Journal of Mathematical Research with Applications
Mar., 2018, Vol. 38, No. 2, pp. 169–182
DOI:10.3770/j.issn:2095-2651.2018.02.007
Http://jmre.dlut.edu.cn

Multipliers on the Dirichlet Space for the Annulus

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Abstract Multipliers on the classic Dirichlet space of the unit disk are much more complex than those on the Hardy space and the Bergman space, many basic problems have not been solved, such as the boundedness, which is still an open problem. The annulus, as a kind of typical complex connected domain, has more complicated function structure. This paper focuses on discussing the invertibility and Fredholmness of multipliers on the Dirichlet space of the annulus. The spectra and essential spectra of multipliers with Laurent polynomials symbols are calculated. In addition, we anwser a problem proposed by Guangfu CAO and Li HE on spectrum and essential spectrum for general multipliers.

Keywords annulus; multiplier; spectra; essential spectra

MR(2010) Subject Classification 31C25; 42B15; 47A10

1. Introduction

Operator theory on classical function spaces studies mainly the structure of the Toeplitz operators and their algebraic properties on the Hardy space, Bergman space or Dirichlet space of the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. It is well known that $\mathbb{D}$ is a classical simple connected domain in the complex plane. Annulus is the another important domain, it is a complex connected domain, the function structure on which is different from the structure of the analytic function on the disk. In addition, the difference between the structure of different spaces is large, and the corresponding structure of their operator and operator algebras also have huge difference. Even in the case of the unit disk, the corresponding problems on the Dirichlet space are much complex than on the Hardy space and Bergman space, some basic problems are still open, such as the boundedness of the multipliers. In recent years, the research on the Dirichlet space and their operators become an active field, for example, Wu [1,2], Cao [3,4], Lu and Sun [5] studied the structure and properties of various operators on the Dirichlet space. [6],[7] discussed some problems of the multipliers on the Hardy-Sobolev space of the unit disk. In this paper, we find a gap of the proof of [6, Lemma 2.1], and give a new proof of the key lemma on the multipliers with Laurent polynomial symbols on the Dirichlet space of the annulus, and calculate the spectra and essential spectra of these mutipliers. In addition, we answer a problem left over by [6].

Received November 21, 2017; Accepted January 13, 2018
Supported by the National Natural Science Foundation of China (Grant No. 11501136), Featured Innovation Project of Guangdong Province (Grant No. 2016KTSCX105) and Youth Project of Guangzhou City (Grant No. 1201630152).
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Assume that $0 < r_0 < 1$, denote by $\mathbb{D}_{r_0}$ the disk centered at zero with radius r_0, $H = \mathbb{D} - \overline{\mathbb{D}_{r_0}}$ the annulus in $\mathbb{C}$, and ∂H its boundary. Let $\mathrm{d}A = \frac{1}{\pi(1-r_0^2)} r\mathrm{d}r\mathrm{d}\theta$ be the Lebesgue area measure of H. The Dirichlet space of H, written as $\mathfrak{D}$, is the set of all analytic functions on H satisfying

$$\|f\|_{\mathfrak{D}} = [\int_H |\frac{\partial(\sum_{k=1}^{+\infty} b_k \frac{1}{z^k})}{\partial z}|^2 \mathrm{d}A + \int_H |\frac{\partial(\sum_{k=1}^{+\infty} a_k z^k)}{\partial z}|^2 \mathrm{d}A + |a_0|^2]^{\frac{1}{2}} < +\infty,$$

where $f(z) = \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=0}^{+\infty} a_k z^k$ is the Laurent polynomial of f on H. Then, $\mathfrak{D}$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{\mathfrak{D}} = \langle \frac{\partial f}{\partial z}, \frac{\partial g}{\partial z}\rangle_{L^2(H,\mathrm{d}A)} + a_0 \cdot \overline{\widetilde{a_0}} = \int_H \frac{\partial f}{\partial z} \cdot \overline{\frac{\partial g}{\partial z}} \mathrm{d}A + a_0 \cdot \overline{\widetilde{a_0}}, \quad \forall f, g \in \mathfrak{D},$$

where $f(z) = \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=0}^{+\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{+\infty} \widetilde{b_k} \frac{1}{z^k} + \sum_{k=0}^{+\infty} \widetilde{a_k} z^k$ are the Laurent polynomials of f and g on H, respectively. Obviously, $\|f\|_{\mathfrak{D}}^2 = \langle f, f\rangle_{\mathfrak{D}}$.

It is not easy to check that $\{z^k\}_{k=-\infty}^{+\infty}$ is an orthogonal basis of $\mathfrak{D}$, and

$$\|z^k\|_{\mathfrak{D}} = \begin{cases} [-k\frac{1-r_0^{-2k}}{(1-{r_0}^2)r_0^{-2k}}]^{\frac{1}{2}}, & k < 0; \\ [k\frac{1-r_0^{2k}}{1-{r_0}^2}]^{\frac{1}{2}}, & k > 0. \end{cases} \tag{1.1}$$

Denote by $K_w(z)$ the reproducing kernel function of $\mathfrak{D}$, then

$$K_w(z) = \sum_{k=1}^{+\infty} \frac{1-{r_0}^2}{k} \cdot \frac{r_0^{2k}}{1-r_0^{2k}} (z\overline{w})^{-k} + \sum_{k=1}^{+\infty} \frac{1-{r_0}^2}{k} \cdot \frac{1}{1-r_0^{2k}} (z\overline{w})^k + 1$$

and

$$\begin{aligned} \|K_w\|_{\mathfrak{D}}^2 &= \langle K_w, K_w\rangle_{\mathfrak{D}} = K_w(w) \\ &= \sum_{k=1}^{+\infty} \frac{1-{r_0}^2}{k} \cdot \frac{r_0^{2k}}{1-r_0^{2k}} \frac{1}{|w|^{2k}} + \sum_{k=1}^{+\infty} \frac{1-{r_0}^2}{k} \cdot \frac{1}{1-r_0^{2k}} |w|^{2k} + 1 \\ &= \sum_{k=1}^{+\infty} \frac{1-{r_0}^2}{k} \cdot \frac{1}{1-r_0^{2k}} (\frac{r_0}{|w|})^{2k} + \sum_{k=1}^{+\infty} \frac{1-{r_0}^2}{k} \cdot \frac{|w|^{2k}}{1-r_0^{2k}} + 1, \end{aligned}$$

which indicates

$$K_w(w) \to \sum_{k=1}^{+\infty} \frac{1-{r_0}^2}{k} \cdot \frac{1}{1-r_0^{2k}} r_0^{2k} + \sum_{k=1}^{+\infty} \frac{1-{r_0}^2}{k} \cdot \frac{1}{1-r_0^{2k}} + 1 = +\infty$$

as $|w| \to 1$, and

$$K_w(w) \to \sum_{k=1}^{+\infty} \frac{1-{r_0}^2}{k} \cdot \frac{1}{1-r_0^{2k}} + \sum_{k=1}^{+\infty} \frac{1-{r_0}^2}{k} \cdot \frac{r_0^{2k}}{1-r_0^{2k}} + 1 = +\infty$$

as $|w| \to r_0$. Then, $k_w(z) = \frac{K_w(z)}{\|K_w\|_{\mathfrak{D}}}$ is the normalized reproducing kernel function. Obviously, k_w converges weakly to 0 in $\mathfrak{D}$ as $|w| \to 1$ or $|w| \to r_0$.

Suppose $\varphi \in \mathfrak{D}$, for $\forall f \in \mathfrak{D}$, define $M_\varphi f = \varphi f$, called the multiplier with symbol φ. In general case, M_φ is the operator densely defined on $\mathfrak{D}$. The boundedness of M_φ is unknown, even in the disk, it is still an open problem. Write

$$\mathcal{M} = \{\varphi \in \mathfrak{D} \mid M_\varphi \text{ is bounded on } \mathfrak{D}\},$$

then $\mathcal{M}$ is the multiplier algebra of $\mathfrak{D}$.

2. Spectral properties of the multipliers

Lemma 2.1 *Assume that $\lambda \in H$. Then, $M_{z-\lambda}$ is lower bounded on $\mathfrak{D}$.*

Proof For $\forall f(z) = \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=0}^{+\infty} a_k z^k \in \mathfrak{D}$, we have

$$\begin{aligned}(z-\lambda)f(z) &= (z-\lambda)(\sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=0}^{+\infty} a_k z^k) \\ &= \sum_{k=1}^{+\infty} b_k \frac{1}{z^{k-1}} - \lambda \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=0}^{+\infty} a_k z^{k+1} - \lambda \sum_{k=0}^{+\infty} a_k z^k \\ &= (\sum_{k=2}^{+\infty} b_k \frac{1}{z^{k-1}} - \lambda \sum_{k=1}^{+\infty} b_k \frac{1}{z^k}) + (b_1 - \lambda a_0) + (\sum_{k=0}^{+\infty} a_k z^{k+1} - \lambda \sum_{k=1}^{+\infty} a_k z^k).\end{aligned}$$

Then,

$$\begin{aligned}\|(z-\lambda)f(z)\|_{\mathfrak{D}}^2 =& \| \sum_{k=2}^{+\infty} b_k \frac{1}{z^{k-1}} - \lambda \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} \|_{\mathfrak{D}}^2 + \\ & \| \sum_{k=0}^{+\infty} a_k z^{k+1} - \lambda \sum_{k=1}^{+\infty} a_k z^k \|_{\mathfrak{D}}^2 + |b_1 - \lambda a_0|^2 \\ \geq & (|\lambda| \| \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} \|_{\mathfrak{D}} - \| \sum_{k=2}^{+\infty} b_k \frac{1}{z^{k-1}} \|_{\mathfrak{D}})^2 + \\ & (\| \sum_{k=0}^{+\infty} a_k z^{k+1} \|_{\mathfrak{D}} - |\lambda| \| \sum_{k=1}^{+\infty} a_k z^k \|_{\mathfrak{D}})^2 + |b_1 - \lambda a_0|^2 \\ =& (I_1 - I_2)^2 + (J_1 - J_2)^2 + |b_1 - \lambda a_0|^2,\end{aligned}$$

where

$$I_1 = |\lambda| \| \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} \|_{\mathfrak{D}}, \quad I_2 = \| \sum_{k=2}^{+\infty} b_k \frac{1}{z^{k-1}} \|_{\mathfrak{D}},$$

$$J_1 = \| \sum_{k=0}^{+\infty} a_k z^{k+1} \|_{\mathfrak{D}}, \quad J_2 = |\lambda| \| \sum_{k=1}^{+\infty} a_k z^k \|_{\mathfrak{D}}.$$

Direct calculation gives

$$\begin{aligned}I_1 =& |\lambda| \| \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} \|_{\mathfrak{D}} = |\lambda| \langle \sum_{k=1}^{+\infty} b_k \frac{1}{z^k}, \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} \rangle_{\mathfrak{D}}^{\frac{1}{2}} \\ =& |\lambda| \langle \frac{\partial[\sum_{k=1}^{+\infty} b_k \frac{1}{z^k}]}{\partial z}, \frac{\partial[\sum_{k=1}^{+\infty} b_k \frac{1}{z^k}]}{\partial z} \rangle_{L^2(H,\mathrm{d}A)}^{\frac{1}{2}} \\ =& |\lambda| [\sum_{k=1}^{+\infty} |b_k|^2 k^2 \int_H \frac{1}{|z|^{2(k+1)}} \mathrm{d}A]^{\frac{1}{2}} \\ =& |\lambda| [\sum_{k=1}^{+\infty} |b_k|^2 k^2 \cdot \frac{1}{-k} \cdot \frac{1 - r_0^{-2k}}{1 - r_0^2}]^{\frac{1}{2}}\end{aligned}$$

$$=|\lambda|[\sum_{k=1}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}]^{\frac{1}{2}}.$$

Similarly,

$$\begin{aligned}I_2=&\|\sum_{k=2}^{+\infty}b_k\frac{1}{z^{k-1}}\|_{\mathfrak{D}}=\langle\sum_{k=2}^{+\infty}b_k\frac{1}{z^{k-1}},\sum_{k=2}^{+\infty}b_k\frac{1}{z^{k-1}}\rangle_{\mathfrak{D}}^{\frac{1}{2}}\\=&[\sum_{k=2}^{+\infty}|b_k|^2(k-1)\cdot\frac{1-r_0^{2(k-1)}}{(1-r_0^2)r_0^{2(k-1)}}]^{\frac{1}{2}},\end{aligned}$$

which makes

$$\begin{aligned}I_1-I_2=&|\lambda|[\sum_{k=1}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}]^{\frac{1}{2}}-[\sum_{k=2}^{+\infty}|b_k|^2(k-1)\cdot\frac{1-r_0^{2(k-1)}}{(1-r_0^2)r_0^{2(k-1)}}]^{\frac{1}{2}}\\=&\frac{|\lambda|}{r_0}[|b_1|^2+\sum_{k=2}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2(k-1)}}]^{\frac{1}{2}}-[\sum_{k=2}^{+\infty}|b_k|^2(k-1)\cdot\frac{1-r_0^{2(k-1)}}{(1-r_0^2)r_0^{2(k-1)}}]^{\frac{1}{2}}\\\geq&\frac{|\lambda|}{r_0}[\sum_{k=2}^{+\infty}|b_k|^2(k-1)\cdot\frac{1-r_0^{2(k-1)}}{(1-r_0^2)r_0^{2(k-1)}}]^{\frac{1}{2}}-[\sum_{k=2}^{+\infty}|b_k|^2(k-1)\cdot\frac{1-r_0^{2(k-1)}}{(1-r_0^2)r_0^{2(k-1)}}]^{\frac{1}{2}}\\=&(\frac{|\lambda|}{r_0}-1)[\sum_{k=2}^{+\infty}|b_k|^2(k-1)\cdot\frac{1-r_0^{2(k-1)}}{(1-r_0^2)r_0^{2(k-1)}}]^{\frac{1}{2}}\end{aligned}$$

where

$$\begin{aligned}&\sum_{k=2}^{+\infty}|b_k|^2(k-1)\cdot\frac{1-r_0^{2(k-1)}}{(1-r_0^2)r_0^{2(k-1)}}=\sum_{k=2}^{+\infty}\frac{k-1}{k}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}\cdot\frac{r_0^2[1-r_0^{2(k-1)}]}{1-r_0^{2k}}\\&\geq\frac{1}{2}\sum_{k=2}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}\cdot r_0^2\cdot\frac{1^{k-1}-r_0^{2(k-1)}}{1^k-r_0^{2k}}\\&=\frac{1}{2}\sum_{k=2}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}\cdot r_0^2\cdot\frac{1+r_0^2+\cdots+r_0^{2(k-2)}}{1+r_0^2+\cdots+r_0^{2(k-1)}}\\&\geq\frac{r_0^2}{2}\sum_{k=2}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}\cdot[1-\frac{1}{1+r_0^2+\cdots+r_0^{2(k-1)}}]\\&\geq\frac{r_0^2}{2}\sum_{k=2}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}\cdot[1-\frac{1}{1+r_0^2}]=\frac{r_0^2}{2}\sum_{k=2}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}\cdot\frac{r_0^2}{1+r_0^2}\\&=\frac{r_0^4}{2(1+r_0^2)}\sum_{k=2}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}=\frac{r_0^4}{2(1+r_0^2)}\|\sum_{k=2}^{+\infty}b_k\frac{1}{z^k}\|_{\mathfrak{D}}^2.\end{aligned}$$

Thus,

$$I_1-I_2\geq(\frac{|\lambda|}{r_0}-1)\frac{r_0^2}{\sqrt{2(1+r_0^2)}}\|\sum_{k=2}^{+\infty}b_k\frac{1}{z^k}\|_{\mathfrak{D}}.\tag{2.1}$$

In addition,

$$J_1-J_2=\|\sum_{k=0}^{+\infty}a_kz^{k+1}\|_{\mathfrak{D}}-|\lambda|\|\sum_{k=1}^{+\infty}a_kz^k\|_{\mathfrak{D}}$$

$$=[\sum_{k=0}^{+\infty}|a_k|^2(k+1)^2\cdot\int_H|z|^{2k}\mathrm{d}A(z)]^{\frac12}-|\lambda|[\sum_{k=1}^{+\infty}|a_k|^2k^2\cdot\int_H|z|^{2(k-1)}\mathrm{d}A(z)]^{\frac12}$$
$$=[\sum_{k=0}^{+\infty}|a_k|^2(k+1)\cdot\frac{1-r_0^{2(k+1)}}{1-r_0^2}]^{\frac12}-|\lambda|[\sum_{k=1}^{+\infty}|a_k|^2k\cdot\frac{1-r_0^{2k}}{1-r_0^2}]^{\frac12}$$
$$\geq(1-|\lambda|)[\sum_{k=1}^{+\infty}|a_k|^2k\cdot\frac{1-r_0^{2k}}{1-r_0^2}]^{\frac12}=(1-|\lambda|)\|\sum_{k=1}^{+\infty}a_kz^k\|_{\mathfrak{D}}.$$

Combining (2.1) and (2.2), we obtain

$$\|(z-\lambda)f\|_{\mathfrak{D}}^2\geq(\frac{|\lambda|}{r_0}-1)^2\frac{r_0^4}{2(1+r_0^2)}\|\sum_{k=2}^{+\infty}b_k\frac{1}{z^k}\|_{\mathfrak{D}}^2+(1-|\lambda|)^2\|\sum_{k=1}^{+\infty}a_kz^k\|_{\mathfrak{D}}^2+|b_1-\lambda a_0|^2. \quad (2.3)$$

We are to show that $M_{z-\lambda}$ is lower bounded on $\mathfrak{D}$. Otherwise, there exist $f_n(z)=\sum_{k=1}^{+\infty}b_k^{(n)}\frac{1}{z^k}+\sum_{k=0}^{+\infty}a_k^{(n)}z^k\in\mathfrak{D}$ such that $\|f_n\|_{\mathfrak{D}}=1$ and $\|M_{z-\lambda}f_n\|\to0$ as $n\to\infty$. Since the unit ball in $\mathfrak{D}$ is weakly compact, without loss of generality, assume $f_n\xrightarrow{w}f$, we have $M_{z-\lambda}f_n\xrightarrow{w}M_{z-\lambda}f$, which implies $M_{z-\lambda}f=0$. Note $\operatorname{Ker}M_{z-\lambda}=\{0\}$, we get $f=0$. This makes $f_n\xrightarrow{w}0$. Hence, $a_k^{(n)}\to0(n\to\infty)$, $b_k^{(n)}\to0(n\to\infty)$ for each $k\in\mathbf{Z}$. Especially, $a_0^{(n)}\to0(n\to\infty)$, $b_1^{(n)}\to0(n\to\infty)$. (2.3) gives

$$\|M_{z-\lambda}f_n\|_{\mathfrak{D}}^2\geq(\frac{|\lambda|}{r_0}-1)^2\frac{r_0^4}{2(1+r_0^2)}\|\sum_{k=2}^{+\infty}b_k^{(n)}\frac{1}{z^k}\|_{\mathfrak{D}}^2+(1-|\lambda|)^2\|\sum_{k=1}^{+\infty}a_k^{(n)}z^k\|_{\mathfrak{D}}^2+|b_1^{(n)}-\lambda a_0^{(n)}|^2$$
$$\geq\min\{(\frac{|\lambda|}{r_0}-1)^2\frac{r_0^4}{2(1+r_0^2)},(1-|\lambda|)^2,1\}(\|\sum_{k=2}^{+\infty}b_k^{(n)}\frac{1}{z^k}\|_{\mathfrak{D}}^2+\|\sum_{k=1}^{+\infty}a_k^{(n)}z^k\|_{\mathfrak{D}}^2+|b_1^{(n)}-\lambda a_0^{(n)}|^2),$$

which indicates

$$\|\sum_{k=2}^{+\infty}b_k^{(n)}\frac{1}{z^k}\|_{\mathfrak{D}}^2+\|\sum_{k=1}^{+\infty}a_k^{(n)}z^k\|_{\mathfrak{D}}^2+|b_1^{(n)}-\lambda a_0^{(n)}|^2\to0$$

as $n\to\infty$. Thus,

$$\|f_n\|_{\mathfrak{D}}=[\|\sum_{k=1}^{+\infty}b_k^{(n)}\frac{1}{z^k}\|_{\mathfrak{D}}^2+\|\sum_{k=1}^{+\infty}a_k^{(n)}z^k\|_{\mathfrak{D}}^2+|a_0^{(n)}|^2]^{\frac12}\to0$$

as $n\to\infty$, which makes contradiction with $\|f_n\|_{\mathfrak{D}}=1$. This completes the proof. □

Lemma 2.2 *Assume* $\lambda\in\mathbb{C}\backslash\bar{H}$. *Then,* $M_{z-\lambda}$ *is invertible on* $\mathfrak{D}$.

Proof Case 1 if $\lambda\notin\bar{\mathbb{D}}$, then $z-\lambda$ is lower bounded. Let

$$\varphi_\lambda(z)=\frac{1}{z-\lambda}=-\frac{1}{\lambda}\sum_{k=0}^{+\infty}(\frac{1}{\lambda})^kz^k.$$

For $\forall f(z)=\sum_{k=1}^{+\infty}b_k\frac{1}{z^k}+\sum_{k=0}^{+\infty}a_kz^k\in\mathfrak{D}$, $\forall k_0\in\mathbf{Z}$, we have

$$z^{k_0}f=z^{k_0}(\sum_{k=1}^{+\infty}b_k\frac{1}{z^k}+\sum_{k=0}^{+\infty}a_kz^k).$$

If $k_0=0$, then $\|z^{k_0}f\|_{\mathfrak{D}}=\|f\|_{\mathfrak{D}}$.

If $k_0 > 0$, then

$$\begin{aligned} z^{k_0} f(z) =& \sum_{k=1}^{+\infty} b_k z^{k_0-k} + \sum_{k=0}^{+\infty} a_k z^{k_0+k} \\ =& \sum_{k=k_0+1}^{+\infty} b_k z^{k_0-k} + (\sum_{k=1}^{k_0-1} b_k z^{k_0-k} + \sum_{k=0}^{+\infty} a_k z^{k_0+k}) + b_{k_0}. \end{aligned}$$

When $k = 0$, we have $\|z^{k_0+k}\|_{\mathfrak{D}}^2 = \|z^{k_0}\|_{\mathfrak{D}}^2 = k_0 \cdot \frac{1-r_0^{k_0}}{1-r_0^2} \leq \frac{k_0}{1-r_0^2}$.

When $k > 0$, we have

$$\begin{aligned} \|z^{k_0+k}\|_{\mathfrak{D}}^2 =& \frac{k_0 + k}{1 - r_0^2} \cdot [1 - r_0^{2(k_0+k)}] \\ \leq& \frac{k_0 \cdot k + k \cdot k_0}{1 - r_0^2} \cdot \frac{1 - r_0^{2k}}{1 - r_0^{2k}} \cdot [1 - r_0^{2(k_0+k)}] \\ =& 2k_0 \frac{k(1 - r_0^{2k})}{1 - r_0^2} \cdot \frac{1 + r_0^2 + \cdots + r_0^{2(k_0+k-1)}}{1 + r_0^2 + \cdots + r_0^{2(k-1)}} \\ =& 2k_0 \frac{k(1 - r_0^{2k})}{1 - r_0^2} \cdot (1 + \frac{r_0^{2k} + \cdots + r_0^{2(k_0+k-1)}}{1 + r_0^2 + \cdots + r_0^{2(k-1)}}) \\ \leq& 2k_0(1 + k_0) \cdot \frac{k(1 - r_0^{2k})}{1 - r_0^2} \\ \leq& 2(k_0 + 1)^2 \cdot \|z^k\|_{\mathfrak{D}}^2. \end{aligned}$$

If $k < k_0$, then

$$\begin{aligned} \|z^{k_0-k}\|_{\mathfrak{D}}^2 =& \frac{k_0 - k}{1 - r_0^2} \cdot [1 - r_0^{2(k_0-k)}] \\ =& (k_0 - k) \cdot \frac{1 - r_0^{2k}}{(1 - r_0^2) r_0^{2k}} \cdot \frac{1 - r_0^{2(k_0-k)}}{1 - r_0^{2k}} \cdot r_0^{2k} \\ \leq& (k_0 \cdot k + k \cdot k_0) \cdot \frac{1 - r_0^{2k}}{(1 - r_0^2) r_0^{2k}} \cdot \frac{1 - r_0^{2(k_0-k)}}{1 - r_0^{2k}} \\ =& 2k_0 k \cdot \frac{1 - r_0^{2k}}{(1 - r_0^2) r_0^{2k}} \cdot \frac{1 + r_0^2 + \cdots + r_0^{2(k_0-k-1)}}{1 + r_0^2 + \cdots + r_0^{2(k-1)}} \\ \leq& 2k_0 k \cdot \frac{1 - r_0^{2k}}{(1 - r_0^2) r_0^{2k}} \cdot \frac{k_0}{1} \\ =& 2k_0^2 \cdot \frac{k(1 - r_0^{2k})}{(1 - r_0^2) r_0^{2k}} = 2k_0^2 \|z^{-k}\|_{\mathfrak{D}}^2. \end{aligned}$$

If $k > k_0$, then

$$\begin{aligned} \|z^{k_0-k}\|_{\mathfrak{D}}^2 =& \frac{k - k_0}{1 - r_0^2} \cdot \frac{1 - r_0^{2(k-k_0)}}{r_0^{2(k-k_0)}} \\ \leq& \frac{2k_0}{1 - r_0^2} k \cdot \frac{1 - r_0^{2k}}{r_0^{2k}} \cdot \frac{r_0^{2k}}{r_0^{2(k-k_0)}} \cdot \frac{1 - r_0^{2(k-k_0)}}{1 - r_0^{2k}} \end{aligned}$$

$$=2k_0 \cdot \frac{k(1-r_0^{2k})}{(1-r_0^2)r_0^{2k}} \cdot r_0^{2k_0} \cdot \frac{1+r_0^2+\cdots+r_0^{2(k-k_0-1)}}{1+r_0^2+\cdots+r_0^{2(k-1)}}$$
$$\leq 2k_0 \cdot \|z^{-k}\|_{\mathfrak{D}}^2.$$

Therefore,

$$\|z^{k_0}f\|_{\mathfrak{D}}^2 = \|\sum_{k=k_0+1}^{+\infty} b_k z^{k_0-k}\|_{\mathfrak{D}}^2 + \|\sum_{k=1}^{k_0-1} b_k z^{k_0-k}\|_{\mathfrak{D}}^2 + \|\sum_{k=0}^{+\infty} a_k z^{k_0+k}\|_{\mathfrak{D}}^2 + |b_{k_0}|^2$$
$$\leq \sum_{k=k_0+1}^{+\infty} |b_k|^2 \cdot 2k_0 \cdot \|z^{-k}\|_{\mathfrak{D}}^2 + \sum_{k=1}^{k_0-1} |b_k|^2 \cdot 2k_0^2 \|z^{-k}\|_{\mathfrak{D}}^2 + \sum_{k=0}^{+\infty} |a_k|^2 \cdot 2(k_0+1)^2 \cdot \|z^k\|_{\mathfrak{D}}^2 + |b_{k_0}|^2$$
$$\leq 2k_0 \|\sum_{k=k_0+1}^{+\infty} b_k z^{-k}\|_{\mathfrak{D}}^2 + 2k_0^2 \|\sum_{k=1}^{k_0-1} b_k z^{-k}\|_{\mathfrak{D}}^2 + 2(k_0+1)^2 [\|\sum_{k=1}^{+\infty} a_k z^k\|_{\mathfrak{D}}^2 + |a_0|^2 \cdot \frac{k_0}{1-r_0^2}] +$$
$$|b_{k_0}|^2 \cdot \frac{1}{k_0} \cdot \frac{k_0(1-r_0^{2k_0})}{(1-r_0^2)r_0^{2k_0}} \cdot \frac{(1-r_0^2)r_0^{2k_0}}{1-r_0^{2k_0}}$$
$$= 2k_0 \|\sum_{k=k_0+1}^{+\infty} b_k z^{-k}\|_{\mathfrak{D}}^2 + 2k_0^2 \|\sum_{k=1}^{k_0-1} b_k z^{-k}\|_{\mathfrak{D}}^2 + \frac{(1-r_0^2)r_0^{2k_0}}{k_0(1-r_0^{2k_0})} \cdot |b_{k_0}|^2 \|z^{-k_0}\|_{\mathfrak{D}}^2 +$$
$$2(k_0+1)^2 [\|\sum_{k=1}^{+\infty} a_k z^k\|_{\mathfrak{D}}^2 + |a_0|^2 \cdot \frac{k_0}{1-r_0^2}]$$
$$\leq 2k_0^2 [\|\sum_{k=k_0+1}^{+\infty} b_k z^{-k}\|_{\mathfrak{D}}^2 + \|\sum_{k=1}^{k_0-1} b_k z^{-k}\|_{\mathfrak{D}}^2 + \|b_{k_0} z^{-k_0}\|_{\mathfrak{D}}^2] + \frac{2(k_0+1)^3}{1-r_0^2} \|\sum_{k=0}^{+\infty} a_k z^k\|_{\mathfrak{D}}^2$$
$$\leq \frac{2(k_0+1)^3}{1-r_0^2} [\|\sum_{k=1}^{+\infty} b_k z^{-k}\|_{\mathfrak{D}}^2 + \|\sum_{k=0}^{+\infty} a_k z^k\|_{\mathfrak{D}}^2]$$
$$= \frac{2(k_0+1)^3}{1-r_0^2} \|f\|_{\mathfrak{D}}^2.$$

That is,

$$\|z^{k_0}f\|_{\mathfrak{D}} \leq \frac{\sqrt{2}}{\sqrt{1-r_0^2}} (k_0+1)^{\frac{3}{2}} \|f\|_{\mathfrak{D}}. \tag{2.4}$$

Since $\sum_{k=0}^{+\infty} (\frac{1}{|\lambda|})^k k^\alpha$ is convergent for arbitrary $\alpha \in \mathbf{R}$ as $|\lambda| > 1$, and

$$\|\varphi_\lambda f\|_{\mathfrak{D}}^2 = \|\frac{1}{\lambda} \sum_{k=0}^{+\infty} (\frac{1}{\lambda})^k z^k f\|_{\mathfrak{D}}^2 \leq \frac{1}{|\lambda} |^2 [\sum_{k=0}^{+\infty} (\frac{1}{|\lambda|})^k \|z^k f\|_{\mathfrak{D}}^2]^2$$
$$\leq \frac{1}{|\lambda} |^2 [\sum_{k=0}^{+\infty} (\frac{1}{|\lambda|})^k \frac{\sqrt{2}}{\sqrt{1-r_0^2}} (k+1)^{\frac{3}{2}}]^2 \|f\|_{\mathfrak{D}}^2,$$

we have $\varphi_\lambda \in \mathcal{M}$ and $M_{\varphi_\lambda} M_{z-\lambda} = M_{z-\lambda} M_{\varphi_\lambda} = I$.

Case 2 If $0 \leq |\lambda| < r_0$, for any $z \in H$, set

$$\varphi_\lambda(z) = \frac{1}{z-\lambda} = \frac{1}{z(1-\frac{\lambda}{z})} = \varphi_1(z) \cdot \varphi_2(z),$$

where $\varphi_1(z)=\frac{1}{z}$, $\varphi_2(z)=\frac{1}{1-\frac{\lambda}{z}}$. Then, for $\forall f(z)=\sum_{k=1}^{+\infty}b_k\frac{1}{z^k}+\sum_{k=0}^{+\infty}a_kz^k\in\mathfrak{D}$, we have

$$\varphi_1 f(z)=\frac{1}{z}(\sum_{k=1}^{+\infty}b_k\frac{1}{z^k}+\sum_{k=0}^{+\infty}a_kz^k)=(\sum_{k=2}^{+\infty}b_{k-1}\frac{1}{z^k}+a_0\frac{1}{z})+\sum_{k=1}^{+\infty}a_{k+1}z^k+a_1,$$

thus

$$\begin{aligned}\|\varphi_1 f\|_{\mathfrak{D}}^2 &=\|\sum_{k=2}^{+\infty}b_{k-1}\frac{1}{z^k}+a_0\frac{1}{z}\|_{\mathfrak{D}}^2+\|\sum_{k=1}^{+\infty}a_{k+1}z^k\|_{\mathfrak{D}}^2+|a_1|^2\\ &=\sum_{k=2}^{+\infty}|b_{k-1}|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}+|a_0|^2\frac{1}{1-r_0^2}\frac{1-r_0^2}{r_0^2}+\sum_{k=1}^{+\infty}|a_{k+1}|^2k\cdot\frac{1-r_0^{2k}}{1-r_0^2}+|a_1|^2\\ &=\sum_{k=1}^{+\infty}|b_k|^2(k+1)\cdot\frac{1-r_0^{2(k+1)}}{(1-r_0^2)r_0^{2(k+1)}}+\frac{|a_0|^2}{r_0^2}+\sum_{k=2}^{+\infty}|a_k|^2(k-1)\cdot\frac{1-r_0^{2(k-1)}}{1-r_0^2}+|a_1|^2\\ &\le\frac{2}{r_0^2}\sum_{k=1}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}\cdot\frac{1-r_0^{2(k+1)}}{1-r_0^{2k}}+\frac{|a_0|^2}{r_0^2}+\sum_{k=2}^{+\infty}|a_k|^2k\cdot\frac{1-r_0^{2k}}{1-r_0^2}+|a_1|^2\\ &\le\frac{4}{r_0^2}\sum_{k=1}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}+\frac{|a_0|^2}{r_0^2}+\sum_{k=1}^{+\infty}|a_k|^2k\cdot\frac{1-r_0^{2k}}{1-r_0^2}\\ &\le\frac{4}{r_0^2}[\sum_{k=1}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}+\sum_{k=1}^{+\infty}|a_k|^2k\cdot\frac{1-r_0^{2k}}{1-r_0^2}+|a_0|^2]\\ &=\frac{4}{r_0^2}\|f\|_{\mathfrak{D}}^2.\end{aligned}$$

This implies $\varphi_1\in\mathcal{M}$.

Furthermore, $\frac{|\lambda|}{|z|}<\frac{|\lambda|}{r_0}<1$ for arbitrary $z\in H$ since $|\lambda|<r_0$, which makes $\varphi_2(z)=\sum_{k=0}^{+\infty}(\frac{\lambda}{z})^k$ is uniformly convergent on H. For arbitrary $k_0\ge 1$ and $f(z)=\sum_{k=1}^{+\infty}b_k\frac{1}{z^k}+\sum_{k=0}^{+\infty}a_kz^k\in\mathfrak{D}$, we have

$$\frac{1}{z^{k_0}}f=\sum_{k=1}^{+\infty}b_k\frac{1}{z^{k+k_0}}+\sum_{k=0}^{k_0-1}a_kz^{k-k_0}+a_{k_0}+\sum_{k=k_0+1}^{+\infty}a_kz^{k-k_0}.$$

Thus,

$$\begin{aligned}&\|\frac{1}{z^{k_0}}f\|_{\mathfrak{D}}^2\\ &=\sum_{k=1}^{+\infty}|b_k|^2(k+k_0)\cdot\frac{1-r_0^{2(k+k_0)}}{(1-r_0^2)r_0^{2(k+k_0)}}+\sum_{k=0}^{k_0-1}|a_k|^2(k_0-k)\cdot\frac{1-r_0^{2(k_0-k)}}{(1-r_0^2)r_0^{2(k_0-k)}}+\\ &\quad|a_{k_0}|^2+\sum_{k=k_0+1}^{+\infty}|a_k|^2(k-k_0)\cdot\frac{1-r_0^{2(k-k_0)}}{1-r_0^2}\\ &\le 2k_0\sum_{k=1}^{+\infty}|b_k|^2k\cdot\frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}}\cdot\frac{1-r_0^{2(k+k_0)}}{(1-r_0^{2k})r_0^{2k_0}}+\sum_{k=1}^{k_0-1}|a_k|^2k\cdot\frac{1-r_0^{2k}}{1-r_0^2}\cdot\frac{k_0-k}{k}\cdot\frac{1-r_0^{2(k_0-k)}}{(1-r_0^2)r_0^{2(k_0-k)}}+\\ &\quad|a_0|^2k_0\cdot\frac{1-r_0^{2k_0}}{(1-r_0^2)r_0^{2k_0}}+|a_{k_0}|^2k_0\cdot\frac{1-r_0^{2k_0}}{1-r_0^2}\cdot\frac{1-r_0^2}{1-r_0^{2k_0}}+\sum_{k=k_0+1}^{+\infty}|a_k|^2k\cdot\frac{1-r_0^{2k}}{1-r_0^2}\cdot\frac{1-r_0^{2(k-k_0)}}{1-r_0^{2k}}\end{aligned}$$

$$\le \frac{2(k_0+1)^2}{r_0^{2k_0}} \sum_{k=1}^{+\infty} |b_k|^2 k \cdot \frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}} + \frac{(k_0+1)^2}{r_0^{2k_0}} \sum_{k=1}^{k_0-1} |a_k|^2 k \cdot \frac{1-r_0^{2k}}{1-r_0^2} + \frac{(k_0+1)^2}{r_0^{2k_0}} |a_0|^2 +$$

$$|a_{k_0}|^2 k_0 \cdot \frac{1-r_0^{2k_0}}{1-r_0^2} + \sum_{k=k_0+1}^{+\infty} |a_k|^2 k \cdot \frac{1-r_0^{2k}}{1-r_0^2}$$

$$\le \frac{2(k_0+1)^2}{r_0^{2k_0}} [\sum_{k=1}^{+\infty} |b_k|^2 k \cdot \frac{1-r_0^{2k}}{(1-r_0^2)r_0^{2k}} + \sum_{k=1}^{+\infty} |a_k|^2 k \cdot \frac{1-r_0^{2k}}{1-r_0^2} + |a_0|^2]$$

$$= \frac{2(k_0+1)^2}{r_0^{2k_0}} \|f\|_{\mathfrak{D}}^2,$$

which implies

$$\|\varphi_2 f\|_{\mathfrak{D}} \le \sum_{k=0}^{+\infty} \|(\frac{\lambda}{z})^k f\|_{\mathfrak{D}} \le \sum_{k=0}^{+\infty} |\lambda|^k \cdot \|\frac{1}{z^k} f\|_{\mathfrak{D}} \le \sqrt{2}[\sum_{k=0}^{+\infty} (\frac{|\lambda|}{r_0})^k (k+1)] \|f\|_{\mathfrak{D}}.$$

Hence, $\varphi_2 \in \mathcal{M}$. This suggests that $\varphi_\lambda(z) = (z-\lambda)^{-1} \in \mathcal{M}$ for $0 \le |\lambda| < r_0$, and M_{φ_λ} is the inverse of $M_{z-\lambda}$. $\square$

Lemma 2.3 *Suppose $p(z)$ is a Laurent polynomial on H, and $p(z)$ has no zero point on ∂H. Then, M_p is lower bounded on $\mathfrak{D}$.*

Proof Assume $p(z) = \sum_{k=1}^{m} b_k \frac{1}{z^k} + \sum_{k=0}^{n} a_k z^k$ where $m, n \in \mathbf{N}$. Then,

$$p(z) = z^{-m}[\sum_{k=1}^{m} b_k z^{m-k} + \sum_{k=0}^{n} a_k z^{k+m}].$$

Without loss of generality, assume $p(z)$ has the decomposition $p(z) = z^{-m} a_n \Pi_{k=1}^{n+m} (z-\lambda_k)$. Since $p(z)$ has no zero point on ∂H, we have $\lambda_k \in H$ or $\lambda_k \in \mathbb{C}\backslash \bar{H}$. If $\lambda_k \in \mathbb{C}\backslash \bar{H}$, then Lemma 2.2 gives that $M_{z-\lambda_k}$ is invertible on $\mathfrak{D}$; If $\lambda_k \in H$, then Lemma 2.1 gives that $M_{z-\lambda_k}$ is lower bounded on $\mathfrak{D}$. Hence, $M_p = M_{z^{-m} a_n \Pi_{k=1}^{n+m} (z-\lambda_k)}$ is lower bounded on $\mathfrak{D}$. The proof is finished here. $\square$

Denote by $\mathfrak{R}(p)(\overline{H})$ the range of p on $\overline{H}$, it is not difficult to get the following conclusion.

Lemma 2.4 *Assume $p(z)$ is a Laurent polynomial on H. Then, $\sigma(M_p) = \mathfrak{R}(p)(\overline{H})$.*

Proof Assume $p(z) = \sum_{k=1}^{m} b_k \frac{1}{z^k} + \sum_{k=0}^{n} a_k z^k$ where $m, n \in \mathbf{N}$. If $\lambda \notin \mathfrak{R}(p)(\overline{H})$, then $\frac{1}{p-\lambda}$ is a bounded analytic function on H. Suppose the decomposition of $p - \lambda$ is

$$p - \lambda = a z^{-m} \Pi_{k=1}^{n+m} (z-\lambda_k),\ \lambda_k \notin \bar{H},$$

where a is the coefficient of the highest order term of $p(z)$. Then,

$$(p-\lambda)^{-1} = \frac{z^m}{a \Pi_{k=1}^{n+m} (z-\lambda_k)}.$$

By Lemma 2.2, we have $\frac{1}{z-\lambda_k} \in \mathcal{M}$ for each $\lambda_k \notin \bar{H}$, which implies $(p-\lambda)^{-1} \in \mathcal{M}$ for $\forall \lambda \notin \bar{H}$. Hence, $\sigma(M_p) \subseteq \mathfrak{R}(p)(\overline{H})$. On the other hand, if $\lambda \in \mathfrak{R}(p)(\overline{H})$, then there is a $\lambda_{i_0} \in \overline{H} (1 \le i_0 \le n+m)$, thus $M_{z-\lambda_{i_0}}$ is not invertible, further $M_{p-\lambda}$ is not invertible. This shows that $\sigma(M_p) = \mathfrak{R}(p)(\overline{H})$, which ends the proof. $\square$

Theorem 2.5 *Assume $p(z)$ is a Laurent polynomial on $\mathbb{H}$. If $\lambda \notin \mathfrak{R}(p)(\partial H)$, then $M_{p-\lambda}$ is a Fredholm operator.*

Proof If $\lambda \notin \mathfrak{R}(p)(\bar{H})$, then Lemma 2.4 shows that $M_{p-\lambda}$ is invertible, the conclusion holds in this case. Without loss of generality, assume $\lambda \notin \mathfrak{R}(p)(\partial H)$, but $\lambda \in \mathfrak{R}(p)(H)$. Then, $p-\lambda$ has no zero point on ∂H. According to Lemma 2.3, $M_{p-\lambda}$ is lower bounded on $\mathfrak{D}$. Note $\dim[\ker M_{p-\lambda}] = 0$, we just need to prove $\dim[\ker(M_{p-\lambda})^*] < \infty$.

Assume $p-\lambda$ can be decomposed as $p(z)-\lambda = az^{-m}\Pi_{i=1}^{n+m}(z-\lambda_i)$, $\lambda_i \notin \partial H$. Then,

$$(M_{p-\lambda})^* = \bar{a}[M_{z^{-m}\Pi_{i=1}^{n+m}(z-\lambda_i)}]^* = \bar{a}[M_{\Pi_{i=1}^{n+m}(z-\lambda_i)}]^* M_{z^{-m}}^*.$$

Firstly, we show $\dim[\ker(M_{z-\lambda_i})^*] < \infty$ and $\dim[\ker(M_{z^{-m}}^*)] < \infty$ for each $1 \le i \le n$. For $\forall f(z) = \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=0}^{+\infty} a_k z^k \in \mathfrak{D}$, if $(M_{z-\lambda_i})^* f = 0$, then $\langle (M_{z-\lambda_i})^* f, z^n\rangle_{\mathfrak{D}} = 0$ for every $n \in \mathbf{Z}$. Note

$$\begin{aligned}\langle (M_{z-\lambda_i})^* f, z^n\rangle_{\mathfrak{D}} =& \langle f, M_{z-\lambda_i} z^n\rangle_{\mathfrak{D}} = \langle f, z^{n+1} - \lambda_i z^n\rangle_{\mathfrak{D}}\\ =& \langle \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=0}^{+\infty} a_k z^k, z^{n+1} - \lambda_i z^n\rangle_{\mathfrak{D}}.\end{aligned}$$

(1) When $n=-1$, we have

$$\begin{aligned}\langle (M_{z-\lambda_i})^* f, z^{-1}\rangle_{\mathfrak{D}} =& \langle \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=1}^{+\infty} a_k z^k + a_0, 1 - \lambda_i z^{-1}\rangle_{\mathfrak{D}}\\ =& \langle \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=1}^{+\infty} a_k z^k, -\lambda_i z^{-1}\rangle_{\mathfrak{D}} + a_0\\ =& \langle b_1 z^{-1}, -\lambda_i z^{-1}\rangle_{\mathfrak{D}} + a_0\\ =& b_1 \overline{\lambda_i} \cdot \frac{1 - r_0^{-2}}{1 - r_0^2} + a_0.\end{aligned}$$

By the fact that $\langle (M_{z-\lambda_i})^* f, z^n\rangle_{\mathfrak{D}} = 0$ ($\forall n \in \mathbf{Z}$), we get $a_0 = -b_1\overline{\lambda_i} \cdot \frac{1-r_0^{-2}}{1-r_0^2}$.

(2) When $n=0$, we have

$$\begin{aligned}\langle (M_{z-\lambda_i})^* f, 1\rangle_{\mathfrak{D}} =& \langle \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=0}^{+\infty} a_k z^k, z - \lambda_i\rangle_{\mathfrak{D}}\\ =& \langle \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=0}^{+\infty} a_k z^k, z\rangle_{\mathfrak{D}} - a_0\overline{\lambda_i}\\ =& \langle a_1 z, z\rangle_{\mathfrak{D}} - a_0\overline{\lambda_i} = a_1 - a_0\overline{\lambda_i}.\end{aligned}$$

By the fact that $\langle (M_{z-\lambda_i})^* f, z^n\rangle_{\mathfrak{D}} = 0$ ($\forall n \in \mathbf{Z}$), we get $a_1 = a_0\overline{\lambda_i}$.

(3) When $n \ge 1$, we have

$$\begin{aligned}\langle (M_{z-\lambda_i})^* f, z^n\rangle_{\mathfrak{D}} =& \langle \sum_{k=1}^{+\infty} b_k \frac{1}{z^k} + \sum_{k=1}^{+\infty} a_k z^k + a_0, z^{n+1} - \lambda_i z^n\rangle_{\mathfrak{D}}\\ =& \langle a_{n+1} z^{n+1}, z^{n+1}\rangle_{\mathfrak{D}} + \langle a_n z^n, -n\lambda_i z^{n-1}\rangle_{\mathfrak{D}}\end{aligned}$$

$$=(n+1)a_{n+1}\cdot\frac{1-r_0^{2(n+1)}}{1-r_0^2}-n\bar{\lambda_i}a_n\cdot\frac{1-r_0^{2n}}{1-r_0^2}.$$

By the fact that $\langle (M_{z-\lambda_i})^*f, z^n\rangle_{\mathfrak{D}}=0$ $(\forall n\in\mathbf{Z})$, we get

$$a_{n+1}=\frac{n\bar{\lambda_i}(1-r_0^{2n})}{(n+1)[1-r_0^{2(n+1)}]}a_n,\quad n\geq 1.$$

(4) When $n\leq -2$, we have

$$\begin{aligned}\langle (M_{z-\lambda_i})^*f, z^n\rangle_{\mathfrak{D}} =&\langle\sum_{k=1}^{+\infty}b_k\frac{1}{z^k}+\sum_{k=1}^{+\infty}a_kz^k+a_0, z^{n+1}-\lambda_iz^n\rangle_{\mathfrak{D}}\\ =&\langle b_{-n-1}z^{n+1}, z^{n+1}\rangle_{\mathfrak{D}}+\langle b_{-n}z^n, -\lambda_iz^n\rangle_{\mathfrak{D}}\\ =&-(n+1)b_{-n-1}\frac{1-r_0^{-2(n+1)}}{(1-{r_0}^2)r_0^{-2(n+1)}}+n\bar{\lambda_i}b_{-n}\frac{1-r_0^{-2n}}{(1-{r_0}^2)r_0^{-2n}}\\ =&(n+1)b_{-n-1}\cdot\frac{1-r_0^{2(n+1)}}{1-r_0^2}-n\bar{\lambda_i}b_{-n}\cdot\frac{1-r_0^{2n}}{1-r_0^2}.\end{aligned}$$

By the fact that $\langle (M_{z-\lambda_i})^*f, z^n\rangle_{\mathfrak{D}}=0$ $(\forall n\in\mathbf{Z})$, we get

$$b_{-n}=\frac{(n+1)[1-r_0^{2(n+1)}]}{n\bar{\lambda_i}(1-r_0^{2n})}b_{-n-1},\quad n\leq -2.$$

That is,

$$b_k=\frac{(k-1)[1-r_0^{2(-k+1)}]}{k\bar{\lambda_i}(1-r_0^{-2k})}b_{k-1},\quad k\geq 2.$$

Combining (1)–(4), we conclude that $\dim[\ker(M_{z-\lambda_i})^*]=1$. Furthermore, $\dim[\ker\Pi_{i=1}^{n+m}M^*_{z-\lambda_i}]$ $<+\infty$, thus $\Pi_{i=1}^{n+m}M_{z-\lambda_i}$ is a Fredholm operator. Note $M_{z^{-m}}$ is invertible by Lemma 2.4, we conclude that $M_{p-\lambda}$ is also a Fredholm operator, and this completes the proof. □

Theorem 2.6 *Assume $p(z)$ is a Laurent polynomial on $\mathbb{H}$. Then, $\sigma_e(M_p)=\mathfrak{R}(p)(\partial H)$.*

Proof On one hand, Theorem 2.5 gives that $M_{p-\lambda}$ is a Fredholm operator if $\lambda\notin\mathfrak{R}(p)(\partial H)$. That is, $\lambda\notin\sigma_e(M_p)$, which indicates $\sigma_e(M_p)\subseteq\mathfrak{R}(p)(\partial H)$. On the other hand, if $\lambda\in\mathfrak{R}(p)(\partial H)$, then there exists a $z_0\in\partial H$ such that $p(z_0)=\lambda$. We still need to show $\lambda\in\sigma_e(M_p)$.

Fetch some sequence $\{z_n\}\subseteq H$ such that $z_n\to z_0(n\to\infty)$. Suppose $K_{z_n}(w)$ is the reproducing kernel function at z_n, and $k_{z_n}(w)=\frac{K_{z_n}(w)}{\|K_{z_n}\|_{\mathfrak{D}}}$. Then, $\|k_{z_n}\|_{\mathfrak{D}}=1$ and $k_{z_n}(w)\xrightarrow{w}0(n\to\infty)$. Note

$$\begin{aligned}|\langle M^*_{p-\lambda}k_{z_n}, f\rangle_{\mathfrak{D}}|=&|\langle k_{z_n}, (p-\lambda)f\rangle_{\mathfrak{D}}|=\frac{|\langle K_{z_n}, (p(z)-p(z_0))f\rangle_{\mathfrak{D}}|}{\|K_{z_n}\|_{\mathfrak{D}}}\\ =&\frac{1}{\|K_{z_n}\|_{\mathfrak{D}}}\cdot|\overline{\langle (p(z)-p(z_0))f, K_{z_n}\rangle_{\mathfrak{D}}}|\\ =&\frac{1}{\|K_{z_n}\|_{\mathfrak{D}}}\cdot|(p(z_n)-p(z_0))f(z_n)|\\ \leq&|(p(z_n)-p(z_0))|\cdot\|f\|_{\mathfrak{D}},\end{aligned}$$

we have

$$\|M_{p-\lambda}^*k_{z_n}\| = \sup_{\|f\|_{\mathfrak{D}}\le 1} |\langle M_{p-\lambda}^*k_{z_n}, f\rangle_{\mathfrak{D}}| \le |(p(z_n)-p(z_0))| \to 0,\quad n\to\infty.$$

Then, $M_{p-\lambda}$ is not a Fredholm operator. That is, $\lambda\in\sigma_e(M_p)$. This means $\mathfrak{R}(p)(\partial H)\subseteq\sigma_e(M_p)$. The proof has been finished here. □

3. Multipliers with general symbols

Theorem 3.1 *Suppose $M_\varphi\in\mathcal{M}(\mathfrak{D})$, then*

(i) $\sigma(M_\varphi)=\overline{\mathfrak{R}(\varphi)(H)}$;

(ii) $\sigma_e(M_\varphi)=\bigcap_{\delta>0}\overline{\mathfrak{R}(\varphi)(H-H_\delta)}$ where $H_\delta=\{z\in H | r_0+\delta<|z|<1-\delta\}$.

Proof Assume $\lambda\in\mathfrak{R}(\varphi)(H)$. Then, $\varphi(z)-\lambda$ has zero points on H. Without loss of generality, assume $z_0\in H$ satisfies $\varphi(z_0)=\lambda$. Then, for arbitrary $f\in\mathfrak{D}$, we have

$$\langle M_{\varphi-\lambda}f, k_{z_0}\rangle = (\varphi(z_0)-\lambda)f(z_0)\frac{1}{\|K_{z_0}\|_{\mathfrak{D}}}=0.$$

That is,

$$\langle f, M_{\varphi-\lambda}^*k_{z_0}\rangle = 0,\quad \forall f\in\mathfrak{D}.$$

Then,

$$M_\varphi^*k_{z_0}=\overline{\varphi(z_0)}k_{z_0}=\overline{\lambda}k_{z_0}.$$

Thus, $\lambda\in\sigma(M_\varphi)$, and $\mathfrak{R}(\varphi)(H)\subset\sigma(M_\varphi)$. Since $\sigma(M_\varphi)$ is closed, we have $\overline{\mathfrak{R}(\varphi)(H)}\subset\sigma(M_\varphi)$.

Conversely, if $\lambda\bar{\in}\overline{\mathfrak{R}(\varphi)(H)}$, without loss of generality, we assume $\lambda=0$, then $|\varphi(z)|$ is lower bounded on H. Thus, there exists a $\delta>0$ such that $|\varphi(z)|\ge\delta>0$ for arbitrary $z\in H$. Let $\psi(z)=\frac{1}{\varphi(z)}$. Then $\psi\in H^\infty$, and we claim that $M_\psi\in\mathcal{M}(\mathfrak{D})$. In fact, there exists a positive constant C such that

$$\begin{aligned}\int_H|\psi' f|^2\mathrm{d}A&=\int_H\frac{|\varphi'(z)|^2}{|\varphi(z)|^4}|f(z)|^2\mathrm{d}A\le\frac{1}{\delta^4}\int_H|\varphi'(z)|^2|f(z)|^2\mathrm{d}A\\&\le\frac{1}{\delta^4}C\|f\|_{\mathfrak{D}}^2,\quad\forall f\in\mathfrak{D}.\end{aligned}$$

This shows M_ψ is a bounded multiplier on $\mathfrak{D}$. Furthermore, $M_\psi M_\varphi=M_\varphi M_\psi=I$, we conclude M_φ is invertible. That is, $0\bar{\in}\sigma(M_\varphi)$. (i) is proved.

To show (ii), without loss of generality, assume $0\in\bigcap_{\delta>0}\overline{\mathfrak{R}(\varphi)(H-H_\delta)}$. Then, there is a sequence $\{z_k\}\subset\mathbb{D}$ such that $|z_k|\to1$ or $|z_k|\to r_0$, and $|\varphi(z_k)|\to0$. Since $M_\varphi^*k_{z_k}=\overline{\varphi(z_k)}k_{z_k}$, we have

$$\|M_\varphi^*k_{z_k}\|=|\varphi(z_k)|\to0.$$

Note $k_{z_k}(w)$ is a unit sequence which weakly converges to 0 as $|z_k|\to1$ or $|z_k|\to r_0$, we conclude that M_φ^* is not a Fredholm operator. This means $0\in\sigma_e(M_\varphi)$. Hence, $\bigcap_{\delta>0}\overline{\mathfrak{R}(\varphi)(H-H_\delta)}\subset\sigma_e(M_\varphi)$.

Conversely, if $0\bar{\in}\bigcap_{\delta>0}\overline{\mathfrak{R}(\varphi)(H-H_\delta)}$, then there exists a $\epsilon_0>0$ and a $\delta_0>0$ such that

$$|\varphi(z)|\ge\epsilon_0,\quad\forall z\in H-H_{\delta_0}.$$

This indicates $\varphi(z)$ has only finite zero points on H. Suppose $\{z_i\}_{i=1}^k \subset H_{\delta_0}$ is the zero point set of φ, let $\varphi_0 = \prod_{i=1}^k (z-z_i)^{k_i}$ where k_i is the repeating number of z_i as the zero point of φ. Then $\psi = \frac{\varphi}{\varphi_0}$ is analytic and has no zero point on H. Obviously, there is an $\epsilon_1 > 0$ and δ_1 with $0 < \delta_1 < \delta_0$ such that

$$|\varphi_0(z)| \geq \epsilon_1, \quad \forall z \in H - H_{\delta_1}.$$

Thus,

$$|\psi(z)| = \frac{|\varphi(z)|}{|\varphi_0(z)|} \leq \frac{|\varphi(z)|}{\epsilon_1}, \quad \forall z \in H - H_{\delta_1},$$

this implies that $\psi \in H^\infty$ by the fact $\varphi \in H^\infty$ and the maximal module principle.

We are to prove that $M_\psi \in \mathcal{M}(\mathfrak{D})$. In fact, for any $f \in \mathfrak{D}$,

$$\begin{aligned}\|M_\psi f\|_{\mathfrak{D}}^2 &\leq \int_H |(\psi f)'|^2 \mathrm{d}A + \int_H |\psi f|^2 \mathrm{d}A \\ &= \int_{H_{\delta_1}} |(\psi f)'|^2 \mathrm{d}A + \int_{H-H_{\delta_1}} |(\psi f)'|^2 \mathrm{d}A + \int_H |\psi f|^2 \mathrm{d}A. \end{aligned} \tag{3.1}$$

Note both φ and ψ' are bounded on H_{δ_1}, we see that

$$\begin{aligned}[\int_{H_{\delta_1}} |(\psi f)'|^2 \mathrm{d}A]^{\frac{1}{2}} =& [\int_{H_{\delta_1}} |\psi' f + \psi f'|^2 \mathrm{d}A]^{\frac{1}{2}} \\ \leq& [\int_{H_{\delta_1}} |\psi' f|^2 \mathrm{d}A]^{\frac{1}{2}} + [\int_{H_{\delta_1}} |\psi f'|^2 \mathrm{d}A]^{\frac{1}{2}} \\ \leq& C_1\{[\int_{H_{\delta_1}} |f|^2 \mathrm{d}A]^{\frac{1}{2}} + [\int_{H_{\delta_1}} |f'|^2 \mathrm{d}A]^{\frac{1}{2}}\} \\ \leq& C_1\{[\int_H |f|^2 \mathrm{d}A]^{\frac{1}{2}} + [\int_H |f'|^2 \mathrm{d}A]^{\frac{1}{2}}\} \\ =& C_2 \|f\|_{\mathfrak{D}}, \end{aligned} \tag{3.2}$$

where C_i $(i = 1, 2)$ are positive constants dependent on δ_1. Furthermore,

$$\begin{aligned} &[\int_{H-H_{\delta_1}} |(\psi f)'|^2 \mathrm{d}A]^{\frac{1}{2}} = [\int_{H-H_{\delta_1}} |\psi' f + \psi f'|^2 \mathrm{d}A]^{\frac{1}{2}} \\ &\leq [\int_{H-H_{\delta_1}} |\psi' f|^2 \mathrm{d}A]^{\frac{1}{2}} + [\int_{H-H_{\delta_1}} |\psi f'|^2 \mathrm{d}A]^{\frac{1}{2}} \\ &\leq [\int_{H-H_{\delta_1}} |\frac{\varphi'\varphi_0 - \varphi_0'\varphi}{\varphi_0^2} f|^2 \mathrm{d}A]^{\frac{1}{2}} + [\int_{H-H_{\delta_1}} |\frac{\varphi}{\varphi_0} f'|^2 \mathrm{d}A]^{\frac{1}{2}} \\ &\leq \frac{1}{\epsilon_1^2}[\int_{H-H_{\delta_1}} |(\varphi'\varphi_0 - \varphi_0'\varphi) f|^2 \mathrm{d}A]^{\frac{1}{2}} + \frac{1}{\epsilon_0}[\int_{H-H_{\delta_1}} |\varphi f'|^2 \mathrm{d}A]^{\frac{1}{2}}. \end{aligned} \tag{3.3}$$

Since φ_0 is a polynomial, there is a positive constant C_3 such that $\max\{\|\varphi_0\|_\infty, \|\varphi_0'\|_\infty\} \leq C_3$, which makes

$$\begin{aligned} &[\int_{H-H_{\delta_1}} |(\varphi'\varphi_0 - \varphi_0'\varphi) f|^2 \mathrm{d}A]^{\frac{1}{2}} \leq C_3[\int_{H-H_{\delta_1}} |\varphi' f|^2 \mathrm{d}A]^{\frac{1}{2}} + [\int_{H-H_{\delta_1}} |\varphi f|^2 \mathrm{d}A]^{\frac{1}{2}} \\ &\leq C_3[\int_H |\varphi' f|^2 \mathrm{d}A]^{\frac{1}{2}} + [\int_H |\varphi f|^2 \mathrm{d}A]^{\frac{1}{2}} \leq C_4 \|f\|_{\mathfrak{D}} \end{aligned} \tag{3.4}$$

where C_4 is a positive constant. Combining (3.1)–(3.4), we have $M_\psi \in \mathcal{M}(\mathfrak{D})$. By the fact that $|\varphi(z)| > \epsilon_0$ for $z \in H - H_{\delta_0}$ and $|\varphi_0(z)| \le \|\varphi_0\|_\infty$, we know that

$$|\psi(z)| = \frac{|\varphi(z)|}{|\varphi_0(z)|} \ge \frac{\epsilon_0}{\|\varphi_0\|_\infty}, \quad \forall z \in H - H_{\delta_0}.$$

Since $\psi(z)$ has no zero point on H, we see that ψ is lower bounded on H. Consequently, M_ψ is invertible on $\mathfrak{D}$, and Lemma 2.4 gives us M_{φ_0} is a Fredholm operator, this indicates $M_\varphi = M_{\varphi_0} M_\psi$ is also a Fredholm operator. That is, $0\bar{\in}\sigma_e(M_\varphi)$. Therefore, $\sigma_e(M_\varphi) \subset \bigcap_{\delta>0} \overline{\mathfrak{R}(\varphi)(H - H_\delta)}$. The proof is completed. □

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