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Extinction for Non-Divergent Parabolic Equations with a Gradient Source

Dengming LIU[1,*], **Chunlai MU**[2]
1. *School of Mathematics and Computational Science, Hunan University of Science and Technology, Hunan* 411201*, P. R. China;*
2. *College of Mathematics and Statistics, Chongqing University, Chongqing* 401331*, P. R. China*

Abstract In this article, we study the extinction and non-extinction behaviors of the solutions for a class of non-divergent parabolic equation with a nonlinear gradient source. Moreover, we discuss the exponential decay estimates of the extinction solutions.

Keywords extinction; non-divergent parabolic equation; gradient source

MR(2010) Subject Classification 35K20; 35K55

1. Introduction

Our main interest in this article is to study the extinction behavior of the following non-divergent parabolic equation with absorption term and nonlinear gradient source

$$\begin{cases} w_t = w^p \Delta w + \lambda \left|\nabla w\right|^q - \delta w, & (x,t) \in \Omega \times (0,+\infty), \\ w(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ w(x,0) = w_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $p \in (-1,0)$, λ and q are positive parameters, $\Omega \subset \mathbf{R}^N$ $(N > 2)$ is an open bounded domain with smooth boundary $\partial\Omega$, and $w_0(x)$ is a nonzero nonnegative function satisfying suitable compatibility conditions. It is well known that model (1.1) with $p = 0$ and $q = 2$ appears in the physical theory of growth and roughening of surfaces [1].

There is an extensive literature on the extinction singularity of the fast diffusion equation

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) + \lambda u^q \tag{1.2}$$

with null Dirichlet boundary condition. For instance, the authors of [2–6] mentioned that the critical extinction exponent of the weak solution to problem (1.2) is $q = m(p-1)$. Recently, Mu et al. [7,8] considered the initial-boundary value problem of the fast diffusion equation

$$u_t = \operatorname{div}(u^\alpha |\nabla u|^{m-1}\nabla u) + \lambda |\nabla u|^q, \tag{1.3}$$

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* Corresponding author
E-mail address: liudengming08@163.com (Dengming LIU); clmu2005@163.com (Chunlai MU)

and proved that the critical extinction exponent of problem (1.3) is $q = m + \alpha$. Especially, the authors of [9] considered problem (1.1) in the special case $\delta = 0$. As far as we know, there is no article on the extinction behavior for problem (1.1). Motivated by the aforementioned works, the aim of this article is to deal with the critical extinction exponent of problem (1.1). In what follows, we work mostly with the equivalent formulation of problem (1.1), obtained by setting $u = w^{1-p}$,

$$\begin{cases} u_t = (1-p)\,\Delta u^{\frac{1}{1-p}} + \frac{\lambda(1-p)q^q}{(q-p)^q}|\nabla u^{\frac{q-p}{q(1-p)}}|^q - \delta\,(1-p)\,u, & (x,t) \in \Omega\times(0,+\infty)\,, \\ u\,(x,t) = 0, & (x,t) \in \partial\Omega\times(0,+\infty)\,, \\ u\,(x,0) = u_0\,(x) = w_0^{1-p}\,(x)\,, & x \in \overline{\Omega}. \end{cases} \tag{1.4}$$

Definition 1.1 *A nonnegative measurable function $u\,(x,t)$ defined in $\Omega\times(0,T)$ is a weak solution of problem (1.4) if $u^{\frac{1}{1-p}} \in L^2\left(0,T;W^{1,2}(\Omega)\right)$, $\left|\nabla u^{\frac{q-p}{q(1-p)}}\right|^q \in L^1\left(0,T;L^1(\Omega)\right)$, $u_t \in L^2\left(0,T;L^2(\Omega)\right)$, $u \in C\left(0,T;L^\infty(\Omega)\right)$, and for any $\zeta \in C_0^\infty(\Omega\times(0,T))$ and $0 < t_1 < t_2 < T$, the following integral identity holds*

$$\begin{aligned} &\int_\Omega u\,(x,t_2)\,\zeta\,(x,t_2)\,\mathrm{d}x + \int_{t_1}^{t_2}\int_\Omega \left[-u\zeta_t + (1-p)\,\nabla u^{\frac{1}{1-p}}\cdot\nabla\zeta\right]\mathrm{d}x\mathrm{d}t \\ &= \int_\Omega u\,(x,t_1)\,\zeta\,(x,t_1)\,\mathrm{d}x + \frac{\lambda\,(1-p)\,q^q}{(q-p)^q}\int_{t_1}^{t_2}\int_\Omega\left|\nabla u^{\frac{q-p}{q(1-p)}}\right|^q\zeta\mathrm{d}x\mathrm{d}t - \delta\,(1-p)\int_{t_1}^{t_2}\int_\Omega u\zeta\mathrm{d}x\mathrm{d}t. \end{aligned}$$

The main results of this article are given by the following theorems.

Theorem 1.2 *Assume that $p \in (-1,0)$ and $q \in \left(p+1, \frac{2}{2-p}\right)$. Then the nonnegative weak solution of problem (1.1) vanishes in finite time if the initial data is sufficiently small. Furthermore, we have*

$$\begin{cases} \|w\|_{2-p} \le e^{-\frac{D_4}{2-p}t}\,\|w_0\|_{2-p}\,, & t \in [0,T_1), \\ \|w\|_{2-p} \le [(\|w(\cdot,T_1)\|_{2-p}^{-p} + D_6)e^{p\delta(t-T_1)} - D_6]^{-\frac{1}{p}}, & t \in [T_1,T_2), \\ \|w\|_{2-p} \equiv 0, & t \in [T_2,+\infty), \end{cases}$$

for $2 < N < \frac{2p-4}{p}$, and

$$\begin{cases} \|w\|_{-\frac{Np}{2}} \le e^{\frac{2D_{10}}{Np}t}\,\|w_0\|_{-\frac{Np}{2}}\,, & t \in [0,T_3), \\ \|w\|_{-\frac{Np}{2}} \le [(\|w(\cdot,T_3)\|_{-\frac{Np}{2}}^{-p} + D_{12})e^{p\delta(t-T_3)} - D_{12}]^{-\frac{1}{p}}, & t \in [T_3,T_4), \\ \|w\|_{-\frac{Np}{2}} \equiv 0, & t \in [T_4,+\infty), \end{cases}$$

for $\frac{2p-4}{p} \le N < +\infty$, where D_4, D_6, D_{10}, D_{12}, T_1, T_2, T_3 and T_4 are positive constants, given in Section 2.

Theorem 1.3 *Assume that $p \in (-1,0)$ and $q \in (0,p+1)$ and λ is sufficiently large. Then problem (1.1) admits at least one non-extinction solution for any nonzero nonnegative initial data.*

Theorem 1.4 *Assume that $p \in (-1,0)$ and $q = p+1$.*

(1) The nonnegative weak solution of problem (1.1) vanishes in finite time if λ is sufficiently

small. Furthermore, we have

$$\begin{cases} \|w\|_{2-p} \le [(\|w_0\|_{2-p}^{-p} + D_{14})e^{p\delta t} - D_{14}]^{-\frac{1}{p}}, & t \in [0, T_5), \\ \|w\|_{2-p} \equiv 0, & t \in [T_5, +\infty), \end{cases}$$

for $2 < N < \frac{2p-4}{p}$, *and*

$$\begin{cases} \|w\|_{-\frac{Np}{2}} \le [(\|w_0\|_{-\frac{Np}{2}}^{-p} + D_{16})e^{p\delta t} - D_{16}]^{-\frac{1}{p}}, & t \in [0, T_6), \\ \|w\|_{-\frac{pN}{2}} \equiv 0, & t \in [T_6, +\infty), \end{cases}$$

for $\frac{2p-4}{p} \le N < +\infty$, *where* D_{14}, D_{16}, T_5 *and* T_6 *are positive constants, given in Section 2.*

(2) Problem (1.1) admits at least one non-extinction solution for any nonzero nonnegative initial data if λ *is sufficiently large.*

2. Proofs of the main results

By energy method, super-solution and sub-solution methods, we will give the proofs of Theorems 1.2, 1.3 and 1.4 in this section.

Proof of Theorem 1.2 Multiplying the first equation in (1.4) by u^s and integrating over Ω by parts, one has

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{s+1}\mathrm{d}x + \frac{4s(s+1)(1-p)^2}{[s(1-p)+1]^2}\int_\Omega |\nabla u^{\frac{s(1-p)+1}{2(1-p)}}|^2\mathrm{d}x + \delta(1-p)(s+1)\int_\Omega u^{s+1}\mathrm{d}x \\ &\quad= \frac{\lambda(s+1)(1-p)2^q}{[s(1-p)+1]^q}\int_\Omega u^{\frac{s(1-p)(2-q)+q-2p}{2(1-p)}}|\nabla u^{\frac{s(1-p)+1}{2(1-p)}}|^q\mathrm{d}x. \end{aligned} \tag{2.1}$$

Since $q \in \left(p+1, \frac{2}{2-p}\right)$, we may employ Young's inequality and Hölder's inequality to estimate the term on the right-hand side of (2.1) as

$$\begin{aligned} &\int_\Omega u^{\frac{s(1-p)(2-q)+q-2p}{2(1-p)}}|\nabla u^{\frac{s(1-p)+1}{2(1-p)}}|^q\mathrm{d}x \\ &\quad\le \epsilon\int_\Omega |\nabla u^{\frac{1+s(1-p)}{2(1-p)}}|^2\mathrm{d}x + C(\epsilon)|\Omega|^{\frac{2+q(p-2)}{(s+1)(1-p)(2-q)}}\left(\int_\Omega u^{s+1}\mathrm{d}x\right)^{\frac{s(1-p)(2-q)+q-2p}{(s+1)(1-p)(2-q)}}. \end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1), we derive that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{s+1}\mathrm{d}x + \Big\{\frac{4s(s+1)(1-p)^2}{[s(1-p)+1]^2} - \frac{\lambda\epsilon(s+1)(1-p)2^q}{[s(1-p)+1]^q}\Big\}\int_\Omega \left|\nabla u^{\frac{s(1-p)+1}{2(1-p)}}\right|^2\mathrm{d}x \\ &\quad\le \delta(1-p)(s+1)\int_\Omega u^{s+1}\mathrm{d}x + \\ &\qquad \frac{\lambda C(\epsilon)(s+1)(1-p)2^q}{[s(1-p)+1]^q|\Omega|^{\frac{2+q(p-2)}{(s+1)(p-1)(2-q)}}}\left(\int_\Omega u^{s+1}\mathrm{d}x\right)^{\frac{s(1-p)(2-q)+q-2p}{(s+1)(1-p)(2-q)}}. \end{aligned} \tag{2.3}$$

Case 1 $2 < N < \frac{2p-4}{p}$. First, we take $s = \frac{1}{1-p}$ and $\epsilon \in \left(0, \frac{1}{\lambda}\right)$ in (2.3). Hölder's inequality and Sobolev embedding inequality lead to

$$\int_\Omega u^{s+1}\mathrm{d}x = \int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x \le |\Omega|^{\frac{4+p(N-2)}{2N}}\left(\int_\Omega u^{\frac{1}{1-p}\cdot\frac{2N}{N-2}}\mathrm{d}x\right)^{\frac{N-2}{2N}\cdot(2-p)}$$

$$\le \kappa_1(p,N)\,|\Omega|^{\frac{4+p(N-2)}{2N}}\Big(\int_\Omega \Big|\nabla u^{\frac{1}{1-p}}\Big|^2\mathrm{d}x\Big)^{\frac{2-p}{2}}, \tag{2.4}$$

where $\kappa_1(p,N)$ denotes the embedding constant. From (2.3) and (2.4), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x + D_1\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x + D_2\Big(\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2}{2-p}} - D_3\Big(\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2(1-p)}{(2-p)(2-q)}} \le 0, \tag{2.5}$$

where $D_1 = \delta(2-p)$, $D_2 = (2-p)(1-\lambda\epsilon)\,\kappa_1^{\frac{2}{p-2}}\,|\Omega|^{\frac{2p-pN-4}{N(2-p)}}$, $D_3 = \lambda C(\epsilon)(2-p)\,|\Omega|^{\frac{2+q(p-2)}{(2-p)(2-q)}}$ are positive constants. Now, we choose $u_0(x)$ to satisfy

$$\Big(\int_\Omega u_0^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2[q-(p+1)]}{(2-p)(2-q)}} \le \frac{D_2}{D_3}.$$

Using [8, Lemma 2.2], we see that there exists a constant $D_4 > D_1$ so that

$$0 \le \int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x \le e^{-D_4 t}\int_\Omega u_0^{\frac{2-p}{1-p}}\mathrm{d}x, \quad t \ge 0. \tag{2.6}$$

Furthermore, from (2.6), it follows that there exists a $T_1 > 0$ such that, for $t \ge T_1$,

$$\begin{aligned}
&D_2\Big(\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2}{2-p}} - D_3\Big(\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2(1-p)}{(2-p)(2-q)}}\\
&= \Big(\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2}{2-p}}\Big[D_2 - D_3\Big(\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2(q-p-1)}{(2-p)(2-q)}}\Big]\\
&\ge \Big(\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2}{2-p}}\underbrace{\Big[D_2 - D_3 e^{-D_4T_1}\Big(\int_\Omega u_0^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2(q-p-1)}{(2-p)(2-q)}}\Big]}_{D_5}.
\end{aligned} \tag{2.7}$$

Exploiting the above inequality to (2.5), we can claim that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x + D_1\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x + D_5\Big(\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2}{2-p}} \le 0. \tag{2.8}$$

Noting that $u = w^{1-p}$, with the help of (2.6), (2.8) and [8, Lemma 2.1], we have

$$\begin{cases}
\|w\|_{2-p} \le e^{-\frac{D_4}{2-p}t}\,\|w_0\|_{2-p}, & 0 \le t < T_1,\\
\|w\|_{2-p} \le [(\|w(\cdot,T_1)\|_{2-p}^{-p} + D_6)e^{p\delta(t-T_1)} - D_6]^{-\frac{1}{p}}, & T_1 \le t < T_2,\\
\|w\|_{2-p} \equiv 0, & T_2 \le t < +\infty,
\end{cases}$$

where

$$D_6 = \frac{D_5}{D_1} \quad\text{and}\quad T_2 = T_1 - \frac{1}{p\delta}\ln(1 + D_6^{-1}\|w(\cdot,T_1)\|_{2-p}^{-p}). \tag{2.9}$$

Case 2 $\frac{2p-4}{p} \le N < +\infty$. Taking $s = \frac{(1-p)(N-2)-N}{2(1-p)}$ and $\epsilon \in \Big(0, \frac{s(1-p)[s(1-p)+1]^{q-2}}{\lambda 2^{q-2}}\Big)$ in (2.3), and using Sobolev embedding inequality, we can claim that

$$\Big(\int_\Omega u^{s+1}\mathrm{d}x\Big)^{\frac{1+s(1-p)}{2(s+1)(1-p)}} = \Big(\int_\Omega u^{\frac{1+s(1-p)}{2(1-p)}\cdot\frac{2N}{N-2}}\mathrm{d}x\Big)^{\frac{N-2}{2N}} \le \kappa_2(p,N)\Big(\int_\Omega \Big|\nabla u^{\frac{1+s(1-p)}{2(1-p)}}\Big|^2\mathrm{d}x\Big)^{\frac{1}{2}}, \tag{2.10}$$

where $\kappa_2(p,N)$ denotes the embedding constant. Combining (2.3) with (2.10) results in

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{s+1}\mathrm{d}x + D_7\int_\Omega u^{s+1}\mathrm{d}x + D_8\Big(\int_\Omega u^{s+1}\mathrm{d}x\Big)^{\frac{1+s(1-p)}{(s+1)(1-p)}} \le D_9\Big(\int_\Omega u^{s+1}\mathrm{d}x\Big)^{\frac{s(1-p)(2-q)+q-2p}{(s+1)(1-p)(2-q)}}, \tag{2.11}$$

here $D_8 = \frac{(s+1)(1-p)\left\{4s(1-p)[1+s(1-p)]^{q-2}-\epsilon\lambda 2^q\right\}}{\kappa_2^2[1+s(1-p)]^q}$, $D_9 = \frac{\lambda C(\epsilon)(s+1)(1-p)2^q}{[1+s(1-p)]^q}|\Omega|^{\frac{2+q(p-2)}{(s+1)(1-p)(2-q)}}$, and $D_7 = \delta(1-p)(s+1)$. Choosing u_0 sufficiently small satisfying

$$\Big(\int_\Omega u_0^{s+1}\mathrm{d}x\Big)^{\frac{2(q-p-1)}{(s+1)(1-p)(2-q)}} < \frac{D_8}{D_9},$$

then [8, Lemma 2.2] tells us that there exists a constant $D_{10} > D_7$ such that

$$0 \le \int_\Omega u^{s+1}\mathrm{d}x \le e^{-D_{10}t}\int_\Omega u_0^{s+1}\mathrm{d}x \tag{2.12}$$

holds for all $t \ge 0$. Furthermore, from (2.12), it follows that there exists a $T_3 > 0$ such that, for all $t \ge T_3$,

$$\begin{aligned} & D_8\Big(\int_\Omega u^{s+1}\mathrm{d}x\Big)^{\frac{1+s(1-p)}{(s+1)(1-p)}} - D_9\Big(\int_\Omega u^{s+1}\mathrm{d}x\Big)^{\frac{s(1-p)(2-q)+q-2p}{(s+1)(1-p)(2-q)}} \\ &= \Big(\int_\Omega u^{s+1}\mathrm{d}x\Big)^{\frac{1+s(1-p)}{(s+1)(1-p)}}\Big[D_8 - D_9\Big(\int_\Omega u^{s+1}\mathrm{d}x\Big)^{\frac{2(q-p-1)}{(1+s)(1-p)(2-q)}}\Big] \\ &\ge \Big(\int_\Omega u^{s+1}\mathrm{d}x\Big)^{\frac{1+s(1-p)}{(s+1)(1-p)}}\underbrace{\Big[D_8 - D_9 e^{-D_{10}T_3}\Big(\int_\Omega u_0^{s+1}\mathrm{d}x\Big)^{\frac{2(q-p-1)}{(1+s)(1-p)(2-q)}}\Big]}_{D_{11}}. \end{aligned} \tag{2.13}$$

Putting the above inequality to (2.11), we can claim that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{s+1}\mathrm{d}x + D_7\int_\Omega u^{s+1}\mathrm{d}x + D_{11}\Big(\int_\Omega u^{s+1}\mathrm{d}x\Big)^{\frac{1+s(1-p)}{(1+s)(1-p)}} \le 0. \tag{2.14}$$

Recalling that $u = w^{1-p}$, $s = \frac{(1-p)(N-2)-N}{2(1-p)}$, in view of (2.12), (2.14) and [8, Lemma 2.1], we have

$$\begin{cases} \|w\|_{-\frac{Np}{2}} \le e^{\frac{2D_{10}}{Np}t}\|w_0\|_{-\frac{Np}{2}}, & 0 \le t < T_3, \\ \|w\|_{-\frac{Np}{2}} \le [(\|w(\cdot,T_3)\|_{-\frac{Np}{2}}^{-p} + D_{12})e^{p\delta(t-T_3)} - D_{12}]^{-\frac{1}{p}}, & T_3 \le t < T_4, \\ \|w\|_{-\frac{Np}{2}} \equiv 0, & T_4 \le t < +\infty, \end{cases}$$

where

$$D_{12} = \frac{D_{11}}{D_7} \quad \text{and} \quad T_4 = T_3 - \frac{1}{p\delta}\ln(1 + D_{12}^{-1}\|w(\cdot,T_3)\|_{-\frac{Np}{2}}^{-p}). \tag{2.15}$$

The proof of Theorem 1.2 is completed. □

Proof of Theorem 1.3 First of all, we let λ_1 be the first eigenvalue of $-\Delta$ on Ω with zero Dirichlet boundary condition, and $\eta_1(x)$ be the corresponding eigenfunction with $\max_{x\in\Omega}\eta_1(x) = 1$. Defining $f(t) = (1-e^{-ct})^{\frac{1-p}{1-q}}$, we can observe that, for any $t > 0$, $f(t) \in (0,1)$ and $f'(t) \le f^{\frac{q-p}{1-p}}(t)$. Put $\underline{v}(x,t) = f(t)\eta_1^{1-p}(x)$. Direct computation yields

$$\begin{aligned} I &:= \int_0^t\int_\Omega \underline{v}_s\zeta + (1-p)\nabla\underline{v}^{\frac{1}{1-p}}\cdot\nabla\zeta - \frac{\lambda(1-p)q^q}{(q-p)^q}|\nabla\underline{v}^{\frac{q-p}{q(1-p)}}|^q\zeta + \delta(1-p)\underline{v}\zeta\mathrm{d}x\mathrm{d}s \\ &< \int_0^t\int_\Omega f^{\frac{q-p}{1-p}}(s)\eta_1^{-p}(x)\zeta(x,s)[1+(\delta+\lambda_1)(1-p) - \lambda(1-p)|\nabla\eta_1(x)|^q]\mathrm{d}x\mathrm{d}s. \end{aligned}$$

It is easy to see that $I < 0$ if $\lambda \ge \frac{|\Omega|[1+(\delta+\lambda_1)(1-p)]}{(1-p)\|\nabla\eta_1\|_q^q}$, which implies that $\underline{v}(x,t)$ is a strict non-extinction weak subsolution of problem (1.4).

On the other hand, noticing that Ω is a bounded domain, without loss of generality, we may assume that $\Omega \subset \left\{x \in \mathbf{R}^N : 0 < x_1 < L\right\}$ for some L. Let $\overline{v}(x) = Ke^{(1-p)x_1}$, where $K > \max\{1, \left\|w_0^{1-p}(x)\right\|_\infty\}$ satisfies

$$\delta K^{\frac{1-q}{1-p}} - K^{\frac{p+1-q}{1-p}} e^{(p+1-q)L} \geq \lambda.$$

Then we can show that $\overline{v}(x)$ is a stationary supersolution of problem (1.4).

Up to now, we find a non-extinction supersolution $\overline{v}$ and a non-extinction weak subsolution $\underline{v}$ of problem (1.4). Moreover, we have $\underline{v} \leq 1 < \overline{v}$. Then by an iterated process, we can conclude that problem (1.4) admits a non-extinction weak solution $u(x,t)$ satisfying $\underline{v}(x,t) \leq u(x,t) \leq \overline{v}(x,t)$. The proof of Theorem 1.3 is completed. □

Proof of Theorem 1.4 (1) **Case 1** For $2 < N < \frac{2p-4}{p}$, similar to the process of the derivation of (2.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x + D_1\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x + D_{13}\Big(\int_\Omega u^{\frac{2-p}{1-p}}\mathrm{d}x\Big)^{\frac{2}{2-p}} \leq 0, \tag{2.16}$$

where $D_{13} = (2-p)\,|\Omega|^{-\frac{p}{2-p}}\,((1-\lambda\epsilon)\kappa_1^{-\frac{2}{2-p}}|\Omega|^{\frac{2p-4}{N(2-p)}} - \lambda C(\epsilon))$. It is easy to verify that $D_{13} > 0$ if $\lambda \in (0, \kappa_1^{-\frac{2}{2-p}}|\Omega|^{\frac{2p-4}{N(2-p)}}(C(\epsilon) + \epsilon\kappa_1^{-\frac{2}{2-p}}|\Omega|^{\frac{2p-4}{N(2-p)}})^{-1})$. Using [8, Lemma 2.1] and recalling that $u = w^{1-p}$ yield

$$\|w\|_{2-p} \leq [(\|w_0\|_{2-p}^{-p} + D_{14})e^{p\delta t} - D_{14}]^{-\frac{1}{p}} \text{ for } 0 \leq t < T_5,$$

and $\|w\|_{2-p} \equiv 0$ for $t \geq T_5$, where

$$D_{14} = \frac{D_{13}}{D_1} \text{ and } T_5 = -\frac{1}{p\delta}\ln(1 + D_{14}^{-1}\|w_0\|_{2-p}^{-p}). \tag{2.17}$$

Case 2 For $\frac{2p-4}{p} \leq N < +\infty$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^{s+1}\mathrm{d}x + D_7\int_\Omega u^{s+1}\mathrm{d}x + D_{15}\Big(\int_\Omega u^{s+1}\mathrm{d}x\Big)^{\frac{s(1-p)+1}{(s+1)(1-p)}} \leq 0, \tag{2.18}$$

where $D_{15} = \frac{4s(s+1)(1-p)^2}{\kappa_2^2[s(1-p)+1]^2} - \frac{\lambda(s+1)(1-p)2^{p+1}}{[s(1-p)+1]^{p+1}}(\frac{\epsilon}{\kappa_2^2} + C(\epsilon)|\Omega|^{\frac{p}{(s+1)(p-1)}})$. If

$$\lambda \in (0, s(1-p)\,\kappa_2^{-2}2^{1-p}\,[s(1-p)+1]^{p-1}\,(\epsilon\kappa_2^{-2} + C(\epsilon)\,|\Omega|^{\frac{p}{(s+1)(p-1)}})^{-1}),$$

then we have $D_{15} > 0$. Using [8, Lemma 2.1], and recalling that $u = w^{1-p}$ and $s = \frac{(1-p)(N-2)-N}{2(1-p)}$ yield

$$\|w\|_{-\frac{Np}{2}} \leq [(\|w_0\|_{-\frac{Np}{2}}^{-p} + D_{16})e^{p\delta t} - D_{16}]^{-\frac{1}{p}} \text{ for } 0 \leq t < T_6,$$

and $\|w\|_{-\frac{Np}{2}} \equiv 0$ for $t \geq T_6$, where

$$D_{16} = \frac{D_{15}}{D_7} \text{ and } T_6 = -\frac{1}{p\delta}\ln(1 + D_{16}^{-1}\|w_0\|_{-\frac{Np}{2}}^{-p}). \tag{2.19}$$

(2) The proof of this part is similar to Theorem 1.3, so we omit the details here. The proof of Theorem 1.4 is completed. □

References

[1] M. KARDAR, G. PARISI, Y. C. ZHANG. *Dynamic scaling of growing interfaces*. Phys. Rev. Lett., 1986, **56**(9): 889–892.

[2] Chunhua JIN, Jingxue YIN, Yuanyuan KE. *Critical extinction and blow-up exponents for fast diffusive polytropic filtration equation with sources*. Proc. Edinb. Math. Soc., 2009, **52**(2): 419–444.

[3] Ya TIAN, Chunlai MU. *Extinction and non-extinction for a p-Laplacian equation with nonlinear source*. Nonlinear Anal., 2008, **69**(8): 2422–2431.

[4] Hongjun YUAN, Songzhe LIAN, Wenjie GAO, et al. *Extinction and positive for the evolution p-Laplacian equation in $\mathbf{R}^N$*. Nonlinear Anal., 2005, **60**(6): 1085–1091.

[5] Jun ZHOU, Chunlai MU. *Critical blow-up and extinction exponents for non-Newton polytropic filtration equation with source*. Bull. Korean Math. Soc., 2009, **46**(6): 1159–1173.

[6] Yifu WANG, Jingxue YIN. *Critical extinction exponents for a polytropic filtration equation with absorption and source*. Math. Meth. Appl. Sci., 2013, **36**(6): 1591–1597.

[7] Dengming LIU, Chunlai MU. *Extinction for a quasilinear parabolic equation with a nonlinear gradient source*. Taiwanese J. Math., 2014, **18**(5): 1329–1343.

[8] Dengming LIU, Chunlai MU. *Extinction for a quasilinear parabolic equation with a nonlinear gradient source and absorption*. J. Appl. Anal. Comput., 2015, **5**(1): 114–137.

[9] Dengming LIU, Chunlai MU. *Critical extinction exponent for a doubly degenerate non-divergent parabolic equation with a gradient source*. Appl. Anal., 2017, online first, DOI: 10.1080/00036811.2017.1359557.

[10] Jingxue YIN, Jing LI, Chunhua JIN. *Non-extinction and critical exponent for a polytropic filtration equation*. Nonlinear Anal., 2009, **71**(8): 347–357.