum_2$, by Lemma 2.9, and $\frac{R-r_n}{R-r}<\frac12$ we have

$$\Big|\sum_2\Big|\le 2\sum_{\frac{R-r_n}{R-r}<\frac12}(\frac{R-r_n}{R-r})^{p+1}=\frac{2}{(R-r)^{p+1}}\sum_{\frac{R-r_n}{R-r}<\frac12}(R-r_n)^{p+1}.$$

If $p=\lambda-1$, then $\sum_{n=1}^{\infty}(R-r_n)^{p+1}$ converges. Hence there exists a constant A_1 such that

$$\Big|\sum_2\Big|<A_1(\frac{1}{R-r})^{p+1}=A_1(\frac{1}{R-r})^{\lambda}.$$

If $p>\lambda-1$, then for any given small ε (>0), $p+1>\lambda+\varepsilon$, So

$$\begin{aligned}\Big|\sum_2\Big|&\le 2\cdot(\frac{1}{R-r})^{\lambda+\varepsilon}\cdot\sum_{\frac{R-r_n}{R-r}<\frac12}\frac{(R-r_n)^{p+1}}{(R-r)^{p+1-\lambda-\varepsilon}}\\&\le 2\cdot(\frac12)^{p+1-\lambda-\varepsilon}\cdot(\frac{1}{R-r})^{\lambda+\varepsilon}\cdot\sum_{n=1}^{\infty}(R-r_n)^{\lambda+\varepsilon}\le A_1(\frac{1}{R-r})^{\lambda+\varepsilon}.\end{aligned}$$

(ii) For $\sum_1$, since $\frac{R-r_n}{R-r}\ge\frac12$, we have

$$\begin{aligned}&\log|E(\frac{R\frac{z_n}{|z_n|}-z_n}{R\frac{z_n}{|z_n|}-z},p)|\le\log(1+\frac{R-r_n}{R-r})+[\frac{R-r_n}{R-r}+(\frac{R-r_n}{R-r})^2+\cdots+(\frac{R-r_n}{R-r})^p]\\&\le 2[\frac{R-r_n}{R-r}+(\frac{R-r_n}{R-r})^2+\cdots+(\frac{R-r_n}{R-r})^p]\\&=2(\frac{R-r_n}{R-r})^p[(\frac{R-r}{R-r_n})^{p-1}+(\frac{R-r}{R-r_n})^{p-2}+\cdots+1].\end{aligned}$$

Hence,

$$\log|E(\frac{R\frac{z_n}{|z_n|}-z_n}{R\frac{z_n}{|z_n|}-z},p)|<2(2^{p-1}+2^{p-2}+\cdots+1)(\frac{R-r_n}{R-r})^p=A_2(\frac{R-r_n}{R-r})^p.$$

Noting that $\lambda+\varepsilon-p>0$, so

$$\begin{aligned}\sum_1&<A_2(\frac{1}{R-r})^p\sum_{\frac{R-r_n}{R-r}\ge\frac12}(R-r_n)^p=A_2(\frac{1}{R-r})^p\sum_{\frac{R-r_n}{R-r}\ge\frac12}(R-r_n)^{p-\lambda-\varepsilon}(R-r_n)^{\lambda+\varepsilon}\\&\le A_2(\frac{1}{R-r})^p(\frac{2}{R-r})^{\lambda+\varepsilon-p}\sum_{n=1}^{\infty}(R-r_n)^{\lambda+\varepsilon}=A_3(\frac{1}{R-r})^{\lambda+\varepsilon}.\end{aligned}$$

From above discussion, we obtain

$$\log|P(z)|\le\max\{A_1,A_3\}(\frac{1}{R-r})^{\lambda+\varepsilon}.$$

Since ε is arbitrary, we get $\sigma_M(P)\le\lambda$.

Therefore, $\lambda(P) = \sigma(P) = \sigma_M(P)$ holds. $\square$

Proof of Theorem 1.16 Set

$$F(z) = \frac{f(z)}{z^m P(z)}, \tag{3.2}$$

where $P(z)$ is the canonical product of non-zero zeros of $f(z)$ in D_R, and m is the order of zeros of $f(z)$ at $z = 0$. Then $F(z)$ is an analytic function in D_R with no zero point in D_R. Hence by Lemma 2.8, we have $F(z) = e^{g(z)}$, where $g(z)$ is an analytic function in D_R. By Corollary 1.12 and Theorem 1.14, we have

$$\sigma(P) = \sigma_M(P) = \lambda(P) = \lambda(f) \leq \sigma(f). \tag{3.3}$$

By (3.3) and Theorem 2.2, we get

$$\sigma_M(F) - 1 \leq \sigma(F) \leq \max\{\sigma(f), \sigma(\frac{1}{P})\} = \max\{\sigma(f), \sigma(P)\} = \sigma(f). \tag{3.4}$$

Hence by (3.4) and Theorem 2.5, we obtain

$$D_M(g) - 1 \leq \sigma_M(F) \leq \sigma(f) + 1.$$

So $D_M(g) \leq \sigma(f) + 2$. Since $\sigma(f) < \infty$, the degree of $g(z)$ is finite. $\square$

Proof of Theorem 1.17 Set

$$F(z) = \frac{f(z)P_2(z)}{z^m P_1(z)}, \tag{3.5}$$

where $P_1(z), P_2(z)$ are respectively the canonical products of non-zero zeros of $f(z)$ and $\frac{1}{f(z)}$ in D_R, and m is the order of zeros or poles of $f(z)$ at $z = 0$. Then $F(z)$ is an analytic function in D_R with no zero point in D_R. Hence by Lemma 2.8, we have $F(z) = e^{g(z)}$, where $g(z)$ is an analytic function in D_R. By Corollary 1.12 and Theorem 1.14, we have

$$\sigma(P_2) = \sigma_M(P_2) = \lambda(P_2) = \lambda(\frac{1}{f}) \leq \sigma(\frac{1}{f}) = \sigma(f) \tag{3.6}$$

and

$$\sigma(\frac{1}{P_1}) = \sigma(P_1) = \sigma_M(P_1) = \lambda(P_1) = \lambda(f) \leq \sigma(f). \tag{3.7}$$

By (3.6), (3.7) and Theorem 2.2, we get

$$\sigma_M(F) - 1 \leq \sigma(F) \leq \max\{\sigma(f), \sigma(P_2), \sigma(\frac{1}{P_1})\} = \sigma(f). \tag{3.8}$$

Hence by (3.8) and Theorem 2.5, we obtain

$$D_M(g) - 1 \leq \sigma_M(F) \leq \sigma(f) + 1.$$

So $D_M(g) \leq \sigma(f) + 2$. Since $\sigma(f) < \infty$, the degree of $g(z)$ is finite. $\square$

4. Concluding remark

We ask the following question: Is it admissible of a meromorphic function f in D_R which has infinitely many zero points in D_R? If it is true, then [12, Theorem 1.5] can be extended to the following conclusion:

For any given sequences $\{a_n\}$ in D_R, there exists a sequence $\{c_n\}$ in D_R which does not intersect with $\{a_n\}$ such that $(\{a_n\}, \{c_n\})$ is not 0-d set of any analytic functions in D_R.

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