Journal of Mathematical Research with Applications
Mar., 2018, Vol. 38, No. 2, pp. 147–161
DOI:10.3770/j.issn:2095-2651.2018.02.005
Http://jmre.dlut.edu.cn

Some Properties of Certain Subclasses of p-Valent Meromorphic Functions Associated with Quasi-Subordination

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Abstract In this paper, we obtain the integral representations and the coefficient estimates for certain new subclasses of p-valent meromorphic functions associated with quasi-subordination. Specially, we obtain the sharp estimates of Fekete-Szegö inequality.

Keywords analytic functions; meromorphic functions; quasi-subordination; integral representation; coefficient estimate; Fekete-Szegö inequality

MR(2010) Subject Classification 30C45

1. Introduction and motivation

Let $\mathcal{A}$ denote the class of functions, which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and normalized by $f(0)=f'(0)-1=0$.

Let Σ_p denote the class of meromorphic functions of the form

$$f(z)=z^{-p}+\sum_{k=1}^{\infty}a_kz^{k-p},\quad p\in\mathbb{N}=\{1,2,\ldots\},\tag{1.1}$$

which are analytic and p-valent in the punctured open unit disk $\mathbb{U}^*=\{z\in\mathbb{C}:0<|z|<1\}=\mathbb{U}\backslash\{0\}$. In particular, we set $\Sigma_1=\Sigma$.

For two analytic functions $f(z)$ and $g(z)$, the function $f(z)$ is subordinate to $g(z)$, written as follows

$$f(z)\prec g(z),\quad z\in\mathbb{U},$$

if there exists an analytic function $\omega(z)$, with $\omega(0)=0$ and $|\omega(z)|<1$ such that $f(z)=g(\omega(z))$ (see [1]). In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, then $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(\mathbb{U})\subset g(\mathbb{U})$.

Received April 7, 2017; Accepted January 13, 2018
Supported by the National Natural Science Foundation of China (Grant No. 11561001), the Natural Science Foundation of Inner Mongolia of China (Grant No. 2014MS0101), the Program for Young Talents of Science and Technology in Universities of Inner Mongolia Autonomous Region (Grant No. NJYT-18-A14) and the Higher School Foundation of Inner Mongolia of China (Grant Nos. 2015NJZY240; 2016NJZY251).
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Let $\mathcal{N}$ be the class of all functions $\phi(z)$ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0)=1$ and $\mathrm{Re}\{\phi(z)\}>0, z\in\mathbb{U}$. Also, let Ω be the class of analytic functions $\omega(z)$, normalized by $\omega(0)=0$, and satisfying the condition $|\omega(z)|<1$.

Cho and Noor [2] introduced the classes $MS(\eta;\phi), MK(\eta;\phi)$ and $MC(\eta,\beta;\phi,\psi)$ of the class Σ for $0\le\eta,\beta<1$ and $\phi,\psi\in\mathcal{N}$ as below

$$MS(\eta;\phi)=\{f(z)\in\Sigma:\frac{1}{1-\eta}(-\frac{zf'(z)}{f(z)}-\eta)\prec\phi(z)\},$$

$$MK(\eta;\phi)=\{f(z)\in\Sigma:\frac{1}{1-\eta}(-\{1+\frac{zf''(z)}{f'(z)}\}-\eta)\prec\phi(z)\},$$

and

$$MC(\eta,\beta;\phi,\psi)=\{f(z)\in\Sigma:\exists g(z)\in MS(\eta;\phi),\ \text{s.t.}\frac{1}{1-\beta}(-\frac{zf'(z)}{g(z)}-\beta)\prec\psi(z)\}.$$

The class $MS(\eta;\phi), MK(\eta;\phi)$ and $MC(\eta,\beta;\phi,\psi)$ include several well-known subclasses of meromorphic starlike, meromorphic convex functions and meromorphic close-to-convex functions as special case.

In 1970, Robertson [3] introduced the concept of quasi-subordination. For two analytic functions $f(z)$ and $g(z)$, the function $f(z)$ is quasi-subordinate to $g(z)$, written as follows

$$f(z)\prec_q g(z),\quad z\in\mathbb{U},$$

if there exist analytic functions $\varphi(z)$ and $\omega(z)$, with $|\varphi(z)|\le 1, \omega(0)=0$ and $|\omega(z)|<1$ such that $f(z)=\varphi(z)g(\omega(z))$. Observe that when $\varphi(z)=1$, then $f(z)=g(\omega(z))$, so that $f(z)\prec g(z)$ in $\mathbb{U}$. Also notice that if $\omega(z)=z$, then $f(z)=\varphi(z)g(z)$ and it is said that $f(z)$ is majorized by $g(z)$ and written $f(z)\ll g(z)$ in $\mathbb{U}$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. See [4–6] for works related to quasi-subordination.

According to the principle of quasi-subordination between analytic functions, we define the following classes.

Definition 1.1 *A function $f(z)\in\Sigma_p$ of the form (1.1) is said to be in the class $MS_{\mu,p,q}(\eta;\phi)$ if and only if*

$$\frac{1}{p-\eta}(-\frac{zf'(z)+\mu z^2f''(z)}{(1-\mu)f(z)+\mu zf'(z)}-\eta)-1\prec_q\phi(z)-1,$$

where $0\le\mu\le1, 0\le\eta<p, \phi(z)\in\mathcal{N}, z\in\mathbb{U}^*$.

We note that

$$MS_{p,q}(\eta;\phi)=MS_{0,p,q}(\eta;\phi)=\{f(z)\in\Sigma_p:\frac{1}{p-\eta}(-\frac{zf'(z)}{f(z)}-\eta)-1\prec_q\phi(z)-1\}$$

and

$$MK_{p,q}(\eta;\phi)=MS_{1,p,q}(\eta;\phi)=\{f(z)\in\Sigma_p:\frac{1}{p-\eta}(-\{1+\frac{zf''(z)}{f'(z)}\}-\eta)-1\prec_q\phi(z)-1\}$$

where $0\le\eta<p, \phi(z)\in\mathcal{N}, z\in\mathbb{U}^*$.

By the well known Alexander equivalence the following relation holds

$$f(z)\in MK_{p,q}(\eta;\phi)\Leftrightarrow zf'(z)\in MS_{p,q}(\eta;\phi). \tag{1.2}$$

Definition 1.2 *A function $f(z) \in \Sigma_p$ of the form (1.1) is said to be in the class $MC_{p,q}(\eta, \beta; \phi, \psi)$ if and only if*

$$\frac{1}{p-\beta}(-\frac{zf'(z)}{g(z)} - \beta) - 1 \prec_q \psi(z) - 1,$$

where $g(z) \in MS_{p,q}(\eta; \phi), 0 \le \eta, \beta < p, \phi(z), \psi(z) \in \mathcal{N}, z \in \mathbb{U}^*$.

Definition 1.3 *A function $f(z) \in \Sigma_p$ of the form (1.1) is said to be in the class $MCK_{p,q}(\eta, \beta; \phi, \psi)$ if and only if*

$$\frac{1}{p-\beta}(-\frac{(zf'(z))'}{g'(z)} - \beta) - 1 \prec_q \psi(z) - 1,$$

where $g(z) \in Mk_{p,q}(\eta; \phi), 0 \le \eta, \beta < p, \phi(z), \psi(z) \in \mathcal{N}, z \in \mathbb{U}^*$.

The bounds for coefficient give information about various geometric properties of the function. A sharp bound of the functional $|a_3 - \lambda a_2^2|$ for univalent functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with real λ $(0 \le \lambda \le 1)$ was obtained by Fekete and Szegö [7]. The functional has since received great attention. Many authors have investigated the bounds for the Fekete-Szegö problem for various classes [8–16]. In particular, some authors start to study the Fekete-Szegö problem for various classes using quasi-subordination [17–22].

In this paper, we determine the integral representations and the coefficient estimates including a Fekete-Szegö inequality of the above defined classes. Our results are new in this direction and they give birth to many corollaries.

We need the following lemmas to prove our main results.

Lemma 1.4 ([23]) *If φ is the function analytic in the open unit disk $\mathbb{U}$, satisfying $|\varphi(z)| \le 1$, and let $\varphi(z) = c_0 + c_1 z + c_2 z^2 + \cdots$. Then $|c_0| \le 1$, and $|c_1| \le 1 - |c_0|^2$.*

Lemma 1.5 ([24]) *If $\omega \in \Omega$, and let $\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots$, then $|\omega_1| \le 1$, and for any natural number $n \ge 2$,$|\omega_n| \le 1 - |\omega_1|^2$.*

Lemma 1.6 ([23]) *If $\omega \in \Omega$, then for any complex number t*

$$|\omega_2 - t\omega_1^2| \le \max\{1, |t|\}.$$

The result is sharp for the functions $\omega(z) = z^2$ or $\omega(z) = z$.

Lemma 1.7 ([25]) *If $\omega \in \Omega$, then*

$$|\omega_2 - t\omega_1^2| \le \begin{cases} -t, & \text{if } t \le -1, \\ 1, & \text{if } -1 \le t \le 1, \\ t, & \text{if } t \ge -1. \end{cases}$$

When $t < -1$ or $t > 1$, equality holds if and only if $\omega(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $\omega(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $\omega(z) = z\frac{\lambda+z}{1+\lambda z}$ $(0 \le \lambda \le 1)$ or one of its rotations while for $t = 1$, equality holds if and only if $\omega(z) = -z\frac{\lambda+z}{1+\lambda z}$ $(0 \le \lambda \le 1)$ or one of its rotations.

Also the sharp upper bound above can be improved as follows when $-1 < t < 1$:

$$|\omega_2 - t\omega_1^2| + (t+1)|\omega_1|^2 \le 1, \quad -1 < t \le 0$$

and

$$|\omega_2 - t\omega_1^2| + (1-t)|\omega_1|^2 \le 1, \quad 0 < t < 1.$$

2. Integral representation

First, we give the integral representation of function in the classes defined in the paper.

Theorem 2.1 *Let* $f(z) \in MS_{p,q}(\eta;\phi)$. *Then*

$$f(z) = z^{-p} \exp\Big\{(\eta - p)\int_0^z \frac{\varphi(\xi)[\phi(\omega(\xi)) - 1]}{\xi}\mathrm{d}\xi\Big\} \tag{2.1}$$

where $|\varphi(z)| \le 1, \omega(z) \in \Omega, \phi(z) \in \mathcal{N}$.

Proof Suppose that $f(z) \in MS_{p,q}(\eta;\phi)$. According to Definition 1.1 and the relationship of quasi-subordination, we have

$$\frac{1}{p-\eta}(-\frac{zf'(z)}{f(z)} - \eta) - 1 = \varphi(z)[\phi(\omega(z)) - 1]. \tag{2.2}$$

From (2.2), we get

$$\frac{f'(z)}{f(z)} = -\frac{p}{z} - \frac{(p-\eta)\varphi(z)[\phi(\omega(z)) - 1]}{z}.$$

Integrating the both sides of the above equality, we obtain

$$\log f(z) = \log z^{-p} - (p-\eta)\int_0^z \frac{\varphi(\xi)[\phi(\omega(\xi)) - 1]}{\xi}.$$

Thus, we complete the proof of Theorem 2.1. □

Theorem 2.2 *Let* $f(z) \in MK_{p,q}(\eta;\phi)$. *Then*

$$f(z) = \int_0^z \frac{1}{t^{p+1}} \exp\Big\{(\eta - p)\int_0^t \frac{\varphi(\xi)[\phi(\omega(\xi)) - 1]}{\xi}\mathrm{d}\xi\Big\}\mathrm{d}t, \tag{2.3}$$

where $|\varphi(z)| \le 1, \omega(z) \in \Omega, \phi(z) \in \mathcal{N}$.

Proof From (1.2), $zf'(z) \in MS_{p,q}(\eta;\phi)$, then by Theorem 2.1 we get

$$zf'(z) = z^{-p} \exp\Big\{(\eta - p)\int_0^z \frac{\varphi(\xi)[\phi(\omega(\xi)) - 1]}{\xi}\mathrm{d}\xi\Big\}$$

or equivalently

$$f'(z) = \frac{1}{z^{p+1}} \exp\Big\{(\eta - p)\int_0^z \frac{\varphi(\xi)[\phi(\omega(\xi)) - 1]}{\xi}\mathrm{d}\xi\Big\}. \tag{2.4}$$

Integrating the both sides of (2.4), we obtain (2.3). Thus, we complete the proof of Theorem 2.2. □

Theorem 2.3 *Let* $f(z) \in MC_{p,q}(\eta,\beta;\phi,\psi)$. *Then*

$$f(z) = \int_0^z \frac{-p + (\beta - p)\varphi_1(t)[\phi(\omega_1(t)) - 1]}{t^{p+1}} \exp\Big\{(\eta - p)\int_0^t \frac{\varphi(\xi)[\phi(\omega(\xi)) - 1]}{\xi}\mathrm{d}\xi\Big\}\mathrm{d}t, \tag{2.5}$$

where $|\varphi(z)|\leq 1, |\varphi_1(z)|\leq 1, \omega(z), \omega_1(z)\in\Omega, \psi(z)\in\mathcal{N}$.

Proof Suppose that $f(z)\in MC_{p,q}(\eta,\beta;\phi,\psi)$. According to Definition 1.2 and the relationship of quasi-subordination, we have

$$\frac{1}{p-\beta}(-\frac{zf'(z)}{g(z)}-\beta)-1=\varphi_1(z)[\psi(\omega_1(z))-1]. \tag{2.6}$$

From (2.6), we get

$$f'(z)=\frac{-p+(\beta-p)\varphi_1(z)[\phi(\omega_1(z))-1]}{z}g(z).$$

Because $g(z)\in MS_{p,q}(\eta;\phi)$, then by (2.1) in Theorem 2.1 we obtain

$$g(z)=z^{-p}\exp\Big\{(\eta-p)\int_0^z\frac{\varphi(\xi)[\phi(\omega(\xi))-1]}{\xi}\mathrm{d}\xi\Big\}.$$

Thus we have

$$f'(z)=\frac{-p+(\beta-p)\varphi_1(z)[\phi(\omega_1(z))-1]}{z^{p+1}}\exp\Big\{(\eta-p)\int_0^z\frac{\varphi(\xi)[\phi(\omega(\xi))-1]}{\xi}\mathrm{d}\xi\Big\}. \tag{2.7}$$

Integrating the both sides of (2.7), we get (2.5). Thus, we complete the proof of Theorem 2.3. □

Theorem 2.4 *Let* $f(z)\in MCK_{p,q}(\eta,\beta;\phi,\psi)$. *Then*

$$f(z)=\int_0^z\frac{1}{s}\int_0^s\frac{-p+(\beta-p)\varphi_2(t)[\psi(\omega_2(t))-1]}{t^{p+1}}\exp\Big\{(\eta-p)\int_0^t\frac{\varphi(\xi)[\phi(\omega(\xi))-1]}{\xi}\mathrm{d}\xi\Big\}\mathrm{d}t\mathrm{d}s, \tag{2.8}$$

where $|\varphi(z)|\leq 1, |\varphi_2(z)|\leq 1, \omega(z), \omega_2(z)\in\Omega, \psi(z)\in\mathcal{N}$.

Proof Suppose that $f(z)\in MCK_{p,q}(\eta,\beta;\phi,\psi)$. According to Definition 1.3 and the relationship of quasi-subordination, we have

$$\frac{1}{p-\beta}(-\frac{(zf'(z))'}{g'(z)}-\beta)-1=\varphi_2(z)[\psi(\omega_2(z))-1]. \tag{2.9}$$

From (2.9), we get

$$(zf'(z))'=[-p+(\beta-p)\varphi_2(z)[\psi(\omega_2(z))-1]]g'(z).$$

Because $g(z)\in MK_{p,q}(\eta;\phi)$, then by (2.5) in Theorem 2.2 we obtain

$$g'(z)=\frac{1}{z^{p+1}}\exp\Big\{(\eta-p)\int_0^z\frac{\varphi(\xi)[\phi(\omega(\xi))-1]}{\xi}d\xi\Big\}.$$

Thus we have

$$(zf'(z))'=\frac{-p+(\beta-p)\varphi_2(z)[\psi(\omega_2(z))-1]}{z^{p+1}}\exp\Big\{(\eta-p)\int_0^z\frac{\varphi(\xi)[\phi(\omega(\xi))-1]}{\xi}\mathrm{d}\xi\Big\}.$$

Integrating the both sides of the above equality, we obtain

$$f'(z)=\frac{1}{z}\int_0^z\frac{-p+(\beta-p)\varphi_2(t)[\psi(\omega_2(t))-1]}{t^{p+1}}\exp\Big\{(\eta-p)\int_0^t\frac{\varphi(\xi)[\phi(\omega(\xi))-1]}{\xi}\mathrm{d}\xi\Big\}\mathrm{d}t. \tag{2.10}$$

Integrating the both sides of (2.10), we obtain (2.8). Thus, we complete the proof of Theorem 2.4. □

3. Coefficient estimate

Throughout, let $f(z) = z^{-p} + a_1 z^{1-p} + a_2 z^{2-p} + \cdots, g(z) = z^{-p} + b_1 z^{1-p} + b_2 z^{2-p} + \cdots$, $\varphi(z) = c_0 + c_1 z + c_2 z^2 + \cdots$, $\varphi_1(z) = d_0 + d_1 z + d_2 z^2 + \cdots$, $\varphi_2(z) = e_0 + e_1 z + e_2 z^2 + \cdots$, $\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots$, $\omega_1(z) = t_1 z + t_2 z^2 + \cdots$, $\omega_2(z) = s_1 z + s_2 z^2 + \cdots$, $\psi(z) = 1 + A_1 z + A_2 z^2 + \cdots$, $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$, $A_1, B_1 \in R$ and $A_1, B_1 > 0$.

Theorem 3.1 *If $f(z)$ given by (1.1) belongs to $MS_{p,q}(\eta, \phi)$, then*

$$|a_1| \le (p - \eta)B_1, \tag{3.1}$$

$$|a_2| \le \frac{p-\eta}{2}[B_1 + \max\{B_1, (p-\eta)B_1^2 + |B_2|\}], \tag{3.2}$$

and, for any complex number λ,

$$|a_2 - \lambda a_1^2| \le \frac{p-\eta}{2}[B_1 + \max\{B_1, (p-\eta)|2\lambda - 1|B_1^2 + |B_2|\}]. \tag{3.3}$$

The result is sharp.

Proof If $f(z) \in MS_{p,q}(\eta, \phi)$, then there exist analytic functions $\varphi(z)$ and $\omega(z)$, with $|\varphi(z)| \le 1, \omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$\frac{1}{p-\eta}(-\frac{zf'(z)}{f(z)} - \eta) - 1 = \varphi(z)(\phi(\omega(z)) - 1). \tag{3.4}$$

Since

$$\frac{1}{p-\eta}(-\frac{zf'(z)}{f(z)} - \eta) - 1 = \frac{1}{\eta - p}a_1 z + (\frac{2}{\eta - p}a_2 - \frac{1}{\eta - p}a_1^2)z^2 + \cdots$$

and

$$\varphi(z)(\phi(\omega(z)) - 1) = B_1 c_0 \omega_1 z + [B_1 c_1 \omega_1 + c_0(B_1\omega_2 + B_2\omega_1^2)]z^2 + \cdots, \tag{3.5}$$

then, comparing both sides of (3.4) we see that

$$a_1 = (\eta - p)B_1 c_0 \omega_1,$$

from which, by the inequality $|c_0| \le 1, |\omega_1| \le 1$, we immediately obtain (3.1). Moreover we have

$$a_2 = \frac{\eta - p}{2}[B_1 c_1 \omega_1 + B_1 c_0 \omega_2 + c_0((\eta - p)B_1^2 c_0 + B_2)\omega_1^2].$$

Further,

$$a_2 - \lambda a_1^2 = \frac{(\eta - p)B_1}{2}[c_1 w_1 + c_0(\omega_2 - ((\eta - p)(2\lambda - 1)B_1 c_0 - \frac{B_2}{B_1})\omega_1^2)],$$

then applying Lemmas 1.4 and 1.5, we get

$$|a_2 - \lambda a_1^2| \le \frac{(p-\eta)B_1}{2}[1 + |\omega_2 - [(\eta - p)(2\lambda - 1)B_1 c_0 - \frac{B_2}{B_1}]\omega_1^2|]. \tag{3.6}$$

Using Lemma 1.6 to (3.6), we obtain

$$|a_2 - \lambda a_1^2| \le \frac{(p-\eta)B_1}{2}[1 + \max\{1, |(\eta - p)(2\lambda - 1)B_1 c_0 - \frac{B_2}{B_1}|\}]. \tag{3.7}$$

Observe that

$$|(\eta - p)(2\lambda - 1)B_1 c_0 - \frac{B_2}{B_1}| \le (p - \eta)|2\lambda - 1|B_1 + \frac{|B_2|}{B_1},$$

and hence we can conclude (3.3). For $\lambda = 0$ in (3.3), we have (3.2). The result is sharp for the functions

$$f(z) = z^{-p} \exp\left\{(\eta - p)\int_0^z \frac{\varphi(\xi)[\phi(\xi) - 1]}{\xi} \mathrm{d}\xi\right\}$$

and

$$f(z) = z^{-p} \exp\left\{(\eta - p)\int_0^z \frac{\varphi(\xi)[\phi(\xi^2) - 1]}{\xi} \mathrm{d}\xi\right\}.$$

Thus we complete the proof of Theorem 3.2. □

Putting $\varphi(z) = 1$ and $\omega(z) = 1$ in Theorem 3.1, we have the following two results.

Corollary 3.2 *If $f(z) \in \sum_p$ satisfies*

$$\frac{1}{p-\eta}(-\frac{zf'(z)}{f(z)} - \eta) \prec \phi(z),$$

then

$$|a_1| \leq (p-\eta)B_1, \quad |a_2| \leq \frac{p-\eta}{2}\max\{B_1, |(p-\eta)B_1^2 - B_2|\},$$

and, for any complex number λ,

$$|a_2 - \lambda a_1^2| \leq \frac{p-\eta}{2}\max\{B_1, |(p-\eta)(2\lambda - 1)B_1^2 - B_2|\}.$$

Corollary 3.3 *If $f(z) \in \Sigma_p$ satisfies*

$$\frac{1}{p-\eta}(-\frac{zf'(z)}{f(z)} - \eta) - 1 \ll \phi(z) - 1,$$

then

$$|a_1| \leq (p-\eta)B_1, \quad |a_2| \leq \frac{p-\eta}{2}[B_1 + (p-\eta)B_1^2 + |B_2|],$$

and, for any complex number λ,

$$|a_2 - \lambda a_1^2| \leq \frac{p-\eta}{2}[B_1 + (p-\eta)|2\lambda - 1|B_1^2 + |B_2|].$$

Theorem 3.4 *If $f(z)$ given by (1.1) belongs to $MS_{p,q}(\eta;\phi)$, then for any real number λ and $c_0 < 0$,*

$$|a_2 - \lambda a_1^2| \leq \begin{cases} \frac{(p-\eta)B_1}{2}[1 + (p-\eta)(2\lambda-1)B_1c_0 + \frac{B_2}{B_1}], & \lambda \leq \sigma_1, \\ (p-\eta)B_1, & \sigma_1 \leq \lambda \leq \sigma_2, \\ \frac{(p-\eta)B_1}{2}[1 - (p-\eta)(2\lambda-1)B_1c_0 - \frac{B_2}{B_1}], & \lambda \geq \sigma_2. \end{cases} \tag{3.8}$$

Further, if $\sigma_1 \leq \lambda \leq \sigma_3$, then

$$|a_2 - \lambda a_1^2| + k_1|a_1|^2 \leq (p-\eta)B_1. \tag{3.9}$$

If $\sigma_3 \leq \lambda \leq \sigma_2$, then

$$|a_2 - \lambda a_1^2| + k_2|a_1|^2 \leq (p-\eta)B_1. \tag{3.10}$$

For any real number λ and $c_0 > 0$,

$$|a_2 - \lambda a_1^2| \leq \begin{cases} \frac{(p-\eta)B_1}{2}[1 - (p-\eta)(2\lambda-1)B_1c_0 - \frac{B_2}{B_1}], & \lambda \leq \sigma_2, \\ (p-\eta)B_1, & \sigma_2 \leq \lambda \leq \sigma_1, \\ \frac{(p-\eta)B_1}{2}[1 + (p-\eta)(2\lambda-1)B_1c_0 + \frac{B_2}{B_1}], & \lambda \geq \sigma_1. \end{cases} \tag{3.11}$$

Further, if $\sigma_2 \le \lambda \le \sigma_3$, then

$$|a_2 - \lambda a_1^2| + k_2|a_1|^2 \le (p-\eta)B_1. \tag{3.12}$$

If $\sigma_3 \le \lambda \le \sigma_1$, then

$$|a_2 - \lambda a_1^2| + k_1|a_1|^2 \le (p-\eta)B_1, \tag{3.13}$$

where

$$B_2 \in R, \sigma_1 = \frac{B_2 - B_1 + (\eta - p)B_1^2 c_0}{2(\eta - p)B_1^2 c_0}, \sigma_2 = \frac{B_2 + B_1 + (\eta - p)B_1^2 c_0}{2(\eta - p)B_1^2 c_0}, \sigma_3 = \frac{B_2 + (\eta - p)B_1^2 c_0}{2(\eta - p)B_1^2 c_0},$$

$$k_1 = \frac{B_1 - B_2 - (p-\eta)(2\lambda - 1)B_1^2 c_0}{2(p-\eta)B_1^2 c_0^2}, k_2 = \frac{B_1 + B_2 + (p-\eta)(2\lambda - 1)B_1^2 c_0}{2(p-\eta)B_1^2 c_0^2}.$$

The result is sharp.

Proof Suppose that $c_0 < 0$. From (3.7), we have

$$|a_2 - \lambda a_1^2| \le \frac{(p-\eta)B_1}{2}[1 + \max\{1, |t|\}],$$

where

$$t = (\eta - p)(2\lambda - 1)B_1 c_0 - \frac{B_2}{B_1}.$$

If $\lambda \le \sigma_1$, then $t \le -1$. Thus, by applying Lemma 1.7, we get the first inequality in (3.8).

Next $\lambda \ge \sigma_2$, then $t \ge 1$. Applying Lemma 1.7, we have the last inequality in (3.8).

Once more $\sigma_1 \le \lambda \le \sigma_2$, then $|t| \le 1$. Thus applying Lemma 1.7, we obtain the middle inequality in (3.8).

By an application of Lemma 1.7 and Theorem 2.1, bounds are sharp as follows. If $\lambda < \sigma_1$ or $\lambda > \sigma_2$, then the equality holds if and only if

$$f(z) = z^{-p}\exp\{(\eta - p)\int_0^z \frac{\varphi(\xi)[\phi(\xi) - 1]}{\xi} \mathrm{d}\xi\}$$

or one of its rotations. When $\sigma_1 < \lambda < \sigma_2$, the equality holds if and only if

$$f(z) = z^{-p}\exp\{(\eta - p)\int_0^z \frac{\varphi(\xi)[\phi(\xi^2) - 1]}{\xi} \mathrm{d}\xi\}$$

or one of its rotations. If $\lambda = \sigma_1$, then the equality holds if and only if

$$f(z) = z^{-p}\exp\{(\eta - p)\int_0^z \frac{\varphi(\xi)[\phi(\xi\frac{\gamma+\xi}{1+\gamma\xi}) - 1]}{\xi} \mathrm{d}\xi\}$$

or one of its rotations. If $\lambda = \sigma_2$, then the equality holds if and only if

$$f(z) = z^{-p}\exp\{(\eta - p)\int_0^z \frac{\varphi(\xi)[\phi(-\xi\frac{\gamma+\xi}{1+\gamma\xi}) - 1]}{\xi} \mathrm{d}\xi\}$$

or one of its rotations.

Last (3.9) and (3.10) are established by an application of Lemma 1.7. Also applying Lemma 1.6, we can prove (3.11)–(3.13) for $c_0 > 0$. Thus we complete the proof of Theorem 3.4. □

In Theorem 3.4, if $\phi(z) = \frac{1+Az}{1+Bz}$ and $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$, respectively, then the following corollaries are obtained.

Corollary 3.5 *Let $-1 \le B < A \le 1$. If $f(z)$ given by (1.1) belongs to $MS_{p,q}(\eta; \frac{1+Az}{1+Bz})$, then for any real number λ and $c_0 < 0$*

$$|a_2 - \lambda a_1^2| \le \begin{cases} \frac{(p-\eta)(A-B)}{2}[1 + (p-\eta)(2\lambda-1)(A-B)c_0 - B], & \lambda \le \sigma_1, \\ (p-\eta)(A-B), & \sigma_1 \le \lambda \le \sigma_2, \\ \frac{(p-\eta)(A-B)}{2}[1 - (p-\eta)(2\lambda-1)(A-B)c_0 + B], & \lambda \ge \sigma_2. \end{cases}$$

Further, if $\sigma_1 \le \lambda \le \sigma_3$, then

$$|a_2 - \lambda a_1^2| + k_1|a_1|^2 \le (p-\eta)(A-B).$$

If $\sigma_3 \le \lambda \le \sigma_2$, then

$$|a_2 - \lambda a_1^2| + k_2|a_1|^2 \le (p-\eta)(A-B).$$

For any real number λ and $c_0 > 0$,

$$|a_2 - \lambda a_1^2| \le \begin{cases} \frac{(p-\eta)(A-B)}{2}[1 - (p-\eta)(2\lambda-1)(A-B)c_0 + B], & \mu \le \sigma_2, \\ (p-\eta)(A-B), & \sigma_2 \le \mu \le \sigma_1, \\ \frac{(p-\eta)(A-B)}{2}[1 + (p-\eta)(2\lambda-1)(A-B)c_0 - B], & \mu \ge \sigma_1. \end{cases}$$

Further, if $\sigma_2 \le \lambda \le \sigma_3$, then

$$|a_2 - \lambda a_1^2| + k_2|a_1|^2 \le (p-\eta)(A-B).$$

If $\sigma_3 \le \lambda \le \sigma_1$, then

$$|a_2 - \lambda a_1^2| + k_1|a_1|^2 \le (p-\eta)(A-B),$$

where

$$B_2 \in R, \sigma_1 = \frac{1}{2} + \frac{B+1}{2(p-\eta)(A-B)c_0}, \sigma_2 = \frac{1}{2} + \frac{B-1}{2(p-\eta)(A-B)c_0}, \sigma_3 = \frac{1}{2} + \frac{B}{2(p-\eta)(A-B)c_0},$$

$$k_1 = \frac{1+B}{2(p-\eta)(A-B)c_0^2} - \frac{2\lambda-1}{2c_0}, k_2 = \frac{1-B}{2(p-\eta)(A-B)c_0^2} + \frac{2\lambda-1}{2c_0}.$$

Corollary 3.6 *Let $0 \le \alpha < 1$. If $f(z)$ given by (1.1) belongs to $MS_{p,q}(\eta; \frac{1+(1-2\alpha)z}{1-z})$, then for any real number λ and $c_0 < 0$*

$$|a_2 - \lambda a_1^2| \le \begin{cases} 2(p-\eta)(1-\alpha)[1 + (p-\eta)(2\lambda-1)(1-\alpha)c_0], & \lambda \le \sigma_1, \\ 2(p-\eta)(1-\alpha), & \sigma_1 \le \lambda \le \sigma_2, \\ -2(p-\eta)^2(1-\alpha)^2(2\lambda-1)c_0, & \lambda \ge \sigma_2. \end{cases}$$

Further, if $\sigma_1 \le \lambda \le \sigma_3$, then

$$|a_2 - \lambda a_1^2| + k_1|a_1|^2 \le (p-\eta)(A-B).$$

If $\sigma_3 \le \lambda \le \sigma_2$, then

$$|a_2 - \lambda a_1^2| + k_2|a_1|^2 \le (p-\eta)(A-B).$$

For any real number λ and $c_0 > 0$,

$$|a_2 - \lambda a_1^2| \le \begin{cases} 2(p-\eta)^2(1-\alpha)^2(2\lambda-1)c_0, & \lambda \le \sigma_2, \\ 2(p-\eta)(1-\alpha), & \sigma_2 \le \lambda \le \sigma_1, \\ 2(p-\eta)(1-\alpha)[1 + (p-\eta)(2\lambda-1)(1-\alpha)c_0, & \lambda \ge \sigma_1. \end{cases}$$

Further, if $\sigma_2 \le \lambda \le \sigma_3$, *then*

$$|a_2 - \lambda a_1^2| + k_2|a_1|^2 \le 2(p-\eta)(1-\alpha).$$

If $\sigma_3 \le \lambda \le \sigma_1$, *then*

$$|a_2 - \lambda a_1^2| + k_1|a_1|^2 \le 2(p-\eta)(1-\alpha),$$

where

$$B_2 \in R, \sigma_1 = \frac{1}{2}, \sigma_2 = \frac{1}{2} - \frac{1}{2(p-\eta)(1-\alpha)c_0}, \sigma_3 = \frac{1}{2} - \frac{1}{4(p-\eta)(1-\alpha)c_0},$$

$$k_1 = -\frac{2\lambda - 1}{2c_0}, k_2 = \frac{1}{2(p-\eta)(1-\alpha)c_0^2} + \frac{2\lambda - 1}{2c_0}.$$

For $p = 1, \alpha = 0, \eta = \frac{1}{3}, c_0 = -1$ in Corollary 3.7, we obtain the following result.

Corollary 3.7 *If* $f(z)$ *given by (1.1) belongs to* $MS_{1,q}(\frac{1}{3}; \frac{1+z}{1-z})$, *then for any real number* λ

$$|a_2 - \lambda a_1^2| \le \begin{cases} \frac{8}{9}(\frac{5}{2} - 2\lambda), & \lambda \le \frac{1}{2}, \\ \frac{4}{3}, & \frac{1}{2} \le \lambda \le \frac{5}{4}, \\ -\frac{8}{9}(1 - 2\lambda), & \lambda \ge \frac{5}{4}. \end{cases}$$

The result is sharp. If $\lambda < \frac{1}{2}$ *or* $\lambda > \frac{5}{4}$, *then the inequality holds if and only if* $f_1(z) = z^{-1}e^{\frac{4}{3}z}$ *or one of its rotation; if* $\frac{1}{2} < \lambda < \frac{5}{4}$, *then the inequality holds if and only if* $f_2(z) = z^{-1}(1+z)^{-\frac{4}{3}}e^{\frac{4}{3}z}$ *or one of its rotation.*

Theorem 3.8 *If* $f(z)$ *given by (1.1) belongs to* $MK_{p,q}(\eta; \phi)$ *for* $p \ge 3$, *then*

$$|a_1| \le \frac{p(p-\eta)}{p-1}B_1, \quad |a_2| \le \frac{p(p-\eta)}{2(p-2)}[B_1 + \max\{B_1, (p-\eta)B_1^2 + |B_2|\}],$$

and, for any complex number λ,

$$|a_2 - \lambda a_1^2| \le \frac{(p(p-\eta)}{2(p-2)}[B_1 + \max\{B_1, \frac{(p-\eta)|2p\lambda(2-p) + (1-p)^2|}{(1-p)^2}B_1^2 + |B_2|\}].$$

The result is sharp.

Proof From (1.2) $zf'(z) \in MS_{p,q}(\eta; \phi)$, then (3.4) becomes

$$\frac{1}{p-\eta}(-\frac{z(zf'(z))'}{zf'(z)} - \eta) - 1 = \varphi(z)(\phi(\omega(z)) - 1)$$

or equivalently

$$\frac{1}{p-\eta}(-1 - \frac{zf''(z)}{f'(z)} - \eta) - 1 = \varphi(z)(\phi(\omega(z)) - 1).$$

Using arguments similar to those in the proof of Theorem 3.1, we can obtain the required estimates. Thus we complete the proof of Theorem 3.8. □

Corollary 3.9 *If* $f(z) \in \sum_p$ *satisfies*

$$\frac{1}{p-\eta}(-1 - \frac{zf''(z)}{f'(z)} - \eta) \prec \phi(z), \quad p \ge 3$$

then

$$|a_1| \le \frac{p(p-\eta)}{p-1}B_1, \quad |a_2| \le \frac{p(p-\eta)B_1}{2(p-2)}\max\{1, |(p-\eta)B_1 - \frac{B_2}{B_1}|\},$$

and, for any complex number λ,

$$|a_2 - \lambda a_1^2| \le \frac{p(p-\eta)B_1}{2(p-2)}\max\{1, |\frac{(p-\eta)[2\lambda p(2-p)+(1-p)^2]}{(1-p)^2}B_1 - \frac{B_2}{B_1}|\}.$$

Corollary 3.10 *If* $f(z) \in \Sigma_p$ *satisfies*

$$\frac{1}{p-\eta}(-1 - \frac{zf''(z)}{f'(z)} - \eta) - 1 \ll \phi(z) - 1, \quad p \ge 3$$

then

$$|a_1| \le \frac{p(p-\eta)B_1}{p-1}, \quad |a_2| \le \frac{p(p-\eta)}{2(p-2)}[B_1 + (p-\eta)B_1^2 + |B_2|],$$

and, for any complex number λ,

$$|a_2 - \lambda a_1^2| \le \frac{p(p-\eta)}{2(p-2)}[B_1 + \frac{(p-\eta)|2\lambda p(2-p)+(1-p)^2|}{(1-p)^2}B_1^2 + |B_2|].$$

Theorem 3.11 *If* $f(z)$ *given by (1.1) belongs to* $MK_{p,q}(\eta;\phi)$ *for* $p \ge 3$, *then for any real number* λ *and* $c_0 \in R, c_0 < 0$,

$$|a_2 - \lambda a_1^2| \le \begin{cases} \frac{p(p-\eta)B_1}{2(p-2)}[1 - \frac{(p-\eta)[2p\lambda(2-p)+(1-p)^2]}{(1-p)^2}B_1c_0 + \frac{B_2}{B_1}], & \mu \le \sigma_1, \\ \frac{p(p-\eta)B_1}{2(p-2)}, & \sigma_1 \le \mu \le \sigma_2, \\ \frac{p(p-\eta)B_1}{2(p-2)}[1 + \frac{(p-\eta)[2p\lambda(2-p)+(1-p)^2]}{(1-p)^2}B_1c_0 - \frac{B_2}{B_1}], & \mu \ge \sigma_2. \end{cases}$$

Further, if $\sigma_1 \le \mu \le \sigma_3$, *then*

$$|a_2 - \lambda a_1^2| + k_1|a_1|^2 \le \frac{p(p-\eta)B_1}{p-2}.$$

If $\sigma_3 \le \mu \le \sigma_2$, *then*

$$|a_2 - \lambda a_1^2| + k_2|a_1|^2 \le \frac{p(p-\eta)B_1}{p-2}.$$

For any real number μ *and* $c_0 \in R, c_0 > 0$,

$$|a_2 - \lambda a_1^2| \le \begin{cases} \frac{p(p-\eta)B_1}{2(p-2)}[1 + \frac{(p-\eta)[2p\lambda(2-p)+(1-p)^2]}{(1-p)^2}B_1c_0 - \frac{B_2}{B_1}], & \mu \le \sigma_2, \\ \frac{p(p-\eta)B_1}{2(p-2)}, & \sigma_2 \le \mu \le \sigma_1, \\ \frac{p(p-\eta)B_1}{2(p-2)}[1 - \frac{(p-\eta)[2p\lambda(2-p)+(1-p)^2]}{(1-p)^2}B_1c_0 + \frac{B_2}{B_1}], & \mu \ge \sigma_1. \end{cases}$$

Further, if $\sigma_2 \le \mu \le \sigma_3$, *then*

$$|a_2 - \lambda a_1^2| + k_2|a_1|^2 \le \frac{p(p-\eta)B_1}{p-2}.$$

If $\sigma_3 \le \mu \le \sigma_1$, *then*

$$|a_2 - \lambda a_1^2| + k_1|a_1|^2 \le \frac{p(p-\eta)B_1}{p-2},$$

where

$$B_2 \in R, \sigma_1 = \frac{(1-p)^2[B_2 - B_1 + (\eta-p)B_1^2c_0]}{2p(p-\eta)(2-p)B_1^2c_0}, \sigma_2 = \frac{(1-p)^2[B_2 + B_1 + (\eta-p)B_1^2c_0]}{2p(p-\eta)(2-p)B_1^2c_0},$$

$$\sigma_3=\frac{(1-p)^2[B_2+(\eta-p)B_1^2c_0]}{2p(p-\eta)(2-p)B_1^2c_0},$$

$$k_1=\frac{p[(1-p)^2(B_1-B_2)+(p-\eta)(2p\lambda(2-p)+(1-p)^2)B_1^2c_0]}{2(p-2)(p-\eta)B_1^2c_0^2},$$

$$k_2=\frac{p[(1-p)^2(B_1+B_2)-(p-\eta)(2p\lambda(2-p)+(1-p)^2)B_1^2c_0]}{2(p-2)(p-\eta)B_1^2c_0^2}.$$

Theorem 3.12 *If $f(z)$ given by (1.1) belongs to $MS_{\mu,p,q}(\eta,\phi)$, then*

$$|a_1|\le\frac{(p-\eta)|1-(1+p)\mu|B_1}{|1-\mu p|},$$

$$|a_2|\le\frac{(p-\eta)|1-(1+p)\mu|B_1}{2|1+(1-p)\mu|}(1+\max\{1,(p-\eta)B_1+\frac{|B_2|}{B_1}\}),$$

and, for any complex number μ,

$$|a_2-\lambda a_1^2|\le\frac{(p-\eta)|1-(1+p)\mu|B_1}{2|1+(1-p)\mu|}(1+\max\{1,\frac{(p-\eta)|(2\lambda-1)(1-\mu p)^2-2\lambda\mu^2|}{(1-\mu p)^2}B_1+\frac{|B_2|}{B_1}\}).$$

The result is sharp.

Proof If $f(z)\in MS_{\mu,p,q}(\eta,\phi)$, then there exist analytic functions $\varphi(z)$ and $\omega(z)$, with $|\varphi(z)|\le 1,\omega(0)=0$ and $|\omega(z)|<1$ such that

$$\frac{1}{p-\eta}(-\frac{zf'(z)+\mu z^2f''(z)}{(1-\mu)f(z)+\mu zf'(z)}-\eta)-1=\varphi(z)(\phi(\omega(z))-1). \tag{3.14}$$

Since

$$\frac{1}{p-\eta}(-\frac{zf'(z)+\mu z^2f''(z)}{(1-\mu)f(z)+\mu zf'(z)}-\eta)-1$$
$$=\frac{1-\mu p}{(\eta-p)[1-(1+p)\mu]}a_1z+[\frac{2[1+(1-p)\mu]}{(\eta-p)[1-(1+p)\mu]}a_2-\frac{(1-\mu p)^2}{(\eta-p)[1-(1+p)\mu]}a_1^2]z^2+\cdots$$

and

$$\varphi(z)(\phi(\omega(z))-1)=B_1c_0\omega_1z+[B_1c_1\omega_1+c_0(B_1\omega_2+B_2\omega_1^2)]z^2+\cdots,$$

then comparing both sides of (3.14), we get

$$a_1=\frac{(\eta-p)[1-(1+p)\mu]}{1-\mu p}B_1c_0\omega_1,$$

$$a_2=\frac{(\eta-p)[1-(1+p)\mu]}{2[1+(1-p)\mu]}[B_1c_1\omega_1+B_1c_0\omega_2+c_0((\eta-p)B_1^2c_0+B_2)\omega_1^2].$$

Further,

$$a_2-\lambda a_1^2$$
$$=\frac{(\eta-p)[1-(1+p)\mu]B_1}{2[1+(1-p)\mu]}[c_1w_1+c_0(\omega_2-(\frac{(\eta-p)[(2\lambda-1)(1-\mu p)^2-2\lambda\mu^2]}{(1-\mu p)^2}B_1c_0-\frac{B_2}{B_1})\omega_1^2)].$$

We can proceed similarly as previous theorems and prove the hypothesis. Thus we complete the proof of Theorem 3.12. □

Theorem 3.13 *If $f(z)$ given by (1.1) belongs to $MC_{p,q}(\eta,\beta;\phi,\psi)$ for $p\geq 3$, then*

$$|a_1|\leq\frac{(p-\beta)A_1+p(p-\eta)B_1}{p-1},$$

$$|a_2|\leq\frac{(p-\beta)A_1}{p-2}[1+\max\{1,\frac{|A_2|}{A_1}\}]+\frac{(p-\beta)(p-\eta)A_1B_1}{p-2}+\frac{p(p-\eta)B_1}{2(p-2)}[1+\max\{B_1,|(p-\eta)B_1c_0-\frac{B_2}{B_1}|\}],$$

and, for any complex number λ

$$|a_2-\lambda a_1^2|\leq\frac{(p-\beta)A_1}{p-2}[1+\max\{1,|\frac{\lambda(\beta-p)(2-p)}{(1-p)^2}A_1d_0-\frac{A_2}{A_1}|\}]+\frac{(p-\beta)(p-\eta)A_1B_1}{(p-2)(1-p)^2}|2p\lambda(2-p)+(1-p)^2|+\frac{p(p-\eta)B_1}{2(p-2)}[1+\max\{1,|\frac{(\eta-p)[2p\lambda(2-p)-(1-p)^2]}{(1-p)^2}B_1c_0-\frac{B_2}{B_1}|\}].$$

The result is sharp.

Proof If $f(z)\in MC_{p,q}(\eta,\beta;\phi,\psi)$, then there exist a function $g(z)\in MS_{p,q}(\eta;\phi)$ and analytic functions $\varphi_1(z)$ and $\omega_1(z)$, with $|\varphi_1(z)|\leq 1,\omega_1(0)=0$ and $|\omega_1(z)|<1$ such that

$$\frac{1}{p-\beta}(-\frac{zf'(z)}{g(z)}-\beta)-1=\varphi_1(z)(\psi(\omega_1(z))-1). \tag{3.15}$$

Since

$$\frac{1}{p-\beta}(-\frac{zf'(z)}{g(z)}-\beta)-1=\frac{(1-p)a_1+pb_1}{\beta-p}z+\frac{(2-p)a_2+pb_2-(1-p)a_1b_1-pb_1^2}{\beta-p}z^2+\cdots$$

and

$$\varphi_1(z)(\phi(\omega_1(z))-1)=A_1d_0t_1z+[A_1d_1t_1+d_0(A_1t_2+A_2t_1^2)]z^2+\cdots,$$

then comparing both sides of (3.15), we get that

$$a_1=\frac{1}{1-p}[(\beta-p)A_1d_0t_1-pb_1], \tag{3.16}$$

$$a_2=\frac{1}{2-p}[(\beta-p)(A_1d_1t_1+A_1d_0t_1b_1+d_0(A_1t_2+A_2t_1^2))-pb_2]. \tag{3.17}$$

Because $g(z)\in MS_{p,q}(\eta;\phi)$, there exist analytic functions $\varphi(z)$ and $\omega(z)$, with $|\varphi(z)|\leq 1,\omega(0)=0$ and $|\omega(z)|<1$ such that

$$\frac{1}{p-\eta}(-\frac{zg'(z)}{g(z)}+\eta)-1=\varphi(z)(\phi(\omega(z))-1).$$

Therefore by Theorem 3.1 we have

$$b_1=(\eta-p)B_1c_0\omega, \tag{3.18}$$

$$b_2=\frac{\eta-p}{2}[B_1c_1\omega_1+B_1c_0\omega_2+c_0((\eta-p)B_1^2c_0+B_2)\omega_1^2]. \tag{3.19}$$

By (3.16)–(3.19), we have

$$a_1=\frac{1}{1-p}[(\beta-p)A_1d_0t_1-p(\eta-p)B_1c_0\omega_1],$$

$$a_2-\lambda a_1^2 =\frac{(\beta-p)A_1}{2-p}\{d_1t_1+d_0[t_2-(\frac{\lambda(\beta-p)(2-p)}{(1-p)^2}d_1t_0-\frac{A_2}{A_1})t_1^2]\}+$$
$$\frac{(\beta-p)(\eta-p)A_1B_1c_0d_0t_1\omega_1}{(p-2)(1-p)^2}[2p\lambda(2-p)+(1-p)^2]-$$
$$\frac{p(\eta-p)B_1}{2(2-p)}\{c_1\omega_1+c_0[\omega_2-(\frac{(\eta-p)[2p\lambda(2-p)-(1-p)^2]}{(1-p)^2}B_1c_0-\frac{B_2}{B_1})\omega_1^2]\}.$$

We can proceed similarly as previous theorems and prove the hypothesis. Thus we complete the proof of Theorem 3.13. □

Theorem 3.14 *If $f(z)$ given by (1.1) belongs to $MCK_{p,q}(\eta,\beta;\phi,\psi)$ for $p\geq 3$, then*

$$|a_1|\leq\frac{p[(p-\beta)A_1+p(p-\eta)B_1]}{(1-p)^2},$$

$$|a_2|\leq\frac{p(p-\beta)A_1}{(2-p)^2}[1+\max\{1,\frac{|A_2|}{A_1}\}]+\frac{p(p-\beta)(p-\eta)A_1B_1}{(2-p)^2}+$$
$$\frac{p^2(p-\eta)B_1}{2(2-p)^2}[1+\max\{1,|(p-\eta)B_1c_0-\frac{B_2}{B_1}|\}],$$

and, for any complex number λ

$$|a_2-\lambda a_1^2|\leq\frac{p(p-\beta)A_1}{(2-p)^2}[1+\max\{1,|\frac{\lambda p(p-\beta)(2-p)^2}{(1-p)^4}A_1e_0-\frac{A_2}{A_1}|\}]+$$
$$\frac{p(p-\beta)(p-\eta)A_1B_1}{(2-p)^2(1-p)^4}|2\lambda p^2(2-p)^2-(1-p)^4|+$$
$$\frac{p^2(p-\eta)B_1}{2(2-p)^2}[1+\max\{1,|\frac{(\eta-p)[2\lambda p^2(2-p)^2-(1-p)^4]}{(1-p)^4}B_1c_0-\frac{B_2}{B_1}|\}].$$

The result is sharp.

Acknowledgements We thank the referees for their time and comments.

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