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Characterizations of Some Operators on the Vanishing Weak Morrey Type Spaces

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Abstract In this paper we mainly give some characterizations for the boundedness of the weight Hardy operator, maximal operator, potential operator and singular integral operator on the vanishing generalized weak Morrey spaces $VW\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ with bounded set Ω.

Keywords weight Hardy operator; maximal operator; potential operator; singular integral operator; weak Morrey space

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1. Introduction

In the paper we are mainly concerned with some characterizations on the generalized Morrey space. Precisely, our aim is to give some properties for the weighted Hardy operator, maximal operator, potential operator and singular operator on the vanishing generalized weak Morrey spaces. It is well known that the classical Morrey spaces named by Morrey were firstly introduced in [1] (or refer to [2]). Later the classical Morrey spaces together with the weighted Lebesgue spaces, were applied to study the local regularity properties of solutions of partial differential equations [3]. In the local Morrey (or Morrey type) spaces and the global Morrey (or Morrey type) spaces the boundedness of various classical operators were largely considered, for example, maximal, potential, singular, Hardy operators and commutators and others, here we may refer to Adams [4], Akbulut et al. [5], Adams and Xiao [6,7], Burenkov et al. [8,9], Guliyev et al. [10,11], Chiarenza and Frasca [12], Kurata et al. [13], Komori and Shirai [14,15], Lukkassen et al. [16], Nakai et al. [17,18], Persson et al. [19,20], Softova [21], Sugano and Tanaka [22] and references therein. However, in the classical harmonic analysis the vanishing Morrey space was firstly introduced by Vitanza [23] to study the regularity results for elliptic partial differential equations, and more than nearly two decades later Ragusa [24] and Samko et al. [25,26] and references therein systematically discussed the boundedness of various classical operators in such these type of spaces. Inspired by the above statements, we continue to study Samko's results from

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[25,26] on the vanishing generalized Morrey spaces, and correspondingly obtain the related results under the weak versions, which are similar to the ones with constant exponent from Kokilashvili et al. [27,28] and Guliyev et al. [29,30]. Since the set Ω is bounded, we do not completely employ the methods of Samko, which is the real hard part of this paper. For the convenience of the better statements about our results, the remainder of this paper is organized as follows. In the rest of this section we will introduce some notation and background about the vanishing generalized Morrey spaces. In Section 2 we will be ready for some necessary lemmas. In Section 3 we will chiefly deal with our main theorems and provide their detailed proofs.

Given a non-empty measurable subset Ω of $\mathbb{R}^N$, let f be a measure function on Ω with the following norms:

$$\|f\|_{L^p(\Omega)} := \Big(\int_\Omega |f(y)|^p \mathrm{d}y\Big)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

$$\|f\|_{L^\infty(\Omega)} := \sup\{\beta : |\{y \in \Omega : |f(y)| \geq \beta\}| > 0\}.$$

For $x \in \Omega$ and $r > 0$, denote by $B(x,r)$ an open ball with x and radius r. Let $\Pi \subseteq \Omega$. The classical Morrey type space $\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ is introduced as the space of all functions f satisfying the next norm

$$\|f\|_{\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)} := \sup_{x\in\Pi, r>0} \varphi(x,r)^{-\frac{1}{p}} \|f\|_{L^p(\widetilde{B}(x,r))} < \infty,$$

where $\widetilde{B}(x,r) = B(x,r) \cap \Omega$ and $1 \leq p < \infty$. Here $\varphi(x,r)$ belongs to the class $\beth = \beth(\Pi \times [0,\ell))$ of all non-negative functions on $\Pi \times [0,\ell)$, which are positive on $\Pi \times (0,\ell)$ with $\ell = \operatorname{diam}\Omega$. Moreover, when $\Pi = \{x_0\}$ and $\Pi = \Omega$, $\mathcal{L}^{p,\varphi}_{\{x_0\}}(\Omega)$ and $\mathcal{L}^{p,\varphi}_{\Omega}(\Omega)$ are called the local generalized Morrey space and the global generalized Morrey space, respectively. If $\varphi(x,r) = r^\lambda$ and $\Pi = \Omega$, then $\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ is exactly the classical Morrey space $\mathcal{L}^{p,\lambda}(\Omega)$ for $0 \leq \lambda \leq N$. For $\lambda = 0$ and $\lambda = N$, we know that $\mathcal{L}^{p,0}(\Omega) = L^p(\Omega)$ and $\mathcal{L}^{p,N}(\Omega) = L^\infty(\Omega)$, respectively. As for $\lambda < 0$ and $\lambda > N$, we know $\mathcal{L}^{p,\lambda}(\Omega) = \Theta$, where Θ is the set of all functions equivalent to 0 on Ω.

Denote by $W\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ the weak Morrey space of all functions $f \in L^p_{\mathrm{loc}}(\Omega)$ via the norm

$$\|f\|_{W\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)} := \sup_{x\in\Pi, r>0} \varphi(x,r)^{-\frac{1}{p}} \|f\|_{WL^p(\widetilde{B}(x,r))} < \infty,$$

where $WL^p(\widetilde{B}(x,r))$ is the weak L^p-space of measurable functions f on $\widetilde{B}(x,r)$ with the norm

$$\begin{aligned}\|f\|_{WL^p(\widetilde{B}(x,r))} &\equiv |f\chi_{B(x,r)}|_{WL^p(\Omega)} := \sup_{t>0} t|\{y \in \widetilde{B}(x,r) : \|f(y)\| > t\}|^{\frac{1}{p}} \\ &= \sup_{t>0} t^{\frac{1}{p}} (f\chi_{\widetilde{B}(x,r)})^*(t) < \infty,\end{aligned}$$

here g^* is the non-increasing rearrangement of the function g.

Moreover, the vanishing generalized Morrey space $V\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ is defined as the spaces of all functions $f \in \mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ such that

$$\lim_{r\to 0} \sup_{x\in\Pi} \varphi(x,r)^{-\frac{1}{p}} \|f\|_{L^p(\widetilde{B}(x,r))} = 0. \tag{1.1}$$

Correspondingly, the vanishing generalized weak Morrey space $VW\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ is defined as the space of all functions $f \in W\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ such that

$$\lim_{r\to 0}\sup_{x\in\Pi}\varphi(x,r)^{-\frac{1}{p}}\|f\|_{WL^p(\widetilde{B}(x,r))}=0. \tag{1.2}$$

Clearly, it is appropriate to impose on $\varphi(x,r)$ with the following extra conditions:

$$\lim_{r\to 0}\sup_{x\in\Pi}\frac{r^N}{\varphi(x,r)}=0 \tag{1.3}$$

and

$$\inf_{\ell\geq r>1}\sup_{x\in\Pi}\varphi(x,r)>0, \tag{1.4}$$

where (1.4) must be imposed when Ω is unbounded. From Eqs. (1.3) and (1.4), we easily know that the bounded functions with compact support belong to $V\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ and $VW\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$.

In the paper we firstly consider the multi-dimensional weighted Hardy operators as follows.

$$\mathrm{H}^{\alpha}_{\omega}f(x)=|x|^{\alpha-N}\omega(|x|)\int_{|y|<|x|}\frac{f(y)\mathrm{d}y}{\omega(|y|)},\quad \mathcal{H}^{\alpha}_{\omega}f(x)=|x|^{\alpha}\omega(|x|)\int_{|y|>|x|}\frac{f(y)\mathrm{d}y}{|y|^N\omega(|y|)},$$

where $\alpha\geq 0$. When $N=1$, the Hardy operators above may be read either as $\mathbb{R}^1$ or $\mathbb{R}^1_+$ with

$$\mathrm{H}^{\alpha}_{\omega}f(x)=x^{\alpha-1}\omega(x)\int_0^x\frac{f(y)\mathrm{d}y}{\omega(y)},\quad \mathcal{H}^{\alpha}_{\omega}f(x)=x^{\alpha}\omega(x)\int_x^\infty\frac{f(y)\mathrm{d}y}{y\omega(y)},\ x>0.$$

If $\omega(t)=t^\beta$, then the operator above is denoted by

$$\mathrm{H}^{\alpha}_{\beta}f(x)=|x|^{\alpha+\beta-N}\int_{|y|<|x|}\frac{f(y)\mathrm{d}y}{|y|^\beta},\quad \mathcal{H}^{\alpha}_{\beta}f(x)=|x|^{\alpha+\beta}\int_{|y|>|x|}\frac{f(y)\mathrm{d}y}{|y|^{\beta+N}},$$

and the one versions

$$\mathrm{H}^{\alpha}_{\beta}f(x)=x^{\alpha+\beta-1}\int_0^x\frac{f(y)\mathrm{d}y}{y^\beta},\quad \mathcal{H}^{\alpha}_{\beta}f(x)=x^{\alpha+\beta}\int_x^\infty\frac{f(y)\mathrm{d}y}{y^{\beta+1}},\ x>0.$$

Besides this we also consider other classical operators, and we list them as follows.

- For $f\in L^1_{\mathrm{loc}}(\Omega)$, the centered Hardy-Littlewood maximal operator $\mathcal{M}f$ of the function f is defined by

$$\mathcal{M}f=\sup_{r>0}\frac{1}{|B(\cdot,r)|}\int_{\widetilde{B}(\cdot,r)}|f(y)|\mathrm{d}y,$$

where the supremum is taken over all the balls $B(\cdot,r)$ in Ω.

- The potential type operator $I^\alpha f$ with order α is denoted by

$$I^\alpha f=\int_\Omega I(\cdot,y)f(y)\mathrm{d}y,\quad 0<\alpha<N,$$

where $I(\cdot,y)=|\cdot-y|^{\alpha-N}$.

- The fractional maximal operator $\mathcal{M}^\alpha f$ with order α of the function f is defined by

$$\mathcal{M}^\alpha f=\sup_{r>0}|B(\cdot,r)|^{\frac{\alpha}{N}-1}\int_{\widetilde{B}(\cdot,r)}|f(y)|\mathrm{d}y,\quad 0\leq\alpha<N,$$

where the supremum is taken over all the balls $B(\cdot,r)$ in Ω.

- The Calderón-Zygmund type singular integral operator is denoted by

$$T_{CZ}f=\int_\Omega K(\cdot,y)f(y)\mathrm{d}y,$$

here $K(\cdot,\cdot)$ is the standard singular kernel. That is to say, $K(x,y)$ is a continuous function on $\{(x,y)\in\Omega\times\Omega : x\neq y\}$ and satisfies the following conditions:

$$|K(x,y)|\leq C|x-y|^{-N}\quad\text{for all}\quad x\neq y,$$

$$|K(x,y)-K(x,z)|\leq C\frac{|y-z|^{\sigma}}{|x-y|^{N+\sigma}},\sigma>0,\quad\text{if}\quad |x-y|>2|y-z|,$$

$$|K(x,y)-K(\xi,y)|\leq C\frac{|x-\xi|^{\sigma}}{|x-y|^{N+\sigma}},\sigma>0,\quad\text{if}\quad |x-y|>2|x-\xi|.$$

Let f be a non-negative function on $[0,\ell]$. If there exists a constant $C\geq 1$ such that $f(x)\leq Cf(y)$ for all $x\leq y$ or $x\geq y$, then f is named almost increasing or decreasing function. Moreover, if f and g are the two almost increasing or decreasing functions and satisfy $c_1 f\leq g\leq c_2 f$ for $c_1,c_2>0$, then they are equivalent.

Definition 1.1 *Let $0<\ell<\infty$.*

- *Denote by $W=W([0,\ell])$ the class of continuous and positive functions $\phi(r)$ on $(0,\ell]$ such that the limit $\lim_{r\to 0}\phi(r)$ exists and is finite;*
- *Denote by $W_0=W_0([0,\ell])$ the class of almost increasing functions $\phi(r)\in W$ on $(0,\ell)$;*
- *Denote by $\overline{W}=\overline{W}([0,\ell])$ the class of functions $\phi(r)\in W$ such that $r^a\phi(r)\in W_0$ for some $a=a(\phi)\in\mathbb{R}$;*
- *Denote by $\underline{W}=\underline{W}([0,\ell])$ the class of functions $\phi(r)\in W$ such that $r^{-b}\phi(r)$ is almost decreasing for some $b\in\mathbb{R}$.*

2. Some necessary Lemmas

In the section we are prepared to provide and prove the related lemmas. At first we give two results being similar to the ones from Persson and Samko [19, Propositions 3.6 and 3.8].

Lemma 2.1 *For $1\leq p<\infty$, $0<s\leq p$ and $0<\ell\leq\infty$, let $\nu(t)\in\overline{W}([0,\ell])$, $\nu(2t)\leq C\nu(t)$, $\frac{\varphi^{\frac{s}{p}}(x,\cdot)}{\nu}\in\underline{W}([0,\ell])$ for $x\in\Pi$. Then*

$$\Big(\int_{|z|<|y|}\frac{|f(z)|^s}{\nu(|z|)}\mathrm{d}z\Big)^{\frac{1}{s}}\leq C\mathcal{D}(|y|)\|f\|_{\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)},\quad 0<|y|\leq\ell,$$

where $C>0$ does not depend on y and f, and

$$\mathcal{D}(r)=\Big(\int_0^r t^{N(1-\frac{s}{p})-1}\frac{\varphi^{\frac{s}{p}}(x,t)}{\nu(t)}\mathrm{d}t\Big)^{\frac{1}{s}}\quad\text{for}\quad x\in\Pi.$$

Lemma 2.2 *For $1\leq p<\infty$ and $0\leq s\leq p$, let $\varphi(r)\geq Cr^N$ and $\nu(t)\in\overline{W}(\mathbb{R}_+)$. Then*

$$\Big(\int_{|z|>|y|}|f(z)|^s\nu(|z|)\mathrm{d}z\Big)^{\frac{1}{s}}\leq C\mathcal{E}(|y|)\|f\|_{\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)},\quad y\neq 0,$$

where $C>0$ does not depend on y and f, and

$$\mathcal{E}(r)=\Big(\int_r^\infty t^{N(1-\frac{s}{p})-1}\varphi^{\frac{s}{p}}(x,t)\nu(t)\mathrm{d}t\Big)^{\frac{1}{s}}\quad\text{for}\quad x\in\Pi.$$

Next we state the lemma below, which is similar to the result from Samko [26, Lemma 3.4]. Because ℓ is finite, here we have to modify slightly the procedure of Samko and give our proof in detail.

Lemma 2.3 *For $1 \le p \le \infty$, $\alpha \in \mathbb{R}$, $x \in \Omega$ and $0 < r < \operatorname{diam}(\Omega)$. Then*

$$\int_{\Omega \setminus B(x,2r)} \frac{|f(z)|}{|x-y|^{N-\alpha}} \mathrm{d}y \le C \int_r^\ell \frac{\|f\|_{L^p(B(x,s))} \mathrm{d}s}{s^{\frac{N}{p}+1-\alpha}},$$

where $C > 0$ does not depend on x, f and r.

Proof For $x \in \Omega$, we may take $\beta > \max\{\frac{N}{p} - \alpha, 0\}$ and specifically proceed as follows:

$$\begin{aligned}
\int_{\Omega \setminus B(x,2r)} \frac{|f(y)|}{|x-y|^{N-\alpha}} \mathrm{d}y &= \frac{\beta 2^\beta}{2^\beta - 1} \int_{\Omega \setminus B(x,2r)} \frac{|f(y)|}{|x-y|^{N-\alpha-\beta}} \Big(\int_{|x-y|}^{2|x-y|} \frac{\mathrm{d}s}{s^{\beta+1}} \Big) \mathrm{d}y \\
&\le C \int_{\Omega \setminus B(x,2r)} \frac{|f(y)|}{|x-y|^{N-\alpha-\beta}} \Big(\int_{|x-y|}^{2\ell} \frac{\mathrm{d}s}{s^{\beta+1}} \Big) \mathrm{d}y \\
&\le C \int_{2r}^{2\ell} \frac{1}{s^{\beta+1}} \Big(\int_{\{y \in \Omega, 2r \le |x-y| \le s\}} \frac{|f(y)| \mathrm{d}y}{|x-y|^{N-\alpha-\beta}} \Big) \mathrm{d}s \\
&\le C \int_r^\ell s^{-\beta-1} \|f\|_{L^p(B(x,s))} \||x-y|^{\alpha-N+\beta}\|_{L^{p'}(B(x,s))} \mathrm{d}s \\
&\le C \int_r^\ell s^{\alpha - \frac{N}{p} - 1} \|f\|_{L^p(B(x,s))} \mathrm{d}s,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, and $C > 0$ does not depend on x, f and r. □

3. Statements of main results

Next we start to state our main theorems and their proofs. Firstly we consider the boundedness of weighted Hardy operator in the weak Morrey type spaces.

Theorem 3.1 *Let $1 \le p, q < \infty$ and $\varphi \in \beth$ satisfy (1.2)–(1.4).*

(I) Suppose that

$$\omega \in \overline{W}([0,\ell]),\ \omega(2t) \le C\omega(t), \frac{\varphi^{\frac{1}{p}}(x,\cdot)}{\omega} \in \underline{W}([0,\ell]).$$

If

$$\sup_{x \in \Pi, r>0} \frac{1}{\varphi(x,r)} \int_{\widetilde{B}(x,r)} \omega^q(|y|)|y|^{q(\alpha-N)} \Big(\int_0^{|y|} \frac{t^{\frac{N}{p'}-1} \varphi^{\frac{1}{p}}(x,t)}{\omega(t)} \mathrm{d}t \Big)^q \mathrm{d}y < \infty, \tag{3.1}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, then the weighted Hardy operator H_ω^α is bounded from $V\mathcal{L}_\Pi^{p,\varphi}(\Omega)$ to $VW\mathcal{L}_\Pi^{q,\varphi}(\Omega)$.

(II) Suppose that

$$\omega \in \overline{W}([0,\ell]) \text{ and } \omega(2t) \le C\omega(t) \text{ or } \frac{1}{\omega} \in \underline{W}([0,\ell]).$$

If

$$\sup_{x \in \Pi, r>0} \frac{1}{\varphi(x,r)} \int_{\widetilde{B}(x,r)} \omega^q(|y|)|y|^{q\alpha} \Big(\int_{|y|}^\ell \frac{t^{-\frac{N}{p}-1} \varphi^{\frac{1}{p}}(x,t)}{\omega(t)} \mathrm{d}t \Big)^q \mathrm{d}y < \infty, \tag{3.2}$$

then the weighted Hardy operator operator $\mathcal{H}_\omega^\alpha$ is bounded from $V\mathcal{L}_\Pi^{p,\varphi}(\Omega)$ to $VW\mathcal{L}_\Pi^{q,\varphi}(\Omega)$.

Proof Note that

$$\|\mathrm{H}_\omega^\alpha f\|_{WL^q(\Omega)} \lesssim \|\mathrm{H}_\omega^\alpha f\|_{L^q(\Omega)}.$$

Set $s=1$ and $\nu(t)=\omega(t)$ in Lemma 2.1. Then

$$|\mathrm{H}_\omega^\alpha f(y)| \le C\omega(|y|)|y|^{\alpha-N}\int_0^{|y|}\frac{t^{\frac{N}{p'}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}\mathrm{d}t\|f\|_{\mathcal{L}_\Pi^{p,\varphi}(\Omega)}$$

for $y\in\widetilde{B}(x,r)$ with $x\in\Pi$, and we obtain

$$\|\mathrm{H}_\omega^\alpha f\|_{WL^q(\widetilde{B}(x,r))}^q \lesssim \int_{\widetilde{B}(x,r)}\omega^q(|y|)|y|^{q(\alpha-N)}\Big(\int_0^{|y|}\frac{t^{\frac{N}{p'}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}\mathrm{d}t\Big)^q\mathrm{d}y. \tag{3.3}$$

That is to say

$$\|\mathrm{H}_\omega^\alpha f\|_{W\mathcal{L}_\Pi^{q,\varphi}(\Omega)}^q \lesssim \sup_{x\in\Pi,r>0}\frac{1}{\varphi(x,r)}\int_{\widetilde{B}(x,r)}\omega^q(|y|)|y|^{q(\alpha-N)}\Big(\int_{|y|}^{\ell}\frac{t^{\frac{N}{p'}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}\mathrm{d}t\Big)^q\mathrm{d}y. \tag{3.4}$$

Hence $\mathrm{H}_\omega^\alpha f\in W\mathcal{L}_\Pi^{q,\varphi}(\Omega)$. On the other hand, by the inequality (3.3) and Eq. (1.3) we get that

$$\lim_{r\to 0}\sup_{x\in\Pi}\varphi^{-\frac{1}{q}}(x,r)\|\mathrm{H}_\omega^\alpha f\|_{WL^q(\widetilde{B}(x,r))}=0,$$

which implies $\mathrm{H}_\omega^\alpha f\in VW\mathcal{L}_\Pi^{q,\varphi}(\Omega)$, i.e., the operator H_ω^α is bounded from $V\mathcal{L}_\Pi^{p,\varphi}(\Omega)$ to $VW\mathcal{L}_\Pi^{q,\varphi}(\Omega)$.

Similarly, once we apply Lemma 2.2 into $\mathcal{H}_\omega^\alpha$, we have that

$$|\mathcal{H}_\omega^\alpha f(y)| \le C\omega(|y|)|y|^{\alpha}\int_{|y|}^{\ell}\frac{t^{-\frac{N}{p}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}\mathrm{d}t\|f\|_{\mathcal{L}_\Pi^{p,\varphi}(\Omega)}$$

for $y\in\widetilde{B}(x,r)$ with $x\in\Pi$, and we know that

$$\|\mathcal{H}_\omega^\alpha f\|_{WL^q(\widetilde{B}(x,r))}^q \lesssim \int_{\widetilde{B}(x,r)}\omega^q(|y|)|y|^{q\alpha}\Big(\int_{|y|}^{\ell}\frac{t^{-\frac{N}{p}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}\mathrm{d}t\Big)^q\mathrm{d}y. \tag{3.5}$$

Therefore,

$$\|\mathcal{H}_\omega^\alpha f\|_{W\mathcal{L}_\Pi^{q,\varphi}(\Omega)}^q \lesssim \sup_{x\in\Pi,r>0}\frac{1}{\varphi(x,r)}\int_{\widetilde{B}(x,r)}\omega^q(|y|)|y|^{q\alpha}\Big(\int_{|y|}^{\ell}\frac{t^{-\frac{N}{p}-1}\varphi^{\frac{1}{p}}(x,t)}{\omega(t)}\mathrm{d}t\Big)^q\mathrm{d}y \tag{3.6}$$

holds, and so follows $\mathcal{H}_\omega^\alpha f\in W\mathcal{L}_\Pi^{q,\varphi}(\Omega)$. Moreover, with the inequality (3.5) and Eq. (1.3) we obtain that

$$\lim_{r\to 0}\sup_{x\in\Pi}\varphi^{-\frac{1}{q}}(x,r)\|\mathcal{H}_\omega^\alpha f\|_{WL^q(\widetilde{B}(x,r))}=0,$$

which implies $\mathcal{H}_\omega^\alpha f\in VW\mathcal{L}_\Pi^{q,\varphi}(\Omega)$. Then we may conclude the operator $\mathcal{H}_\omega^\alpha$ is also bounded from $V\mathcal{L}_\Pi^{p,\varphi}(\Omega)$ to $VW\mathcal{L}_\Pi^{q,\varphi}(\Omega)$. $\Box$

Now we recall the definition of p-admissible singular operator. A sublinear operator T, that is to say $|T(f+g)|\le|Tf|+|Tg|$, is called p-admissible singular operator if it satisfies the next two conditions:

- T satisfies the size conditions of the form as

$$\chi_{B(x,r)}(z)|T(f\chi_{\mathbb{R}^N\setminus B(x,2r)})(z)| \le C\chi_{B(x,r)}(z)\int_{\mathbb{R}^N\setminus B(x,2r)}\frac{|f(y)|\mathrm{d}y}{|y-z|^N} \tag{3.7}$$

for $x \in \mathbb{R}^N$ and $r > 0$;

- T is bounded in $L^p(\mathbb{R}^N)$.

For two similar concepts: Φ-admissible singular operator and (Φ, Ψ)-admissible potential operator, we refer to [31] and references therein. Here we remark that the maximal operator $\mathcal{M}$ and the Calderón-Zygmund type singular integral operator T_{CZ} with standard kernel are p-admissible singular operators.

Theorem 3.2 *Let $\varphi \in \beth$ satisfy (1.2)–(1.4). Every sublinear p-admissible singular operator T is bounded from the vanishing generalized Morrey space $V\mathcal{L}_{\Pi}^{p,\varphi}(\Omega)$ to the vanishing generalized Morrey space $VW\mathcal{L}_{\Pi}^{p,\varphi}(\Omega)$, if the quantity*

$$C_\delta := \int_\delta^\ell \frac{\sup_{x\in\Pi} \varphi^{\frac{1}{p}}(x,t)\mathrm{d}t}{t^{\frac{N}{p}+1}} < \infty \tag{3.8}$$

for each $\delta > 0$ and

$$\int_r^\ell \frac{\varphi^{\frac{1}{p}}(x,t)\mathrm{d}t}{t^{\frac{N}{p}+1}} \leq C_0 \frac{\varphi^{\frac{1}{p}}(x,r)}{r^{\frac{N}{p}}}, \tag{3.9}$$

where C_0 does not depend on $x \in \Pi$ and $r > 0$.

Proof For arbitrary $x \in \Pi$, let $\widetilde{B}(x,r) = B(x,r) \cap \Omega$ for the ball $B(x,r)$ centered at x and of radius r, where $r < \operatorname{diam}(\Omega)/2$. Now we write $f = f_1 + f_2$, where $f_1 = f\chi_{\widetilde{B}(x,2r)}$ and $f_2 = f\chi_{\Omega\setminus\widetilde{B}(x,2r)}$. Therefore, we know that

$$\begin{aligned}
\|Tf\|_{WL^p(\widetilde{B}(x,r))} :=& \sup_{t>0} t|\{y \in \widetilde{B}(x,r) : |Tf(y)| > t\}|^{\frac{1}{p}} \\
\lesssim& \sup_{t>0} t|\{y \in \widetilde{B}(x,r) : |Tf_1(y)| > t/2\}|^{\frac{1}{p}} + \\
& \sup_{t>0} t|\{y \in \widetilde{B}(x,r) : |Tf_2(y)| > t/2\}|^{\frac{1}{p}} \\
\approx& \|Tf_1\|_{WL^p(\widetilde{B}(x,r))} + \|Tf_2\|_{WL^p(\widetilde{B}(x,r))}.
\end{aligned}$$

From the boundedness of T in $L^p(\mathbb{R}^N)$ it naturally follows that

$$\|Tf_1\|_{WL^p(\widetilde{B}(x,r))} \leq \|Tf_1\|_{L^p(\widetilde{B}(x,r))} \leq \|Tf_1\|_{L^p(\mathbb{R}^N)} \leq C\|f_1\|_{L^p(\mathbb{R}^N)} = C\|f\|_{L^p(\widetilde{B}(x,2r))}.$$

Since

$$\|f\|_{L^p(\widetilde{B}(x,2r))} \lesssim r^{\frac{N}{p}} \int_{2r}^\ell t^{-\frac{N}{p}-1}\|f\|_{L^p(\widetilde{B}(x,t))}\mathrm{d}t,$$

we declare that

$$\|Tf_1\|_{WL^p(\widetilde{B}(x,r))} \lesssim r^{\frac{N}{p}} \int_r^\ell \frac{\varphi^{\frac{1}{p}}(x,t)}{t^{1+\frac{N}{p}}}\mathrm{d}t \tag{3.10}$$

holds. On the other hand, for $z \in \widetilde{B}(x,r)$ we have

$$|Tf_2(z)| \leq C \int_{\Omega\setminus\widetilde{B}(x,2r)} \frac{|f(y)|\mathrm{d}y}{|y-z|^N}.$$

Observe that the inequality $\frac{|y-z|}{2} \leq |x-y| \leq \frac{3|y-z|}{2}$ holds for $z \in B(x,r)$ and $y \in \Omega \setminus B(x,2r)$. Therefore,

$$\|Tf_2\|_{WL^p(\widetilde{B}(x,r))} \leq C \int_{\Omega\setminus\widetilde{B}(x,2r)} \frac{|f(y)|\mathrm{d}y}{|x-y|^N} \parallel \chi_{\widetilde{B}(x,r)}\|_{WL^{p(\cdot)}(\Omega)}.$$

Since $\|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)} \sim r^{\frac{N}{p}}$, by the Hölder inequality or Lemma 2.3 it follows

$$\|Tf_2\|_{WL^p(\widetilde{B}(x,r))} \leq Cr^{\frac{N}{p}} \int_r^{\ell} t^{-\frac{N}{p}-1}\|f\|_{L^p(\widetilde{B}(x,t))}\mathrm{d}t.$$

So

$$\|Tf_2\|_{L^p(\widetilde{B}(x,r))} \lesssim r^{\frac{N}{p}} \int_r^{\ell} \frac{\varphi^{\frac{1}{p}}(x,t)}{t^{1+\frac{N}{p}}}\mathrm{d}t. \tag{3.11}$$

By the inequalities (3.10) and (3.11) we see

$$\|Tf\|_{WL^p(\widetilde{B}(x,r))} \lesssim r^{\frac{N}{p}} \int_r^{\ell} \frac{\varphi^{\frac{1}{p}}(x,t)}{t^{1+\frac{N}{p}}}\mathrm{d}t.$$

Together with the inequalities (3.8) and (3.9), it easily follows that the p-admissible singular operator T is bounded from the vanishing generalized Morrey space $V\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ to the vanishing generalized Morrey space $VW\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$. $\square$

Because the maximal operator $\mathcal{M}$ and the Calderón-Zygmund type singular integral operator T_{CZ} with standard kernel are p-admissible singular operators, by Theorem 3.2 we may obtain the following corollary.

Corollary 3.3 *Let $\varphi(x,t)$ satisfy (1.2)–(1.4), (3.8) and (3.9). Then the maximal operator $\mathcal{M}$ and the Calderón-Zygmund type singular integral operator T_{CZ} with standard kernel are bounded from the vanishing generalized Morrey space $V\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ to the vanishing generalized Morrey space $VW\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$.*

Theorem 3.4 *Let $0<\alpha<N$, $1<p<\frac{N}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{N}$ and $\varphi,\psi\in\beth$ satisfy (1.2)–(1.4). If the quantity*

$$C_\delta := \int_\delta^{\ell} \frac{\sup_{x\in\Pi}\varphi^{\frac{1}{p}}(x,t)\mathrm{d}t}{t^{1+\frac{N}{q}}} < \infty \tag{3.12}$$

for each $\delta>0$ and

$$\int_r^{\ell} \frac{\varphi^{\frac{1}{p}}(x,t)\mathrm{d}t}{t^{1+\frac{N}{q}}} \leq C\frac{\psi^{\frac{1}{q}}(x,r)}{r^{\frac{N}{q}}}, \tag{3.13}$$

where C does not depend on $x\in\Pi$ and $r>0$, then the operators $\mathcal{M}^\alpha$ and I^α are bounded from the vanishing generalized Morrey space $V\mathcal{L}^{p,\varphi}_{\Pi}(\Omega)$ to the vanishing generalized weak Morrey space $VW\mathcal{L}^{q,\psi}_{\Pi}(\Omega)$.

Proof Since $\mathcal{M}^\alpha f \leq CI^\alpha(|f|)$, here we only have to consider the case for I^α. As the same methods in Theorem 3.2, we also split the function f into the forms $f=f_1+f_2$ so that

$$I^\alpha f = I^\alpha f_1 + I^\alpha f_2.$$

Clearly, we see that

$$\|I^\alpha f\|_{WL^q(\widetilde{B}(x,r))} \lesssim \|I^\alpha f_1\|_{WL^q(\widetilde{B}(x,r))} + \|I^\alpha f_2\|_{WL^q(\widetilde{B}(x,r))}$$

for each $x\in\Pi$. By the classical Sobolev theorem we get

$$\|I^\alpha f_1\|_{WL^q(\widetilde{B}(x,r))} \lesssim \|I^\alpha f_1\|_{L^q(\widetilde{B}(x,r))} \leq \|I^\alpha f_1\|_{L^q(\Omega)} \leq C\|f_1\|_{L^p(\Omega)} = C\|f\|_{L^p(\widetilde{B}(x,2r))}.$$

Therefore,

$$\|I^\alpha f_1\|_{WL^q(\widetilde{B}(x,r))} \lesssim r^{\frac{N}{q}} \int_r^\ell \frac{\varphi^{\frac{1}{p}}(x,t)}{t^{1+\frac{N}{q}}}\mathrm{d}t. \tag{3.14}$$

Since the inequality $\frac{|y-z|}{2} \le |x-y| \le \frac{3|y-z|}{2}$ holds for $z \in B(x,r)$ and $y \in \Omega \setminus B(x,2r)$, we infer that

$$\|I^\alpha f_2\|_{L^q(\widetilde{B}(x,r))} \le Cr^{\frac{N}{q}} \int_{\Omega\setminus\widetilde{B}(x,2r)} \frac{|f(y)|\mathrm{d}y}{|x-y|^{N-\alpha}}.$$

By Lemma 2.3 it follows that

$$\|I^\alpha f_2\|_{L^q(\widetilde{B}(x,r))} \le Cr^{\frac{N}{q}} \int_r^\ell t^{-\frac{N}{q}-1}\|f\|_{L^p(\widetilde{B}(x,t))}\mathrm{d}t.$$

Hence

$$\|I^\alpha f_2\|_{L^q(\widetilde{B}(x,r))} \lesssim r^{\frac{N}{q}} \int_r^\ell \frac{\varphi^{\frac{1}{p}}(x,t)}{t^{1+\frac{N}{q}}}\mathrm{d}t. \tag{3.15}$$

From the inequalities (3.14) and (3.15), we obtain that

$$\|I^\alpha f\|_{WL^q(\widetilde{B}(x,r))} \lesssim r^{\frac{N}{q}} \int_r^\ell \frac{\varphi^{\frac{1}{p}}(x,t)}{t^{1+\frac{N}{q}}}\mathrm{d}t.$$

In view of (1.2)–(1.4), and the inequalities (3.12) and (3.13), it follows that the potential operator I^α is bounded from the vanishing generalized Morrey space $V\mathcal{L}_\Pi^{p,\varphi}(\Omega)$ to another vanishing generalized Morrey space $V\mathcal{L}_\Pi^{p,\psi}(\Omega)$. □

Corollary 3.5 *Let $0<\alpha,\lambda<N$, $1\le p<\frac{N-\lambda}{\alpha}$ and $\frac{1}{q} \le \frac{1}{p}-\frac{\alpha}{N}$. Then the operators $\mathcal{M}^\alpha$ and I^α are bounded from the vanishing generalized Morrey space $V\mathcal{L}_\Pi^{p,\lambda}(\Omega)$ to the vanishing generalized weak Morrey space $VW\mathcal{L}_\Pi^{q,\mu}(\Omega)$, where $\frac{\mu}{q} \le \frac{\lambda}{p}$.*

Proof Let $\varphi(x,r)=r^\lambda$ and $\psi(x,r)=r^\mu$ in Theorem 3.4. Then we know

$$\begin{aligned}\|I^\alpha f\|_{W\mathcal{L}_\Pi^{q,\mu}(\Omega)} &\le C \sup_{x\in\Pi,\ell>r>0} r^{\frac{N-\mu}{q}} \int_r^\ell t^{-\frac{N}{q}-1}\|f\|_{L^p(\widetilde{B}(x,t))}\mathrm{d}t \\ &\le C\|f\|_{\mathcal{L}_\Pi^{p,\lambda}(\Omega)} \sup_{x\in\Pi,\ell>r>0} r^{\frac{N-\mu}{q}} \int_r^\ell t^{\frac{\lambda}{p}-\frac{N}{q}-1}\mathrm{d}t \\ &\le C\|f\|_{\mathcal{L}_\Pi^{p,\lambda}(\Omega)}\end{aligned}$$

and

$$\lim_{r\to 0}\sup_{x\in\Pi} r^{\frac{-\mu}{q}}\|I^\alpha f\|_{WL^q(\widetilde{B}(x,r))} \le C\lim_{r\to 0}\sup_{x\in\Pi} r^{-\frac{\lambda}{p}}\|f\|_{L^p(\widetilde{B}(x,r))} = 0.$$

Therefore, Corollary 3.5 holds. □

Corollary 3.6 *Let $0<\alpha,\lambda<N$ and $1\le p<\frac{N-\lambda}{\alpha}$. Then the operators $\mathcal{M}^\alpha$ and I^α are bounded from the vanishing generalized Morrey space $V\mathcal{L}_\Pi^{p,\lambda}(\Omega)$ to the vanishing generalized weak Morrey space $VW\mathcal{L}_\Pi^{q,\mu}(\Omega)$, where $\frac{Np}{N-\alpha p}<q$ and $\frac{N-\mu}{q}=\frac{N-\lambda}{p}-\alpha$.*

Proof By the classical Sobolev theorem we know that

$$\|I^\alpha f\|_{W\mathcal{L}_\Pi^{q,\mu}(\Omega)} \le C\sup_{x\in\Pi,\ell>r>0} r^{-\frac{\mu}{q}}\|f\|_{L^p(\widetilde{B}(x,r))} \le C\|f\|_{\mathcal{L}_\Pi^{p,\lambda}(\Omega)} \sup_{x\in\Pi,\ell>r>0} r^{\frac{\lambda}{p}-\frac{\mu}{q}}$$

$$= C\|f\|_{\mathcal{L}_{\Pi}^{p,\lambda}(\Omega)} \sup_{x\in\Pi,\ell>r>0} r^{\frac{N}{p}-\frac{N}{q}-\alpha} \le C\|f\|_{\mathcal{L}_{\Pi}^{p,\lambda}(\Omega)}$$

and

$$\lim_{r\to 0}\sup_{x\in\Pi} r^{\frac{-\mu}{q}}\|I^\alpha f\|_{WL^q(\widetilde{B}(x,r))} \le C\lim_{r\to 0} r^{\frac{\lambda}{p}-\frac{\mu}{q}}\sup_{x\in\Pi} r^{-\frac{\lambda}{p}}\|f\|_{L^p(\widetilde{B}(x,r))} = 0.$$

Hence, Corollary 3.6 follows. □

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