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# Mixture Representation of Residual Lifetimes of Living Components in Failed Coherent Systems

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**Abstract** In this paper we consider a coherent system consisting of $n$ components with independent and identically distributed components. We obtain a mixture representation of the reliability function of the residual lifetime of living components of a coherent system when the system has failed. Based on the concept of signature, stochastic comparisons on the residual life of two systems with stochastically ordered signatures are conducted.

**Keywords** Coherent system; residual lifetime; stochastic ordering; signature; increasing failure rate

**MR(2010) Subject Classification** 62N05; 60E15

## 1. Introduction

Coherent systems are very important in reliability theory and survival analysis. A system is said to be coherent if it has no irrelevant components and the structure function of the system is monotone in every component. Samaniego [1] introduced a concept of "signature" of a coherent system, which can represent the lifetime distribution function of a coherent system with $n$ independent and identically distributed component lifetimes as a mixture of the distribution function of the ordered lifetimes of its components. Let $X_1, \dots, X_n$ be absolutely continuous random variables with continuous distribution function $F$, representing the lifetimes of $n$ independent and identically distributed components of a coherent system. Let $T = \tau(X_1, \dots, X_n)$ be the lifetime of the system, where $\tau$ is a structural function [2]. Samaniego defined the signature of the system as a probability vector $\boldsymbol{s} = (s_1, s_2, \dots, s_n)$ with

$$s_i = P(T = X_{i:n}), \quad i = 1, \dots, n,$$

where $X_{i:n}$ denotes the $i$th ordered lifetime of the components lifetimes $X_1, \dots, X_n$, and $\sum_{i=1}^n s_i = 1$. It can be shown that $s_i = \frac{A_i}{n!}$, the probability that the $i$th failure of components causes the failure of the system, where $A_i$ is the number of permutations of the component lifetimes for which a particular ordered components failure is fatal to the system. As the signature vector does not depend on the common distribution function of $X_i$, the distribution function of $T$ can be

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represented as a mixture of the distribution functions of $X_{1:n}, \ldots, X_{n:n}$ with weights $s_1, \ldots, s_n$. That is,

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t),$$

where $\bar{F}_T(t) = P(T > t)$ is the survival function of the system lifetime.

In the literature, many authors paid their attentions to the residual lifetime of coherent systems when the component lifetimes are assumed to be independent and identical distribution or exchangeable dependent components. Interested reader can refer to [3-23] and the references therein.

In practice, many coherent systems with $n$ components have the property that some components would never fail even the system has failed. Such systems have the signature vector as follows:

$$\boldsymbol{s} = (s_1, \ldots, s_i, 0, \ldots, 0) \tag{1.1}$$

with $s_i > 0$ for $1 < i < n$. The components with lifetimes $X_{j:n}$ $(j = i+1, \ldots, n)$ must be alive even when the system failed. Hence, after the failure of the system, the surviving components may be removed from the system and then can be used for other systems or testing purposes. Thus, it could be important for system designers to get some useful information on the reliability properties of the surviving components in the used coherent system. Therefore, many authors investigated the stochastic properties of surviving components when the system has failed at some time. Suppose, for example, consider a coherent system with 4 exchangeable components, whose signature is $\boldsymbol{s} = (\frac{1}{2}, \frac{1}{2}, 0, 0)$. Then the system would fail upon the second component failure, but clearly there are still two surviving components.

Balakrishnan and Asadi [24] have discussed the residual lifetime of the living components of system having signature as (1.1), under the condition that the system is working at time $t$; that is, $(X_{k:n} - t \mid T > t)$, $k = i+1, \ldots, n$. Kelkin Nama and Asadi [25] have considered another case, given that the system exactly failed at time $t$, that is, $(X_{k:n} - t \mid T = t)$, $k = i+1, \ldots, n$.

Recently, Goliforushani, Asadi and Balakrishnan [26] have demonstrated a kind of conditional random variable $(X_{k:n} - t \mid T < t < X_{k:n})$, $k = i+1, \ldots, n$, which represents the residual lifetimes of the components of the system with signature vector of (1.1), under the condition that the system has failed by time $t$.

In fact, the event "$(T < t < X_{k:n})$" and event "$(T < t)$" have the same meaning for the coherent systems with the signature form in (1.1). Of course, their representations of the reliability functions are different and hence distribution and density functions. Based on this, this paper considers the conditional random variable $(X_{k:n} - t \mid T < t)$, $k = i+1, \ldots, n$, which shows in fact the residual lifetime of the surviving components with lifetime $X_{k:n}$, $k = i+1, \ldots, n$ when the system has failed before time $t$.

The rest of the paper is organized as follows. In Section 2, we first introduce some notations and review some of the concepts that will be used later in the paper. In Section 3, in the case when the components lifetimes are independent and identically distributed, we obtain a mixture representation of the remaining surviving components and discuss its stochastic properties. In Section 4, we conduct some stochastic comparisons of the residual lifetime between two systems with respect to some orders.

## 2. Notations and definitions

In this section we recall several criteria which are useful in the main results presented in the paper. Let $X$ and $Y$ be two random variables with respective distribution functions $F(x)$ and $G(x)$, density functions $f(x)$ and $g(x)$, and reliability functions $\bar{F}(x)$ and $\bar{G}(x)$, respectively.

**Definition 2.1** *The random variable $X$ is said to have increasing (decreasing) failure rate, IFR (DFR), if the hazard rate (or failure rat) $r_X(t) = \frac{f(t)}{\bar{F}(t)}$ is increasing (decreasing) in $t$.*

**Definition 2.2** *$X$ is said to be smaller than $Y$ in the*

*(a) Usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x$.*

*(b) Hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{F}(x)/\bar{G}(x)$ is decreasing in $x$.*

*(c) Reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $F(x)/G(x)$ is decreasing in $x$.*

*(d) Likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f(x)/g(x)$ is decreasing in the union of their supports.*

**Definition 2.3** *For two discrete distributions $\mathbf{s} = (s_{1,1}, \ldots, s_{1,n})$ and $\mathbf{s} = (s_{2,1}, \ldots, s_{2,n})$, $\mathbf{s}_1$ is said to be smaller than $\mathbf{s}_2$ in the*

*(a) Usual stochastic order (denoted by $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$) if $\sum_{j=i}^{n} s_{1,j} \leq \sum_{j=i}^{n} s_{2,j}$ for all $i = 1, 2, \ldots, n$.*

*(b) Hazard rate order (denoted by $\mathbf{s}_1 \leq_{hr} \mathbf{s}_2$) if $\sum_{j=i}^{n} s_{1,j} / \sum_{j=i}^{n} s_{2,j}$ is decreasing in $i$.*

*(c) Reversed hazard rate order (denoted by $\mathbf{s}_1 \leq_{rh} \mathbf{s}_2$) if $\sum_{j=1}^{i} s_{1,j} / \sum_{j=1}^{i} s_{2,j}$ is decreasing in $i$.*

*(d) Likelihood ratio order (denoted by $\mathbf{s}_1 \leq_{lr} \mathbf{s}_2$) if $s_{1,i}/s_{2,i}$ is decreasing in $i$.*

For more comprehensive discussions of the the properties and other details of these stochastic orders, the reader can refer to [27].

## 3. Mixture representation of residual lifetimes of the living components

Let $T$ be the lifetime of a coherent system with $n$ i.i.d components, and let $X_1, \ldots, X_n$ be the component lifetimes having a common continuous distribution $F$. Furthermore, we assume that the signature of the system has the form of (1.1); that is,

$$\mathbf{s} = (s_1, \ldots, s_i, 0, \ldots, 0), \quad s_i > 0, \ 1 \leq i < n.$$

Firstly, we give the reliability function of conditional random variables $(X_{k:n} - t \mid T < t)$ in the following theorem.

**Theorem 3.1** *For $1 \le i \le k \le n$ and $0 \le x \le t$,*

$$P(X_{k:n} - t > x \mid T < t) = \sum_{j=1}^{i} s_j(t) P(X_{k:n} - t > x \mid X_{j:n} < t). \tag{3.1}$$

**Proof** For all $0 \le x \le t$ and $1 \le i < k \le n$,

$$\begin{aligned} P(X_{k:n} - t > x \mid T < t) =& \frac{P(X_{k:n} - t > x, T < t)}{P(T < t)} \\ =& \frac{\sum_{j=1}^{i} P(T = X_{j:n}) P(X_{k:n} - t > x, T < t \mid T = X_{j:n})}{\sum_{m=1}^{i} P(T = X_{m:n}) P(T < t \mid T = X_{m:n})} \\ =& \frac{\sum_{j=1}^{i} s_j P(X_{k:n} - t > x, X_{j:n} < t)}{\sum_{m=1}^{i} s_m P(X_{m:n} < t)} \\ =& \frac{\sum_{j=1}^{i} s_j P(X_{k:n} - t > x \mid X_{j:n} < t) P(X_{j:n} < t)}{\sum_{m=1}^{i} s_m P(X_{m:n} < t)} \\ =& \sum_{j=1}^{i} s_j(t) P(X_{k:n} - t > x \mid X_{j:n} < t), \end{aligned}$$

where the function $s_j(t) = s_j P(X_{j:n} < t) / \sum_{m=1}^{i} s_m P(X_{m:n} < t)$ for $j = 1, \dots, i$. □

**Remark 3.2** Expression in (3.1) indicates that, the residual lifetime of the $k$th component, given that the system failed at time $t$, can be represented as a mixture of the residual lifetime of the $k$th component lifetime given that $n-j+1$-out-of-$n$ system has failed at time $t$ with weight coefficients $s_j(t)$ for $j = 1, \dots, i$. For instance, consider the system with lifetime $T = \min(X_{2:3}, X_4)$, whose signature is $\boldsymbol{s} = (1/4, 3/4, 0, 0)$. Given that the system has failed at some time $t$, then, by some computations, the coefficient vector is given by

$$\boldsymbol{s}(t) = (\frac{1 - \frac{3}{2}F(t) + F^2(t) - \frac{1}{4}F^3(t)}{1 + 3F(t) - 5F^2(t) + 2F^3(t)}, \frac{\frac{9}{2}F(t) - 6F^2(t) + \frac{9}{4}F^3(t)}{1 + 3F(t) - 5F^2(t) + 2F^3(t)}, 0, 0),$$

where $F(t)$ is the distribution of components lifetime. Hence, from Theorem 3.1 we have

$$\begin{aligned} P(X_{3:4} - t > x \mid T < t) =& \sum_{j=1}^{2} s_j(t) P(X_{3:4} - t > x \mid X_{j:4} < t) \\ =& \frac{1 - \frac{3}{2}F(t) + F^2(t) - \frac{1}{4}F^3(t)}{1 + 3F(t) - 5F^2(t) + 2F^3(t)}) P(X_{3:4} - t > x \mid X_{1:4} < t) + \\ & \frac{\frac{9}{2}F(t) - 6F^2(t) + \frac{9}{4}F^3(t)}{1 + 3F(t) - 5F^2(t) + 2F^3(t)} P(X_{3:4} - t > x \mid X_{2:4} < t). \end{aligned}$$

Furthermore, it is noted that $\lim_{t\to\infty} \boldsymbol{s}(t) = (1/4, 3/4, 0, 0)$ and $\boldsymbol{s}(0) = (1, 0, 0, 0)$.

**Theorem 3.3** *Let $\boldsymbol{s}(t)$ be the coefficient vector in (3.1) of a coherent system with $n$ i.i.d. components. Then, for all $0 \le t_1 \le t_2$, $\boldsymbol{s}(t_1) \le_{lr} \boldsymbol{s}(t_2)$.*

**Proof** It suffices to prove that, for all $j = 1, 2, \dots, n$ and $0 \le t_1 \le t_2$,

$$\frac{\sum_{l=j}^{n} s_l P(X_{l:n} < t_1)}{\sum_{m=1}^{n} s_m P(X_{m:n} < t_1)} \le \frac{\sum_{l=j}^{n} s_l P(X_{l:n} < t_2)}{\sum_{m=1}^{n} s_m P(X_{m:n} < t_2)}$$

which is equivalent to

$$\sum_{l=j}^{n}\sum_{m=1}^{n} s_l s_m [F_{X_{l:n}}(t_2)F_{X_{m:n}}(t_1) - F_{X_{l:n}}(t_1)F_{X_{l:n}}(t_2)] \geq 0.$$

After some algebraic computations, the above inequality can be written as

$$\sum_{l=j}^{n}\sum_{m=1}^{j-1} s_l s_m [F_{X_{l:n}}(t_2)F_{X_{m:n}}(t_1) - F_{X_{l:n}}(t_1)F_{X_{l:n}}(t_2)] \geq 0.$$

It is well known that $X_{l:n} \leq_{rh} X_{m:n}$ and for all $l \leq m$, that is $F_{X_{l:n}}(t)/F_{X_{m:n}}(t)$ is decreasing in $t \geq 0$, where $F_{X_{l:n}}$ and $F_{X_{m:n}}$ are respective distribution functions of the order statistics $X_{l:n}$ and $X_{m:n}$. So, the bracket in summation above is nonnegative. Hence the result follows immediately. □

**Theorem 3.4** *If the distribution $F$ is absolutely continuous, then for $1 \leq j < k \leq n$ and $t \geq 0$,*

$$(X_{k:n} - t \mid X_{j-1:n} < t) \geq_{lr} (X_{k:n} - t \mid X_{j:n} < t).$$

**Proof** Let $X_{j-1,k,n,(t)} = (X_{k:n} - t \mid X_{j-1:n} < t)$, $X_{j,k,n,(t)} = (X_{k:n} - t \mid X_{j:n} < t)$, $a_j(t) = 1/P(X_{j:n} < t)$. Let $f_{X_{j-1,k,n,(t)}}(x)$ and $f_{X_{j,k,n,(t)}}(x)$ denote respective density functions. Then,

$$\begin{aligned}
&P(X_{k:n} - t > x \mid X_{j:n} < t) \\
&= a_j(t)P(X_{k:n} - t > x, X_{j:n} < t) \\
&= a_j(t)\sum_{p=j}^{n}\binom{n}{p}F(t)^p \sum_{q=0}^{k-j-1}\binom{n-p}{q}(\bar{F}(t) - \bar{F}(t+x))^q \bar{F}(t+x)^{n-p-q} \\
&= a_j(t)\sum_{p=j}^{n}\binom{n}{p}F(t)^p \bar{F}(t)^{n-p} \sum_{q=0}^{k-j-1}\binom{n-p}{q}(1 - \bar{F}_t(x))^q \bar{F}_t(x)^{n-p-q} \\
&= a_j(t)\sum_{p=j}^{n}\binom{n}{p}F(t)^p \bar{F}(t)^{n-p} \int_{1-\bar{F}_t(x)}^{1}(k-j-1)\binom{n-p}{k-j-1}u^{k-j-2}(1-u)^{n-p-k+j+1}\mathrm{d}u.
\end{aligned}$$

From the above expression, it follows

$$\begin{aligned}
&f_{X_{j,k,n,(t)}}(x) \\
&= a_j(t)\sum_{p=j}^{n}\binom{n}{p}F(t)^p \bar{F}(t)^{n-p}\frac{f(t+x)}{1-F(t)}(k-j-1)\times \\
&\binom{n-p}{k-j-1}(1-\bar{F}_t(x))^{k-j-2}\bar{F}_t(x)^{n-p-k+j+1} \\
&= a_j(t)f(t+x)(1-\bar{F}_t(x))^{k-j-2}\sum_{p=j}^{n}(k-j-1)\binom{n}{p}\times \\
&\binom{n-p}{k-j-1}F(t)^p\bar{F}(t)^{n-p-1}\bar{F}_t(x)^{n-p-k+j+1} \\
&= a_j(t)(k-j-1)f(t+x)(\bar{F}(t) - \bar{F}(t+x))^{k-j-2}\sum_{p=j}^{n}\binom{n}{p}\times
\end{aligned}$$

$$\binom{n-p}{k-j-1}F(t)^p(\bar{F}(t+x))^{n-p-k+j+1}$$

In a similar manner,

$$f_{X_{j-1,k,n,(t)}}(x)$$
$$= a_{j-1}(t)(k-j)f(t+x)(\bar{F}(t)-\bar{F}(t+x))^{k-j-1}\sum_{p=j-1}^{n}\binom{n}{p}\binom{n-p}{k-j}F(t)^p(\bar{F}(t+x))^{n-p-k+j}.$$

Hence, we readily have

$$\frac{f_{X_{j,k,n,(t)}}(x)}{f_{X_{j-1,k,n,(t)}}(x)}$$
$$= \frac{a_j(t)(k-j-1)f(t+x)(\bar{F}(t)-\bar{F}(t+x))^{k-j-2}\sum_{p=j}^{n}\binom{n}{p}\binom{n-p}{k-j-1}F(t)^p(\bar{F}(t+x))^{n-p-k+j+1}}{a_{j-1}(t)(k-j)f(t+x)(\bar{F}(t)-\bar{F}(t+x))^{k-j-1}\sum_{p=j-1}^{n}\binom{n}{p}\binom{n-p}{k-j}F(t)^p(\bar{F}(t+x))^{n-p-k+j}}$$
$$= \frac{a_j(t)(k-j-1)}{a_{j-1}(t)(k-j)}\left(\frac{\bar{F}(t+x)}{\bar{F}(t)-\bar{F}(t+x)}\right)\frac{\sum_{p=j}^{n}\binom{n}{p}\binom{n-p}{k-j-1}(\frac{F(t)}{\bar{F}(t+x)})^p}{\sum_{p=j-1}^{n}\binom{n}{p}\binom{n-p}{k-j}(\frac{F(t)}{\bar{F}(t+x)})^p}$$
$$= \frac{a_j(t)(k-j-1)}{a_{j-1}(t)(k-j)}H(x\mid t).$$

Now, we only need to prove that $H(x\mid t)$ is a decreasing function of $x$. Let

$$H(x\mid t) = B(t,x)\Phi(A(t,x)),$$

where

$$\Phi(u)=\frac{\sum_{p=j}^{n}\binom{n}{p}\binom{n-p}{k-j-1}u^p}{\sum_{p=j-1}^{n}\binom{n}{p}\binom{n-p}{k-j}u^p},\quad A(t,x)=\frac{F(t)}{\bar{F}(t+x)},\ \ B(t,x)=\frac{\bar{F}(t+x)}{\bar{F}(t)-\bar{F}(t+x)}.$$

Then,

$$\frac{\partial}{\partial u}\Phi(u) = \frac{p_1\sum_{p_1=j}^{n}\binom{n}{p_1}\binom{n-p_1}{k-j-1}u^{p_1-1}(\sum_{p_2=j}^{n}\binom{n}{p_2}\binom{n-p_2}{k-j}u^{p_2}+\binom{n}{j-1}\binom{n-j+1}{k-j+1}u^{j-1})}{(\sum_{p=j-1}^{n}\binom{n}{p}\binom{n-p}{k-j}u^p)^2} - \frac{\sum_{p_1=j}^{n}\binom{n}{p_1}\binom{n-p_1}{k-j-1}u^{p_1}(p_2\sum_{p_2=j}^{n}\binom{n}{p_2}\binom{n-p_2}{k-j}u^{p_2-1}(j-1)\binom{n}{j-1}\binom{n-j+1}{k-j+1}u^{j-2})}{(\sum_{p=j-1}^{n}\binom{n}{p}\binom{n-p}{k-j}u^p)^2}.$$

The numerator of the above fraction is equal to

$$\sum_{p_1=j}^{n}\sum_{p_2=j}^{n}\binom{n}{p_1}\binom{n-p_1}{k-j-1}\binom{n}{p_2}\binom{n-p_2}{k-j}u^{p_1+p_2-1}(p_1-p_2)+$$
$$\sum_{p_1=j}^{n}\binom{n}{p_1}\binom{n-p_1}{k-j-1}\binom{n}{j-1}\binom{n-j+1}{k-j+1}u^{p_1+j-2}(p_1-j+1).$$

The second term is obviously non-negative. Hence, we just need to prove that the first term is non-negative. It is noted that,

$$\sum_{p_1=j}^{n}\sum_{p_2=j}^{n}\binom{n}{p_1}\binom{n}{p_2}\binom{n-p_1}{k-j-1}\binom{n-p_2}{k-j}u^{p_1+p_2-1}(p_1-p_2)$$

$$
\begin{aligned}
&=\sum_{p_1=j}^{n}\sum_{p_2=j}^{p_1}\binom{n}{p_1}\binom{n}{p_2}\binom{n-p_1}{k-j-1}\binom{n-p_2}{k-j}u^{p_1+p_2-1}(p_1-p_2)+\\
&\quad\sum_{p_1=j}^{n}\sum_{p_2=p_1}^{n}\binom{n}{p_1}\binom{n}{p_2}\binom{n-p_1}{k-j-1}\binom{n-p_2}{k-j}u^{p_1+p_2-1}(p_1-p_2)\\
&=\sum_{p_1=j}^{n}\sum_{p_2=j}^{p_1}\binom{n}{p_1}\binom{n}{p_2}\binom{n-p_1}{k-j-1}\binom{n-p_2}{k-j}u^{p_1+p_2-1}(p_1-p_2)+\\
&\quad\sum_{p_2=j}^{n}\sum_{p_1=j}^{p_2}\binom{n}{p_1}\binom{n}{p_2}\binom{n-p_1}{k-j-1}\binom{n-p_2}{k-j}u^{p_1+p_2-1}(p_1-p_2)\\
&=\sum_{p_1=j}^{n}\sum_{p_2=j}^{p_1}\binom{n}{p_1}\binom{n}{p_2}\binom{n-p_1}{k-j-1}\binom{n-p_2}{k-j}u^{p_1+p_2-1}(p_1-p_2)+\\
&\quad\sum_{p_1=j}^{n}\sum_{p_2=j}^{p_1}\binom{n}{p_1}\binom{n}{p_2}\binom{n-p_1}{k-j}\binom{n-p_2}{k-j-1}u^{p_1+p_2-1}(p_2-p_1)\\
&=\sum_{p_1=j}^{n}\sum_{p_2=j}^{p_1}\binom{n}{p_1}\binom{n}{p_2}u^{p_1+p_2-1}(p_1-p_2)\left[\binom{n-p_1}{k-j-1}\binom{n-p_2}{k-j}-\binom{n-p_1}{k-j}\binom{n-p_2}{k-j-1}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
&\binom{n-p_1}{k-j-1}\binom{n-p_2}{k-j}-\binom{n-p_1}{k-j}\binom{n-p_2}{k-j-1}\\
&=\frac{(n-p_1)!(n-p_2)!}{(k-j-1)!(k-j)!}\Big[\frac{1}{(n-p_1-k+j+1)!(n-p_2-k+j)!}-\\
&\quad\frac{1}{(n-p_2-k+j+1)!(n-p_1-k+j)!}\Big].
\end{aligned}
$$

From $p_1 \le p_2$, we have

$$
\frac{(n-p_2-k+j+1)!}{(n-p_1-k+j+1)!} \ge \frac{(n-p_2-k+j)!}{(n-p_1-k+j)!}.
$$

So the first term is also non-negative, thus $\Phi(u)$ is an increasing function of $u$. On the other hand, $A(t,x) = F(t)/\bar{F}(t+x)$ is a decreasing function in $x$, and so $\Phi(A(t,x))$ is a decreasing function in $x$. Also

$$
\begin{aligned}
\frac{\partial}{\partial x}B(t,x) =&\frac{-f(t+x)(F(t+x)-F(t)-f(t+x)(1-F(t+x)))}{(\bar{F}(t)-\bar{F}(t+x))^2}\\
=&\frac{-f(t+x)(1-F(t))}{(\bar{F}(t)-\bar{F}(t+x))^2} \le 0.
\end{aligned}
$$

Thus, $B(t,x)$ is a decreasing function in $x$, and then $H(x \mid t) = B(t,x)\Phi(A(t,x))$ is an increasing function in $x$. This completes the proof of the theorem. □

The following lemma is useful in proving Theorem 3.6.

**Lemma 3.5** ([27]) *Let $\alpha$ and $\beta$ be real valued functions such that $\alpha$ is nonnegative and $\alpha$, $\beta/\alpha$ are decreasing. Assume $X_i$ has the distribution function of $F_i$, $i = 1,2$. If $X_1 \le_{rh} X_2$, then*

$$
\frac{\int_{-\infty}^{\infty}\beta(x)\mathrm{d}F_1(x)}{\int_{-\infty}^{\infty}\alpha(x)\mathrm{d}F_1(x)} \ge \frac{\int_{-\infty}^{\infty}\beta(x)\mathrm{d}F_2(x)}{\int_{-\infty}^{\infty}\alpha(x)\mathrm{d}F_2(x)}.
$$

**Theorem 3.6** *Let $T_1, T_2$ be the lifetimes of two coherent structure of size $n$, both based on i.i.d components lifetimes $X_1, \ldots, X_n$ with common distribution function $F$. Assume that the two systems have signatures $s_1(t) = (s_{1,1}, \ldots, s_{1,j}, 0, \ldots, 0)$ and $s_2(t) = (s_{2,1}, \ldots, s_{2,j}, 0, \ldots, 0)$ with $s_{1,j}(t) > 0$, $s_{2,j}(t) > 0$. Let $s_1(t)$ and $s_2(t)$ be corresponding coefficient vectors as in Theorem 3.1.*

*(a) If $\boldsymbol{s}_1(t) \leq_{st} \boldsymbol{s}_2(t)$, then $(X_{k:n} - t \mid T_1 < t) \geq_{st} (X_{k:n} - t \mid T_2 < t)$;*

*(b) If $\boldsymbol{s}_1(t) \leq_{rh} \boldsymbol{s}_2(t)$, then $(X_{k:n} - t \mid T_1 < t) \geq_{hr} (X_{k:n} - t \mid T_2 < t)$;*

*(c) If $\boldsymbol{s}_1(t) \leq_{lr} \boldsymbol{s}_2(t)$, then $(X_{k:n} - t \mid T_1 < t) \geq_{lr} (X_{k:n} - t \mid T_2 < t)$.*

**Proof** (a) By Theorem 3.4, $P(X_{k:n} - t \mid X_{j:n} < t)$ is decreasing in $j, j = 1, \ldots, i$. According to (3.1), for any $t \geq 0$ and $x \geq 0$,

$$\begin{aligned} P(X_{k:n} - t > x \mid T_1 < t) =& \sum_{j=1}^{i} s_{1,j}(t) P(X_{k:n} - t > x \mid X_{j:n} < t) \\ \geq& \sum_{j=1}^{i} s_{2,j}(t) P(X_{k:n} - t > x \mid X_{j:n} < t) \\ =& P(X_{k:n} - t > x \mid T_2 < t). \end{aligned}$$

The last inequality follows from Shaked and Shanthikumar [27, 1.A.7] . This completes the proof of part (a).

(b) Let $\bar{H}_{j,k,n,(t)}(x)$ $(1 \leq j < k \leq n)$ be the reliability function of the random variable $(X_{k:n} - t \mid X_{j:n} < t)$. Then, by (3.1),

$$P(X_{k:n} - t > x \mid T_1 < t) = \sum_{j=1}^{i} s_{1,j}(t) \bar{H}_{j,k,n,(t)}(x);$$

$$P(X_{k:n} - t > x \mid T_2 < t) = \sum_{j=1}^{i} s_{2,j}(t) \bar{H}_{j,k,n,(t)}(x).$$

In order to obtain the result, we only prove

$$\frac{P(X_{k:n} - t > x \mid T_1 < t)}{P(X_{k:n} - t > x \mid T_2 < t)} = \frac{\sum_{j=1}^{i} s_{1,j}(t) \bar{H}_{j,k,n,(t)}(x)}{\sum_{j=1}^{i} s_{2,j}(t) \bar{H}_{j,k,n,(t)}(x)}$$

is increasing in $x$. That is, for all $x_2 \geq x_1 \geq 0$,

$$\frac{\sum_{j=1}^{i} s_{1,j}(t) \bar{H}_{j,k,n,(t)}(x_2)}{\sum_{j=1}^{i} s_{2,j}(t) \bar{H}_{j,k,n,(t)}(x_2)} \geq \frac{\sum_{j=1}^{i} s_{1,j}(t) \bar{H}_{j,k,n,(t)}(x_1)}{\sum_{j=1}^{i} s_{2,j}(t) \bar{H}_{j,k,n,(t)}(x_1)}.$$

Let $\alpha(j) = \bar{H}_{j,k,n,(t)}(x_1)$ and $\beta(j) = \bar{H}_{j,k,n,(t)}(x_2)$. Then by Theorem 3.4, $\alpha(j)$, $\alpha(j)/\beta(j)$ both are decreasing in $j$ for all $t \geq 0$. Hence from Lemma 3.5, the desired result follows.

(c) Let $f_{X_{j,k,n,(t)}}(x)$ be the density function of the random variable $(X_{k:n} - t \mid X_{j:n} < t)$, $\sum_{j=1}^{i} s_{1,j}(t) f_{X_{j,k,n,(t)}}(x)$ and $\sum_{j=1}^{i} s_{2,j}(t) f_{X_{j,k,n,(t)}}(x)$ be the density functions of $(X_{k:n} - t \mid T_1 < t)$ and $(X_{k:n} - t \mid T_2 < t)$, respectively. We only need to prove that

$$\frac{\sum_{j=1}^{i} s_{1,j}(t) f_{X_{j,k,n,(t)}}(x)}{\sum_{j=1}^{i} s_{2,j}(t) f_{X_{j,k,n,(t)}}(x)}$$

is increasing in $x$. Obviously, this is equivalent to, for all $x_2 \geq x_1 \geq 0$,

$$g(x_1, x_2) = \sum_{j=1}^{i}\sum_{l=1}^{i} s_{1,j}s_{2,l}\left[f_{X_{j,k,n,(t)}}(x_2)f_{X_{l,k,n,(t)}}(x_1) - f_{X_{j,k,n,(t)}}(x_1)f_{X_{l,k,n,(t)}}(x_2)\right] \geq 0.$$

Since $s_1(t) \leq_{lr} s_2(t)$ and $(X_{k:n} - t \mid X_{j-1:n} < t) \geq_{lr} (X_{k:n} - t \mid X_{j:n} < t)$,

$$\begin{aligned}
g(x_1, x_2) =& \sum_{j=1}^{i}\sum_{l=1}^{i} s_{1,j}s_{2,l}\left[f_{X_{j,k,n,(t)}}(x_2)f_{X_{l,k,n,(t)}}(x_1) - f_{X_{j,k,n,(t)}}(x_1)f_{X_{l,k,n,(t)}}(x_2)\right] \\
=& \sum_{j=1}^{i}\sum_{l=j}^{i} s_{1,j}s_{2,l}\left[f_{X_{j,k,n,(t)}}(x_2)f_{X_{l,k,n,(t)}}(x_1) - f_{X_{j,k,n,(t)}}(x_1)f_{X_{l,k,n,(t)}}(x_2)\right] + \\
& \sum_{j=1}^{i}\sum_{l=1}^{j} s_{1,j}s_{2,l}\left[f_{X_{j,k,n,(t)}}(x_2)f_{X_{l,k,n,(t)}}(x_1) - f_{X_{j,k,n,(t)}}(x_1)f_{X_{l,k,n,(t)}}(x_2)\right] \\
=& \sum_{j=1}^{i}\sum_{l=j}^{i} s_{1,j}s_{2,l}\left[f_{X_{j,k,n,(t)}}(x_2)f_{X_{l,k,n,(t)}}(x_1) - f_{X_{j,k,n,(t)}}(x_1)f_{X_{l,k,n,(t)}}(x_2)\right] + \\
& \sum_{j=1}^{i}\sum_{l=j}^{i} s_{1,l}s_{2,j}\left[f_{X_{l,k,n,(t)}}(x_2)f_{X_{j,k,n,(t)}}(x_1) - f_{X_{l,k,n,(t)}}(x_1)f_{X_{j,k,n,(t)}}(x_2)\right] \\
=& \sum_{j=1}^{i}\sum_{l=j}^{i} s_{1,j}s_{2,l}\left(\frac{s_{2,l}}{s_{1,l}} - \frac{s_{2,j}}{s_{1,j}}\right)\left[f_{X_{j,k,n,(t)}}(x_2)f_{X_{l,k,n,(t)}}(x_1) - f_{X_{j,k,n,(t)}}(x_1)f_{X_{l,k,n,(t)}}(x_2)\right] \\
\geq& 0.
\end{aligned}$$

This validates the desired result. □

Now, we show if the residual lifetimes of the components of the system are IFR (increasing failure rate), then $P(X_{k:n} - t > x \mid T < t)$ is a decreasing function of time $t$. In order to get this result, we need the following result.

**Theorem 3.7** *For all $0 \leq x \leq t$ and $1 \leq j < k \leq n$,*

$$P(X_{k:n} - t > x \mid X_{j:n} < t) = \frac{\sum_{l=j}^{n}\binom{n}{l}\phi_X(t)^l P(X_{k-l:n-l}^t > x)}{\sum_{m=j}^{n}\binom{n}{m}\phi_X(t)^m},$$

*where $\phi_X(t) = F(t)/\bar{F}(t)$ and $X_{k-l:n-l}^t$ denotes the $(k-l)$th order statistic among $(n-l)$ i.i.d random variables with common distribution function $\bar{F}_t(x) = \bar{F}(t+x)/\bar{F}(t)$.*

**Proof** For $0 \leq x \leq t$ and $1 \leq j < k \leq n$,

$$\begin{aligned}
P(X_{k:n} - t > x \mid X_{j:n} < t) =& \frac{P(X_{k:n} - t > x, X_{j:n} < t)}{P(X_{j:n} < t)} \\
=& \frac{\sum_{l=j}^{n} P(X_{k:n} - t > x, X_{l:n} < t < X_{l+1:n})}{P(X_{j:n} < t)} \\
=& \sum_{l=j}^{n} P(X_{k:n} - t > x \mid X_{l:n} < t < X_{l+1:n}) \frac{P(X_{l:n} < t < X_{l+1:n})}{P(X_{j:n} < t)}
\end{aligned}$$

$$=\sum_{l=j}^{n}\bar{H}_{k,l,l,(t)}(x)M_{l,j}^{n}(t),$$

where $\bar{H}_{k,l,l,(t)}(x)=P(X_{k:n}-t>x\mid X_{l:n}<t<X_{l+1:n})$, $M_{l,j}^{n}(t)=\frac{P(X_{l:n}<t<X_{l+1:n})}{P(X_{j:n}<t)}$. It is noted,

$$\begin{aligned}\bar{H}_{k,l,l,(t)}(x)=&P(X_{k:n}-t>x\mid X_{l:n}<t<X_{l+1:n})\\=&\frac{\binom{n}{l}F(t)^{l}\sum_{m=n-k+1}^{n-l}\binom{n-l}{m}\left[F(t+x)-F(t)\right]^{n-l-m}(1-F(t+x))^{m}}{\binom{n}{l}F(t)^{l}\bar{F}(t)^{n-l}}\\=&\sum_{m=n-k+1}^{n-l}\binom{n-l}{m}\bar{F}_{t}(x)^{m}(1-\bar{F}_{t}(x))^{n-l-m}\\=&P(X_{k-l:n-l}^{t}>x),\end{aligned}$$

and

$$\begin{aligned}M_{l,j}^{n}(t)=&\frac{P(X_{l:n}<t<X_{l+1:n})}{P(X_{j:n}<t)}=\frac{\binom{n}{l}F(t)^{l}\bar{F}(t)^{n-l}}{\sum_{m=j}^{n}\binom{n}{m}F(t)^{m}\bar{F}(t)^{n-m}}\\=&\frac{\binom{n}{l}(\phi_{X}(t))^{l}}{\sum_{m=j}^{n}\binom{n}{m}(\phi_{X}(t))^{m}}.\end{aligned}$$

So, we have

$$P(X_{k:n}-t>x\mid X_{j:n}<t)=\frac{\sum_{l=j}^{n}\binom{n}{l}\phi_{X}(t)^{l}P(X_{k-l:n-l}^{t}>x)}{\sum_{m=j}^{n}\binom{n}{m}\phi_{X}(t)^{m}}.$$

This completes the desired result. □

These two lemmas are very useful later.

**Lemma 3.8** ([25]) *If $X$ is IFR, then $P(X_{k-l:n-l}^{t}>x)$ is stochastically decreasing in $t>0$, where $X_{k-l:n-l}^{t}$ denotes the $(k-l)$th order statistic among $(n-l)$ i.i.d random variables with common distribution function $\bar{F}_{t}(x)=\bar{F}(t+x)/\bar{F}(t)$.*

**Lemma 3.9** ([25]) *For $k>l\geq m$, we have*

$$P(X_{k-l:n-l}^{t}>x)\leq_{lr}P(X_{k-m:n-m}^{t}>x),$$

*where $X_{k-m:n-m}^{t}$ denotes the $(k-m)$th order statistic among $(n-m)$ i.i.d random variables with common distribution function $\bar{F}_{t}(x)=\bar{F}(t+x)/\bar{F}(t)$.*

**Theorem 3.10** *If $X$ is IFR, then $P(X_{k:n}-t>x\mid X_{j:n}<t)$ $(j<k)$ is stochastically decreasing in $t>0$.*

**Proof** Note that $\frac{\partial\phi_{X}(t)}{\partial t}=\frac{r(t)}{\bar{F}(t)}$,

$$\begin{aligned}\frac{\partial\bar{H}_{j,k,n,(t)}(x)}{\partial t}=&\frac{\frac{r(t)}{\bar{F}(t)}l\sum_{l=j}^{n}\binom{n}{l}\phi_{X}(t)^{l-1}P(X_{k-l:n-l}^{t}>x)\sum_{m=j}^{n}\binom{n}{m}\phi_{X}(t)^{m}}{(\sum_{m=j}^{n}\binom{n}{m}\phi_{X}(t)^{m})^{2}}+\\&\frac{\sum_{l=j}^{n}\binom{n}{l}\phi_{X}(t)^{l}\frac{\partial}{\partial t}P(X_{k-l:n-l}^{t}>x)\sum_{m=j}^{n}\binom{n}{m}\phi_{X}(t)^{m}}{(\sum_{m=j}^{n}\binom{n}{m}\phi_{X}(t)^{m})^{2}}-\end{aligned}$$

$$\frac{\frac{r(t)}{\bar{F}(t)}\sum_{l=j}^{n}\binom{n}{l}\phi_X(t)^l P(X^t_{k-l:n-l}>x)m\sum_{m=j}^{n}\binom{n}{m}\phi_X(t)^{m-1}}{(\sum_{m=j}^{n}\binom{n}{m}\phi_X(t)^m)^2}.$$

The numerator of the above fraction is rewritten as

$$\frac{r(t)}{\bar{F}(t)}(C+D), \tag{3.2}$$

where

$$\begin{aligned} C =& l\sum_{l=j}^{n}\binom{n}{l}\phi_X(t)^{l-1}P(X^t_{k-l:n-l}>x)\sum_{m=j}^{n}\binom{n}{m}\phi_X(t)^m - \\ & \sum_{l=j}^{n}\binom{n}{l}\phi_X(t)^l P(X^t_{k-l:n-l}>x)m\sum_{m=j}^{n}\binom{n}{m}\phi_X(t)^{m-1}, \end{aligned}$$

and

$$D = \frac{\bar{F}(t)}{r(t)}\sum_{l=j}^{n}\binom{n}{l}\phi_X(t)^l\frac{\partial}{\partial t}P(X^t_{k-l:n-l}>x)\sum_{m=j}^{n}\binom{n}{m}\phi_X(t)^m.$$

Then it suffices to prove that the term $C$ and $D$ in (3.2) are both negative. By Lemma 3.8 and the condition that $X$ is IFR, $P(X^t_{k-l:n-l}>x)$ is decreasing in $t>0$. Thus, we have $\frac{\partial}{\partial t}P(X^t_{k-l:n-l}>x)\leq 0$ for all $t>0$, and hence $D\leq 0$. Next, we need to prove $C$ is also negative. It can be shown that,

$$\begin{aligned} C =& l\sum_{l=j}^{n}\binom{n}{l}\phi_X(t)^{l-1}P(X^t_{k-l:n-l}>x)\sum_{m=j}^{n}\binom{n}{m}\phi_X(t)^m - \\ & \sum_{m=j}^{n}\binom{n}{m}\phi_X(t)^m P(X^t_{k-m:n-m}>x)l\sum_{l=j}^{n}\binom{n}{l}\phi_X(t)^{l-1} \\ =& \sum_{l=j}^{n}\sum_{m=j}^{n}\binom{n}{l}\binom{n}{m}l\phi_X(t)^{l+m-1}\left[P(X^t_{k-l:n-l}>x)-P(X^t_{k-m:n-m}>x)\right] \\ =& \sum_{l=j}^{n}\sum_{m=j}^{l}\binom{n}{l}\binom{n}{m}l\phi_X(t)^{l+m-1}\left[P(X^t_{k-l:n-l}>x)-P(X^t_{k-m:n-m}>x)\right] + \\ & \sum_{l=j}^{n}\sum_{m=l}^{n}\binom{n}{l}\binom{n}{m}l\phi_X(t)^{l+m-1}\left[P(X^t_{k-l:n-l}>x)-P(X^t_{k-m:n-m}>x)\right]. \end{aligned}$$

By interchanging the order of summations, the second term can be represented as

$$\begin{aligned} & \sum_{m=j}^{n}\sum_{l=j}^{m}\binom{n}{l}\binom{n}{m}m\phi_X(t)^{l+m-1}\left[P(X^t_{k-m:n-m}>x)-P(X^t_{k-l:n-l}>x)\right] \\ =& \sum_{l=j}^{n}\sum_{m=j}^{l}\binom{n}{l}\binom{n}{m}m\phi_X(t)^{l+m-1}\left[P(X^t_{k-m:n-m}>x)-P(X^t_{k-l:n-l}>x)\right]. \end{aligned}$$

It follows that

$$C = \sum_{l=j}^{n}\sum_{m=j}^{l}\binom{n}{l}\binom{n}{m}(l-m)\phi_X(t)^{l+m-1}\left[P(X^t_{k-l:n-l}>x)-P(X^t_{k-m:n-m}>x)\right].$$

By Lemma 3.9, for all $l \geq m$,

$$P(X^t_{k-l:n-l} > x) - P(X^t_{k-m:n-m} > x) \leq 0.$$

Thus the desired result follows. □

**Theorem 3.11** *If $X$ is IFR, then $(X_{k:n} - t \mid T < t)$ is stochastically decreasing in $t > 0$.*

**Proof** From Theorem 3.4, we know that for all $x > 0$ and $t > 0$, $P(X_{k:n} - t > x \mid X_{j:n} < t)$ is a decreasing function of $t > 0$, and so

$$P(X_{k:n} - t > x \mid T < t) = \sum_{j=1}^{i} s_j(t) P(X_{k:n} - t > x \mid X_{j:n} < t)$$

is a decreasing function of $t$. □

## 4. Stochastic comparisons between the live components

In this section, we consider two coherent systems having the same structure with $n$ i.i.d components with their lifetimes denoted by $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. Let $F$ and $G$ be continuous distribution functions of $X_1$ and $Y_1$, respectively. Suppose $T_1$ and $T_2$ are the corresponding lifetimes of the two coherent systems. We wish to show that if $X_1 \leq_{hr} Y_1$, then the remaining components of the system with lifetime $T_1$ are more reliable than the remaining components of the system with lifetime $T_2$. For establishing this result, we first need the following lemma.

**Lemma 4.1** If $X_1 \leq_{hr} Y_1$ *and* $1 < l < k < n$, *then* $P(X^t_{k-l:n-l} > x) \leq_{st} P(Y^t_{k-l:n-l} > x)$.

**Proof** The distribution function of $P(X^t_{k-l:n-l} > x)$ can be rephrased as

$$P(X^t_{k-l:n-l} > x) = \int_{1-\bar{F}_t(x)}^{1} (k-l)\binom{n-l}{k-l} u^{k-l-1}(1-u)^{n-k+l} \mathrm{d}u.$$

The assumption $X_1 \leq_{hr} Y_1$ implies that $\bar{F}_t(x) < \bar{G}_t(x)$, so $1 - \bar{F}_t(x) > 1 - \bar{G}_t(x)$. Hence, $P(X^t_{k-l:n-l} > x) \leq P(Y^t_{k-l:n-l} > x)$. This proof is completed. □

**Theorem 4.2** If $X_1 \leq_{hr} Y_1$, *then* $(X_{k:n} - t \mid X_{j:n} < t) \leq_{st} (Y_{k:n} - t \mid Y_{j:n} < t)$.

**Proof** For $1 \leq j < k \leq n$ and $0 \leq t \leq x$, we have

$$\begin{aligned}
&P(X_{k:n} - t > x \mid X_{j:n} < t) - P(Y_{k:n} - t > x \mid Y_{j:n} < t) \\
&= \frac{\sum_{l=j}^{n} \binom{n}{l} \phi_X(t)^l P(X^t_{k-l:n-l} > x)}{\sum_{m=j}^{n} \binom{n}{m} \phi_X(t)^m} - \frac{\sum_{l=j}^{n} \binom{n}{l} \phi_Y(t)^l P(X^t_{k-l:n-l} > x)}{\sum_{m=j}^{n} \binom{n}{m} \phi_Y(t)^m} \\
&= \frac{\sum_{l=j}^{n} \binom{n}{l} \phi_X(t)^l \sum_{m=j}^{n} \binom{n}{m} \phi_Y(t)^l P(X^t_{k-l:n-l} > x)}{\sum_{m=j}^{n} \binom{n}{m} \phi_X(t)^m \sum_{m=j}^{n} \binom{n}{m} \phi_Y(t)^m} - \\
&\quad \frac{\sum_{m=j}^{n} \binom{n}{m} \phi_X(t)^m \sum_{l=j}^{n} \binom{n}{l} \phi_Y(t)^l P(Y^t_{k-l:n-l} > x)}{\sum_{m=j}^{n} \binom{n}{m} \phi_X(t)^m \sum_{m=j}^{n} \binom{n}{m} \phi_Y(t)^m} \\
&= \frac{\sum_{l=j}^{n} \binom{n}{l} \phi_X(t)^l \sum_{m=j}^{n} \binom{n}{m} \phi_Y(t)^l (P(X^t_{k-l:n-l} > x) - P(Y^t_{k-m:n-m} > x))}{\sum_{m=j}^{n} \binom{n}{m} \phi_X(t)^m \sum_{m=j}^{n} \binom{n}{m} \phi_Y(t)^m}.
\end{aligned}$$

By Lemmas 3.9 and 4.1, we know

$$P(X_{k-l:n-l}^t > x) - P(Y_{k-m:n-m}^t > x) \leq 0.$$

Hence, the last term is negative. This completes the proof. □

**Theorem 4.3** *Let $X_1, \ldots, X_n$ be the independent and identically distributed random variable with a common distribution function $F$ and let $Y_1, \ldots, Y_n$ be the independent and identically distributed random variable with a common distribution function $G$. Denote by $T_1 = \tau_1(X_1, \ldots, X_n)$ and $T_2 = \tau_2(Y_1, \ldots, Y_n)$ the lifetimes of two coherent systems with coefficient vectors $\boldsymbol{s}_1(t) = (s_{1,1}(t), \ldots, s_{1,j}(t), 0, \ldots, 0)$ and $\boldsymbol{s}_2(t) = (s_{2,1}(t), \ldots, s_{2,j}(t), 0, \ldots, 0)$, respectively. If $X_1 \leq_{hr} Y_1$, then*

$$(X_{k:n} - t \mid T_1 < t) \leq_{st} (Y_{k:n} - t \mid T_2 < t).$$

**Proof** If $X_1 \leq_{hr} Y_1$, then from Theorem 4.2, we get that

$$(X_{k:n} - t \mid X_{j:n} < t) \leq_{st} (Y_{k:n} - t \mid Y_{j:n} < t).$$

On the other hand, the two systems have the same signature while the coefficient vectors as in term (3.1) are different because of different components. Therefore, by Theorem 3.1, it is noted that

$$\begin{aligned} P(X_{k:n} - t > x \mid T_1 < t) =& \sum_{j=1}^{i} s_{1,j}(t) P(X_{k:n} - t > x \mid X_{j:n} < t) \\ \leq& \sum_{j=1}^{i} s_{1,j}(t) P(Y_{k:n} - t > x \mid Y_{j:n} < t) \\ =& \sum_{j=1}^{i} s_{2,j}(t) P(Y_{k:n} - t > x \mid Y_{j:n} < t) \\ =& P(Y_{k:n} - t > x \mid T_2 < t). \end{aligned}$$

This completes the proof of the theorem. □

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